ON A CERTAIN CLASS OF WEINGARTEN SURFACES IN *Sol* SPACE

ZLATKO ERJAVEC

ABSTRACT. In this paper a certain class of Weingarten surfaces in *Sol* geometry is considered. The theorem that the only non-planar ruled Weingarten surface composed from vertical geodesics are surfaces $r(u,v) = (ae^{ku}, be^{-ku}, v)$ is proved.

1. INTRODUCTION

A **Weingarten surface** or a W surface is a surface satisfying the Jacobi equation

$$\Phi(K,H) = \det \begin{pmatrix} K_u & K_v \\ H_u & H_v \end{pmatrix} = 0,$$

where K is Gaussian curvature and H is mean curvature of the surface.

If a surface satisfies a linear equation with respect to K and H

$$aK + bH = c,$$

$a, b, c \in \mathbb{R}$, not all zero, then the surface is called **linear Weingarten surface** or LW-surface.

It is clear that surface with constant Gauss curvature or constant mean curvature is a Weingarten surface. Therefore, Weingarten surfaces can be regarded as generalization of surfaces of constant Gauss and constant mean curvature.

The study of Weingarten surfaces was initiated by J. Weingarten in 1861. E. Beltrami and U. Dini few years later proved that the only non-developable Weingarten ruled surface in Euclidean 3-space is a helicoidal ruled surface. In the last decade several papers on Weingarten surfaces in different 3-dimensional spaces have appeared. Some results on W-surfaces can be found in [2], [3] and [8].

Motivated by the fact that there are no results about Weingarten surfaces in *Sol* geometry, we examine a class of ruled Weingarten surface in *Sol* geometry.

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The *Sol* geometry is one of the eight homogeneous Thurston 3-geometries

$$E^3, S^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \widetilde{SL(2,\mathbb{R})}, Nil, Sol.$$

More about curves and surfaces in *Sol* geometry can be found in [1], [5], [6] and [7].

In this paper we examine ruled Weingarten surfaces in *Sol* space generated by vertical geodesics (Proposition 3.1) and prove that the only non-planar surfaces of this type are surfaces $r(u,v) = (ae^{ku}, be^{-ku}, v)$ (Theorem 3.3).

2. The Sol geometry

The *Sol* geometry is a geometry of 3-dimensional *Sol* space, the space $\mathbb{R}^3$ equipped with the metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \tag{2.1}$$

As we mentioned the *Sol* geometry is one of the 3-dimensional homogeneous geometries. Generally, the Riemannian manifold (M, g) is called homogeneous if for any $x, y \in M$ there exists an isometry $\Phi : M \to M$ such that $y = \Phi(x)$. For more about other 3-dim homogeneous geometries see [9].

The *Sol* space is also a Lie group with the multiplication

$$(x, y, z) * (a, b, c) = (x + e^{-z}a, y + e^{z}b, z + c).$$

Given metric is left-invariant with respect to this operation,. It is worth to mention that in contrast to other homogeneous geometries in *Sol* geometry there are no rotations and the corresponding isometry group is 3 dimensional, the lowest dimension among homogeneous geometries.

A left orthonormal frame $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ in *Sol* is given by

$$\mathbf{e_1} = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e_2} = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e_3} = \frac{\partial}{\partial z}. \tag{2.2}$$

The Levi-Civita connection $\overline{\nabla}$ (in terms of the orthonormal frame), is given by

$$\begin{array}{lll} \overline{\nabla}_{\mathbf{e_1}}\mathbf{e_1} = -\mathbf{e_3} & \overline{\nabla}_{\mathbf{e_1}}\mathbf{e_2} = 0 & \overline{\nabla}_{\mathbf{e_1}}\mathbf{e_3} = \mathbf{e_1} \\ \overline{\nabla}_{\mathbf{e_2}}\mathbf{e_1} = 0 & \overline{\nabla}_{\mathbf{e_2}}\mathbf{e_2} = \mathbf{e_3} & \overline{\nabla}_{\mathbf{e_2}}\mathbf{e_3} = -\mathbf{e_2} \\ \overline{\nabla}_{\mathbf{e_3}}\mathbf{e_1} = 0 & \overline{\nabla}_{\mathbf{e_3}}\mathbf{e_2} = 0 & \overline{\nabla}_{\mathbf{e_3}}\mathbf{e_3} = 0. \end{array} \tag{2.3}$$

3. The Weingarten ruled surface in Sol geometry

In this section we consider ruled surface $r(u,v) = (x(u), y(u), v)$ generated by vertical geodesics $c(t) = (x_0, y_0, t)$. Unlike the usual where the investigation of surfaces in a space begins with a surface that is a graph of the function $z = z(x,y)$, here we start with other type of surface taking in account specificity of Sol metric. Even though the chosen type of cylindrical surface is perhaps the simplest to consider in *Sol*, calculations are not trivial and require the use of a computer algebra system.

First, we determine the Gauss curvature and mean curvature of the given surface.

Proposition 3.1. *The Gauss curvature K and the mean curvature H of the ruled surface $r(u,v) = (x(u), y(u), v)$ are given by*

$$K = \frac{-4x_u^2 y_u^2}{W^4}, \tag{3.1}$$

$$H = \frac{x_{uu}y_u - x_u y_{uu}}{2W^3}, \tag{3.2}$$

where $x_u = \frac{\partial x}{\partial u}$, $y_u = \frac{\partial y}{\partial u}$, $x_{uu} = \frac{\partial^2 x}{\partial u^2}$, $y_{uu} = \frac{\partial^2 y}{\partial u^2}$ and $W = \sqrt{x_u^2 e^{2v} + y_u^2 e^{-2v}}$.

Proof. The tangent vectors to the surface $r(u,v) = \big(x(u), y(u), v\big)$ in the base of ambient space *Sol* are

$$r_u = (x_u, y_u, 0) = x_u e^v \mathbf{e_1} + y_u e^{-v}\mathbf{e_2}, \quad r_v = (0,0,1) = \mathbf{e_3}.$$

The coefficients of the first fundamental form are

$$E = x_u^2 e^{2v} + y_u^2 e^{-2v}, \quad F = 0, \quad G = 1. \tag{3.3}$$

The normal vector is given by $n = \frac{1}{W}(y_u e^{-v}\mathbf{e_1} - x_u e^v \mathbf{e_2})$ and covariant derivations of tangent vectors are

$$\begin{aligned}
\overline{\nabla}_{r_u} r_u &= x_{uu}e^v\mathbf{e_1} + y_{uu}e^{-v}\mathbf{e_2} + \big(y_u^2 e^{-2v} - x_u^2 e^{2v}\big)\mathbf{e_3}\\
\overline{\nabla}_{r_u} r_v &= \overline{\nabla}_{r_u} r_v = x_u e^v \mathbf{e_1} - y_u e^{-v}\mathbf{e_2}\\
\overline{\nabla}_{r_v} r_v &= 0
\end{aligned}$$

The coefficients of the second fundamental form are

$$L = \frac{1}{W}(x_{uu}y_u - x_u y_{uu}), \quad M = \frac{2x_u y_u}{W}, \quad N = 0. \tag{3.4}$$

Therefore, knowing

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN - 2FM + GL}{2W^2},$$

we obtain the equations (3.1) and (3.2). $\square$

Remark 3.2. In [7] the author investigated minimal ruled surfaces in *Sol* geometry and obtained the same expressions for the first and the second fundamental form. In the same paper he also proved the following statement: "The ruled minimal surfaces composed from vertical geodesics are the surfaces of the form $r(s,t) = (as+b, s, t)$ or $r(s,t) = (s, as+b, t)$."

Next, we give a characterization of Weingarten surfaces of type $r(u,v) = \big(x(u), y(u), v\big)$ in *Sol* geometry.

Theorem 3.3. *A ruled surface $r(u,v) = \big(x(u), y(u), v\big)$ in Sol space is a Weingarten surface if it is either*

(1) *a plane parallel to the z axis*
(2) *a cylindrical surface $r(u,v) = (ae^{ku}, be^{ku}, v), \quad a, b, k \in \mathbb{R}$.*

Proof. Using Proposition 3.1 we have:

$$\begin{aligned}
K_u &= \frac{8x_u y_u}{W^6}(x_{uu}y_u - x_u y_{uu})(x_u^2 e^{2v} - y_u^2 e^{-2v}),\\
K_v &= \frac{16x_u^2 y_u^2}{W^6}(x_u^2 e^{2v} - y_u^2 e^{-2v}),\\
H_u &= \frac{1}{2W^5}\big[W^2(x_{uuu}y_u - x_u y_{uuu}) - 3(x_{uu}y_u - x_u y_{uu})(x_u x_{uu} e^{2v} + y_u y_{uu} e^{-2v})\big],\\
H_v &= \frac{-3}{2W^5}(x_{uu}y_u - x_u y_{uu})(x_u^2 e^{2v} - y_u^2 e^{-2v}).
\end{aligned}$$

¿From the condition $K_u H_v - K_v H_u = 0$, it follows

$$8x_u y_u (x_u^4 e^{4v} - y_u^4 e^{-4v})\big[3(x_u^2 y_{uu}^2 - x_{uu}^2 y_u^2) + 2x_u y_u (x_{uuu}y_u - x_u y_{uuu})\big] = 0. \quad (3.5)$$

Further, from the equation (3.5) we have two cases:

Case 1. Suppose $x_u y_u = 0$. Hence $x(u) = c_1$ or $y(u) = c_2,\ c_1, c_2 \in \mathbb{R}$.

Corresponding surfaces are planes parallel to the xz-plane and yz-plane with equations

$$r(u,v) = \big(c_1, y(u), v\big) \quad \text{and} \quad r(u,v) = \big(x(u), c_2, v\big), \quad \text{respectively.}$$

These planes represent minimal totally geodesics surfaces in *Sol* space ($K = 0$ and $H = 0$).

Case 2. Suppose $x_u y_u \neq 0$.

Since $x_u y_u \neq 0$ it follows $(x_u^4 e^{4v} - y_u^4 e^{-4v}) \neq 0$ $(\forall v \neq 0)$ and hence must be

$$3(x_u^2 y_{uu}^2 - x_{uu}^2 y_u^2) + 2x_u y_u (x_{uuu} y_u - x_u y_{uuu}) = 0 \tag{3.6}$$

or equivalently

$$3(x_u y_{uu} - x_{uu} y_u)(x_u y_{uu} + x_{uu} y_u) + 2x_u y_u (x_{uu} y_u - x_u y_{uu})' = 0 \tag{3.7}$$

This equation is obviously satisfied for $x_u y_{uu} - x_{uu} y_u = 0$.

¿From $x_u y_{uu} - x_{uu} y_u = 0$ we have $\frac{y_{uu}}{y_u} = \frac{x_{uu}}{x_u}$ which after integration give $\ln y_u = \ln(a x_u)$. After some manipulations and the second integration we obtain

$$y(u) = a x(u) + b, \; a, b \in \mathbb{R}. \tag{3.8}$$

The obtained surface is a plain parallel to the z axis

$$r(u, v) = \big(x(u), a x(u) + b, v\big),$$

and represents a ruled minimal surface for which hold $H = 0$ and $K = \frac{-4a^2 e^{4v}}{(a^2 + e^{4v})^2} \neq \text{const}$.

Further, we could say that the equation (3.6) is satisfied for

$$(x_u^2 y_{uu}^2 - x_{uu}^2 y_u^2) = 0 \quad \text{and} \quad (x_{uuu} y_u - x_u y_{uuu}) = 0,$$

or equivalently

$$(x_u y_{uu} - x_{uu} y_u)(x_u y_{uu} + x_{uu} y_u) = 0 \tag{3.9}$$

$$\text{and} \quad \frac{\partial}{\partial u}(x_u y_{uu} - x_{uu} y_u) = 0. \tag{3.10}$$

Remark 3.4. We point out that we don't have a classification of given type of surfaces because we didn't prove equivalence of the equation (3.6) with equations (3.9) and (3.10).

¿From the equation (3.9) we again have two cases:

Case 2a.

We have already examined the case when the equation $x_u y_{uu} - x_{uu} y_u = 0$ implies the condition (3.10). Remember that we have obtained planes parallel to the z axis.

Case 2b.

¿From $x_u y_{uu} + x_{uu} y_u = 0$ we have $\frac{y_{uu}}{y_u} = -\frac{x_{uu}}{x_u}$ which after integration give

$$x_u y_u = \text{const.} \tag{3.11}$$

On the other hand, if we insert $x_u y_{uu} = -x_{uu} y_u$ in $\frac{\partial}{\partial u}(x_u y_{uu} - x_{uu} y_u) = 0$, we have

$$x_{uu} y_u = \text{const} \tag{3.12}$$

Combining the equations (3.11) and (3.12), it follows $x_{uu} = k x_u,\ k \in \mathbb{R}$. Solving this differential equation we obtain $x(u) = a e^{ku}$.
Hence $y(u) = b e^{-ku}$ and finally

$$r(u,v) = \big(a e^{ku}, b e^{-ku}, v\big),\ \ a, b, k \in \mathbb{R}. \tag{3.13}$$

□

Figure 1 shows ruled Weingarten surface (3.13) for $a = b = k = 1$.

FIGURE 1. $r(u,v) = \big(e^u, e^{-u}, v\big)$

Corollary 3.5. *Planes parallel to the xz-plane or yz-plane are only linear Weingarten surfaces of type $r(u,v) = \big(x(u), y(u), v\big)$ in Sol space.*

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University of Zagreb, Faculty of Organization and Informatics, Pavlinska 2, HR-42000 Varaždin, Croatia
E-mail address: zlatko.erjavec@foi.hr