On certain classes of Weingarten surfaces in *Sol* space

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Introduction

A **Weingarten surface** or a W surface is a surface satisfying the Jacobi equation

$$\Phi(K,H) = \det \begin{pmatrix} K_u & K_v \\ H_u & H_v \end{pmatrix} = 0,$$

where K is Gaussian curvature and H is mean curvature of the surface. If a surface satisfies a linear equation with respect to K and H $aK + bH = c$, $a, b, c \in \mathbb{R}$, not all zero, then the surface is called **linear Weingarten surface** or LW-surface. It is clear that surface with constant Gauss curvature or constant mean curvature is a Weingarten surface. Therefore, Weingarten surfaces can be regarded as generalization of surfaces of constant Gauss and constant mean curvature.

The study of Weingarten surfaces was initiated by J. Weingarten in 1861. E. Beltrami and U. Dini few years later proved that the only non-developable Weingarten ruled surface in Euclidean 3-space is a helicoidal ruled surface. In the last decade several papers on Weingarten surfaces in different 3-dimensional spaces have appeared. Some results on W-surfaces can be found in [2], [3] and [8].

Motivated by the fact that there are no results about Weingarten surfaces in *Sol* geometry, we examine two classes of ruled Weingarten surface in *Sol* geometry. The *Sol* geometry is one of the eight homogeneous Thurston 3-geometries

$$E^3,\ S^3,\ H^3,\ S^2\times\mathbb{R},\ H^2\times\mathbb{R},\ \widetilde{SL(2,\mathbb{R})},\ Nil,\ Sol.$$

More about curves and surfaces in *Sol* geometry can be found in [1], [5], [6] and [7].

The *Sol* geometry

The *Sol* geometry is a geometry of 3-dimensional *Sol* space, the space $\mathbb{R}^3$ equipped with the metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

As we mentioned the *Sol* geometry is one of the 3-dimensional homogeneous geometries. Generally, the Riemannian manifold (M,g) is called homogeneous if for any $x, y \in M$ there exists an isometry $\Phi : M \to M$ such that $y = \Phi(x)$. The *Sol* space is also a Lie group with the multiplication

$$(x,y,z) * (a,b,c) = (x + e^{-z}a,\, y + e^{z}b,\, z + c).$$

Given metric is left-invariant with respect to this operation,. It is worth to mention that in contrast to other homogeneous geometries in *Sol* geometry there are no rotations and the corresponding isometry group is 3 dimensional, the lowest dimension among homogeneous geometries. A left orthonormal frame $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ in *Sol* is given by

$$\mathbf{e_1} = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e_2} = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e_3} = \frac{\partial}{\partial z}.$$

The Levi-Civita connection $\overline{\nabla}$ (in terms of the orthonormal frame), is given by

$$\begin{array}{lll}
\overline{\nabla}_{\mathbf{e_1}}\mathbf{e_1} = -\mathbf{e_3} & \overline{\nabla}_{\mathbf{e_1}}\mathbf{e_2} = 0 & \overline{\nabla}_{\mathbf{e_1}}\mathbf{e_3} = \mathbf{e_1} \\
\overline{\nabla}_{\mathbf{e_2}}\mathbf{e_1} = 0 & \overline{\nabla}_{\mathbf{e_2}}\mathbf{e_2} = \mathbf{e_3} & \overline{\nabla}_{\mathbf{e_2}}\mathbf{e_3} = -\mathbf{e_2} \\
\overline{\nabla}_{\mathbf{e_3}}\mathbf{e_1} = 0 & \overline{\nabla}_{\mathbf{e_3}}\mathbf{e_2} = 0 & \overline{\nabla}_{\mathbf{e_3}}\mathbf{e_3} = 0.
\end{array}$$

The Weingarten surfaces of type $r(u,v) = (x(u), y(u), v)$

Proposition

The Gauss curvature K and the mean curvature H of the ruled surface $r(u,v) = (x(u), y(u), v)$ are given by

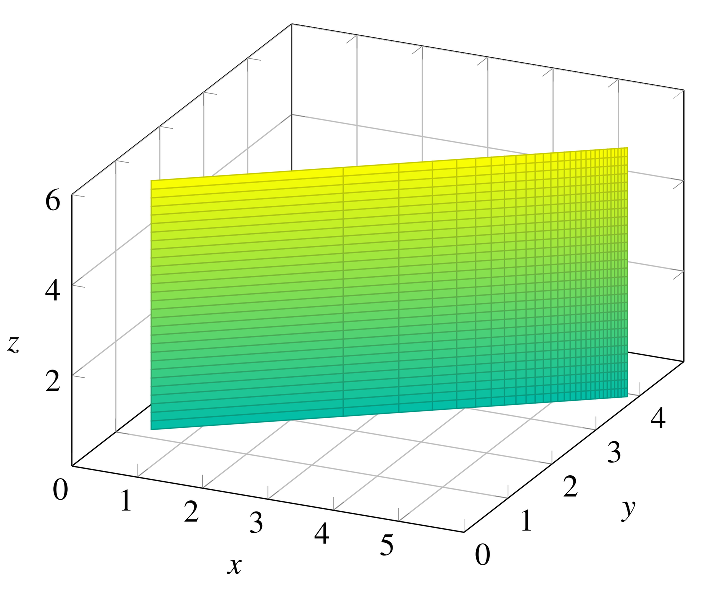
$$K = \frac{-4x_u^2y_u^2}{W^4}, \qquad H = \frac{x_{uu}y_u - x_uy_{uu}}{2W^3}$$

where $x_u = \frac{\partial x}{\partial u}$, $y_u = \frac{\partial y}{\partial u}$, $x_{uu} = \frac{\partial^2 x}{\partial u^2}$, $y_{uu} = \frac{\partial^2 y}{\partial u^2}$ *and* $W = \sqrt{x_u^2e^{2v} + y_u^2e^{-2v}}$.

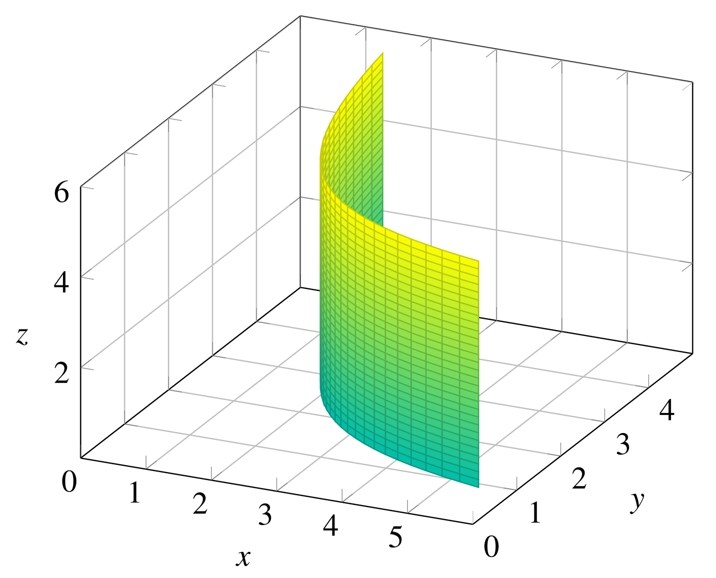
Theorem

A ruled surface $r(u,v) = \big(x(u), y(u), v\big)$ in Sol space is a Weingarten surface if it is either

- *a plane parallel to the z axis*
  - $r(u,v) = \big(a, y(u), v\big)$, $\quad r(u,v) = \big(x(u), b, v\big)$, $\quad a, b \in \mathbb{R}$
  - $r(u,v) = \big(x(u), ax(u) + b, v\big)$, $\quad a, b \in \mathbb{R}$
- *a cylindrical surface $r(u,v) = (ae^{ku}, be^{-ku}, v)$,* $\quad a, b, k \in \mathbb{R}$.



$r(u,v) = \big(\ln u,\ \frac{1}{2}\ln u + 1,\ v\big)$



$r(u,v) = \big(e^u,\ 5e^{-u},\ v\big)$

The Weingarten surfaces of type $r(u,v) = (u, v, f(u,v))$

Proposition

The Gauss curvature K and the mean curvature H of the surface $r(u,v) = (u, v, f(u,v))$ are given by

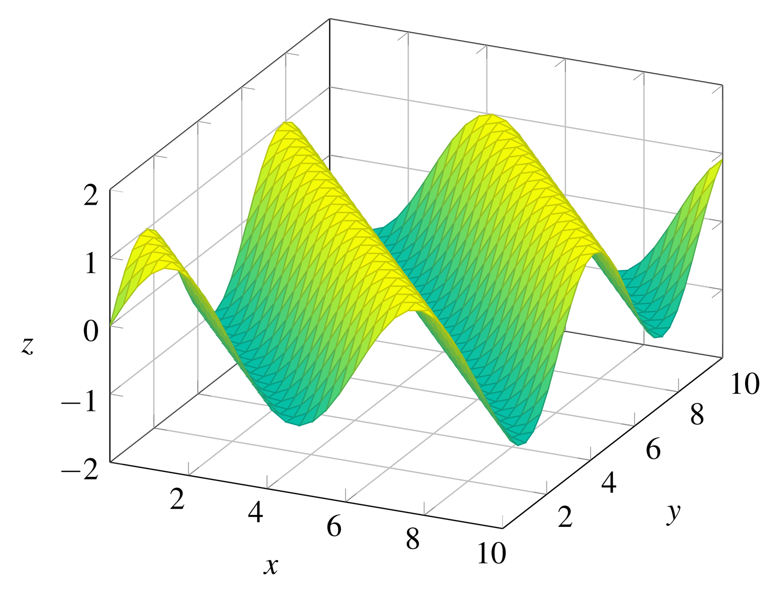
$$K = \frac{(-2f_u^2 + f_{uu} - e^{2f})(2f_v^2 + f_{vv} + e^{-2f}) - f_{uv}^2}{W^4},$$

$$H = \frac{f_{uu}(f_v^2 + e^{-2f}) + f_{vv}(f_u^2 + e^{2f}) + f_v^2e^{2f} - f_u^2e^{-2f} - 2f_uf_vf_{uv}}{2W^3}$$

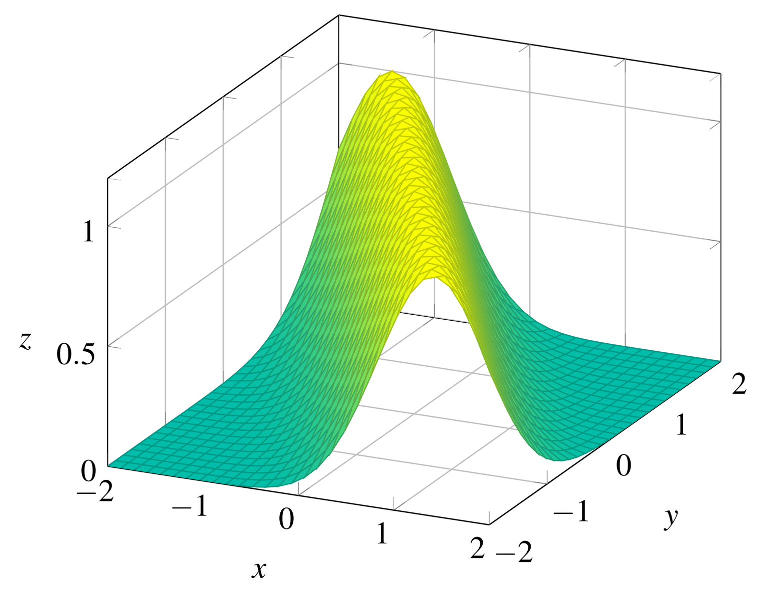
where $W = \sqrt{1 + e^{2f}f_v^2 + e^{-2f}f_u^2}$, $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, $f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{vv} = \frac{\partial^2 f}{\partial v^2}$ *and* $f_{uv} = \frac{\partial^2 f}{\partial u \partial v}$.

Proposition

Suppose that $g(u,v) = au + bv + c$ for some $a, b, c \in \mathbb{R}$. If h is smooth real function of real variable such that $f = h \circ g$ is defined, then $r(u,v) = (u, v, f(u,v))$ is a Weingarten surface in Sol space.



$r(u,v) = \big(u,\ v,\ \sin(u+v)\big)$



$r(u,v) = \big(u,\ v,\ 2^{-(1.4u+v)^2}\big)$

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