Harmonic evolutes of timelike ruled surfaces in Minkowski space

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Minkowski 3-space

Definition

A Minkowski 3-space is a real affine space whose underlying vector space $\mathbb{R}^3$ is endowed with a pseudo-scalar product, that is, with a non-degenerate indefinite symmetric bilinear form.

If $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, we define this form by

$$< x, y >_1 := -x_1 y_1 + x_2 y_2 + x_3 x_3 \tag{1}$$

and denote the vector space by $\mathbb{R}^3_1$.

Harmonic evolute in $\mathbb{R}^3_1$

Definition

The harmonic evolute of a surface S is the locus of points $\bar{p}$ which are harmonic conjugates of a point $p \in S$ with respect to centers of curvature p_1, p_2 of S

$$(p_1, p_2; p, \bar{p}) = \frac{p_1 p \cdot p_2 \bar{p}}{p_2 p \cdot p_1 \bar{p}} = -1. \tag{2}$$

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- The harmonic evolute of a surface S can be parametrized by

$$\bar{\mathbf{f}}(u, v) = \mathbf{f}(u, v) + \frac{\epsilon}{H(u, v)} n(u, v), \quad \epsilon \in \{1, -1\}. \tag{3}$$

Ruled surface in $\mathbb{R}^3_1$

Definition

A ruled surface in 3-dimensional Minkowski space $\mathbb{R}^3_1$ is a surface parametrized by

$$\boldsymbol{f}(u, v) = c(u) + ve(u), \tag{4}$$

where $c(u)$ is a base curve and $e(u)$ a non-vanishing vector field along c which generates the rulings.

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1. skew ruled surfaces $\quad (K \neq 0)$
2. developable surfaces $\quad (K = 0)$

Developable surfaces

- one principal curvature is 0, therefore the corresponding center of curvature is a point at infinity
- the point of a harmonic evolute is a symmetric point to $p \in S$ with respect to the remaining center of curvature p_1 (p_1 is the mid-point)
- developable surface are divided into cylindrical surfaces, conical surfaces and tangent surfaces

Cylindrical surfaces

Definition

A ruled surface is called a cylindrical surface if it can be parameterized by

$$\mathbf{f}(u,v) = c(u) + ve, \quad e \in \mathbb{R}^3_1, \quad e^2 = \epsilon = \{1,-1\}, \quad c' \cdot e = 0, \tag{5}$$

where $c(u)$ is base curve.

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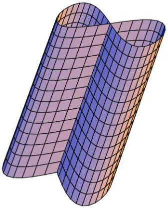

The mean curvature of cylindrical surfaces are given by

$$H(u) = \frac{\epsilon\, k_1}{2} \qquad (6)$$

where k_1 is the principal curvature.

Cylindrical surfaces

The harmonic evolute of cylindrical surface S is given by

$$\bar{\mathbf{f}}(u,v) = (c(u) + \frac{2}{k_1(u)}N(u)) + ve \tag{7}$$

where $N(u)$ is the principal normal of the curve $c(u)$.

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Proposition

The harmonic evolute of cylindrical surface in Minkowski space is again a cylindrical surface with parallel rulings.

Conical surfaces

Definition

A ruled surface is called a conical surface if it can be parameterized by

$$\mathbf{f}(u, v) = p + ve(u), \quad e \cdot e' = 0, \tag{8}$$

where $p \in \mathbb{R}^3_1$ is fixed (it can be interpreted as the vertex of the cone).

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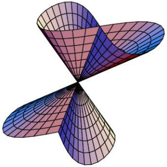

The mean curvature of conical surfaces are given by

$$H(u) = \frac{-\epsilon\, det(e, e', e'')}{2|v|\|e'\|^3}. \tag{9}$$

Conical surfaces

The harmonic evolute of a conical surface S is given by

$$\bar{\mathbf{f}}(u,v) = p + v(e(u) + \frac{2\|e'\|^2}{det(e,e',e'')} \cdot e(u) \times_1 e'(u)), \tag{10}$$

where $e(u) \times_1 e'(u)$ is the cross product in Minkowski space.

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where $e(u) \times_1 e'(u)$ is the cross product in Minkowski space.

Proposition

The harmonic evolute of conical surface in Minkowski space is again a conical surface with the same vertex.

Tangent surfaces

Definition

A ruled surface is called a tangent surface if it can be parameterized by

$$\mathbf{f}(u,v) = c(u) + vc'(u), \quad \|c'(u)\| = \epsilon = \{1,-1\} \quad c'(u)\cdot c''(u) = 0, \tag{11}$$

where $p \in \mathbb{R}^3_1$ is fixed (it can be interpreted as the vertex of the cone).

- the mean curvature of tangent surfaces are given by

$$H(u) = -(\operatorname{sing} v)\frac{\epsilon\,\tau(u)}{2v\kappa(u)}B(u) \quad \epsilon = \{1,-1\} \tag{12}$$

Tangent surfaces

The harmonic evolute of tangent surface S is given by

$$\bar{\mathbf{f}}(u,v) = c(u) + v\left(c'(u) + \frac{\kappa(u)}{\tau(u)}B(u)\right), \tag{13}$$

where $\kappa(u)$ is the flexion, $\tau(u)$ is the torsion of $c(u)$ and $B(u)$ is the binormal of the curve c.

Tangent surfaces

The harmonic evolute of S is again a ruled surface with rulings

$$\bar{e}(u) = c'(u) + \frac{\epsilon\kappa(u)}{\tau(u)} B(u). \tag{14}$$

Because the parameter of distribution of $\bar{S}$ does not vanish since

$$det(c', \bar{e}, \bar{e}') = -\epsilon\frac{2\kappa(u)^2}{\tau(u)} \neq 0, \quad \epsilon = \{1, -1\}, \tag{15}$$

the harmonic evolute is skew ruled surface.

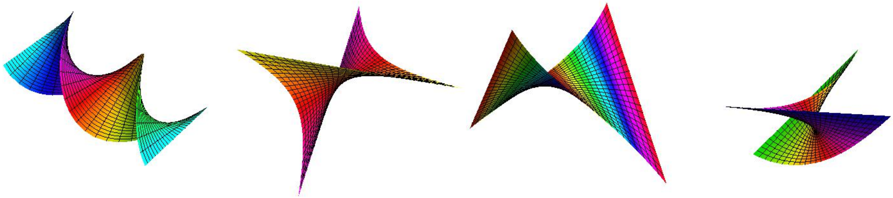
Theorem

The harmonic evolute of a developable surface in Minkowski 3-space is a ruled surface.
In particular, harmonic evolutes of cylindrical and conical surfaces are developable surfaces, cylindrical and conical respectively, whereas harmonic evolutes of tangent surfaces are skew ruled surfaces.

Example

Minimal surfaces

Harmonic evolute of a minimal surface degenerates to a point at infinity.



Example

The harmonic evolute of a regular surface in Euclidean space is always a surface or a point (for spheres = totally umbilical surface).

But in Minkowski space it can also be a curve (for quasi-umbilical surfaces).

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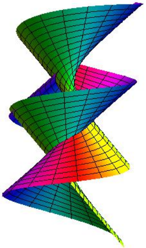
But in Minkowski space it can also be a curve (for quasi-umbilical surfaces).



Example

A harmonic evolute of a ruled helicoidal surface given by this parametrization

$$\mathbf{f}(u, v) = (hu + v, v\cos u, v\sin u) \quad h > 0 \qquad (16)$$

degenerates to a curve (helix).

The first case

- suppose that the axis of revolution is a z-axis

$$\begin{pmatrix} \operatorname{ch} v & \operatorname{sh} v & 0 \\ \operatorname{sh} v & \operatorname{ch} v & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

- without loss of generality, we may assume that the curve c is parametrized by the arc-length and is lying in the yz-plane or in the xz-plane

- the curve c is parametrized either by

$$c(u) = (0, f(u), g(u)) \quad \text{or} \quad c(u) = (f(u), 0, g(u)) \tag{18}$$

where $f(u)$ is a positive function of class C^1 and $g(u)$ is a function of class C^2 on $I = \langle a, b \rangle$

The helicoidal surfaces can be parametrized by

Type I

$$\mathbf{f}(u,v) = (f(u)\,\mathrm{sh}\,v, f(u)\,\mathrm{ch}\,v, g(u) + c\,v), \quad f(u) > 0, \quad c \in \mathbb{R}^+$$

or

Type II

$$\mathbf{f}(u,v) = (f(u)\,\mathrm{ch}\,v, f(u)\,\mathrm{sh}\,v, g(u) + c\,v), \quad f(u) > 0, \quad c \in \mathbb{R}^+$$

- The harmonic evolute of a helicoidal surface *type* I is given by

$$\bar{\mathbf{f}}(u,v) = \begin{pmatrix} \operatorname{ch} v & \operatorname{sh} v & 0 \\ \operatorname{sh} v & \operatorname{ch} v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\epsilon\, c f'(u)}{H(u)W(u)} \\ f(u)\left(1 + \frac{\epsilon\, g'(u)}{H(u)W(u)}\right) \\ g(u) - \frac{\epsilon\, f(u) f'(u)}{H(u)W(u)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c\, v \end{pmatrix} \tag{19}$$

where $W(u)^2 = -c^2 f'(u)^2 + f(u)^2$, $\epsilon = \{1, -1\}$ and $H(u)$ is the mean curvature.

- The harmonic evolute of a helicoidal surface *type* II is given by

$$\bar{\mathbf{f}}(u,v) = \begin{pmatrix} \operatorname{ch} v & \operatorname{sh} v & 0 \\ \operatorname{sh} v & \operatorname{ch} v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(u)\left(1 + \frac{\epsilon\, g'(u)}{H(u)W(u)}\right) \\ -\frac{\epsilon\, c f'(u)}{H(u)W(u)} \\ g(u) + \frac{\epsilon \cdot f(u) f'(u)}{H(u)W(u)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c\,v \end{pmatrix} \tag{20}$$

where $W(u)^2 = c^2 f'(u)^2 - \epsilon\, f(u)^2$, $\epsilon = \{1, -1\}$ and $H(u)$ is the mean curvature.

The secand case

- suppose that the axis of revolution is the x-axis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix} \tag{21}$$

- without loss of generality we may assume that the curve c is parametrized by the arc-length and is lying in the xy-plane

- The curve c is parametrized either by

$$c(u) = (g(u), f(u), 0), \tag{22}$$

where $f(u)$ is a positive function of class C^1 and $g(u)$ is a function of class C^2 on $I = \langle a, b\rangle$.

The secand case

Type III

$$\mathbf{f}(u,v) = (g(u) + c\,v, f(u)\cos v, f(u)\sin v,), \quad f(u) > 0 \quad c \in \mathbb{R}^+$$

The harmonic evolute of a helicoidal surface *type* III is given by

$$\bar{\mathbf{f}}(u,v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix} \begin{pmatrix} g(u) - \frac{\epsilon\, f(u) f'(u)}{H(u)W(u)} \\ f(u)(1 - \frac{\epsilon\, g'(u)}{H(u)W(u)}) \\ -\frac{\epsilon\, c f'(u)}{H(u)W(u)} \end{pmatrix} + \begin{pmatrix} c\,v \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

where $W(u)^2 = -\epsilon f(u)^2 + c^2 f'(u)^2$, $\epsilon = \{1, -1\}$ and $H(u)$ is the mean curvature.

The third case

- Suppose that the axis of revolution is the line spanned by (1, 1, 0)

$$\begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \tag{24}$$

- Since the surface S is non-degenerate, we can assume without loss of generality that the curve c is parametrized by the arc-length and is lying in the xy-plane.

- The curve c is parametrized either by

$$c(u) = (f(u), g(u), 0), u \in I \tag{25}$$

where $f(u)$ and $g(u)$ are functions on I, such that $f(u) \neq g(u)$, for each $u \in I$.

The third case

Type IV

$$\mathbf{f}(u,v) = ((1+\frac{v^2}{2})f(u) - \frac{v^2}{2}g(u) + v\,c,\ \frac{v^2}{2}f(u) + (1-\frac{v^2}{2})g(u) + v\,c,\ (f(u)-g(u))v) \tag{26}$$

The harmonic evolute a helicoidal surface of *type* IV is given by

$$\bar{\mathbf{f}}(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} f(u) + \frac{\epsilon(-g'(u)f(u)+g'(u)g(u))}{H(u)W(u)} \\ g(u) + \frac{\epsilon(-f'(u)f(u)+f'(u)g(u)}{H(u)W(u)} \\ \frac{-c\,\epsilon}{H(u)W(u)} \end{pmatrix} + \begin{pmatrix} c\,v \\ c\,v \\ 0 \end{pmatrix} \tag{27}$$

where $W(u)^2 = (f(u)+g(u))^2 - \epsilon^2(v(f(u)-g(u))+c)^2$, and $H(u)$ is the mean curvature.

Theorem

The harmonic evolute of a helicoidal surface is a coaxial helicoidal surface.

Special case of helicoidal surfaces are rotation surfaces when is $c = 0$.

Proposition

The harmonic evolute of a rotation surface is a coaxial rotation surface.

Example

A time-like helicoidal surface

Red is the a time-like helicoidal surface and green is its harmonic evolute.

$$\mathbf{f}(u,v) = (u, uv, -\frac{uv^2}{2} + v), \quad u > 0, \quad v \in \mathbb{R} \tag{28}$$

$$\bar{\mathbf{f}}(u,v) = (u + \frac{uv^2 - 2v}{2H}, uv + \frac{uv-1}{H}, v - \frac{uv^2}{2} + \frac{u}{H}), \tag{29}$$

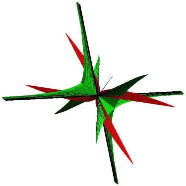
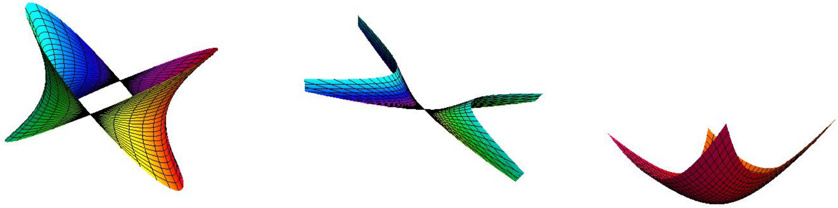


Figure 6. Time-like helicoidal surface of its harmonic evolute

Example

Spheres

The harmonic evolutes of spheres degenerate to a point.

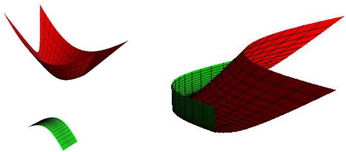


Example

Torus

Red is the torus and green is its harmonic evolute.

In Euclidean space, we have the following result.



Theorem

The harmonic evolute of a ring-torus is a rotation quadratic surface if and only $R = \sqrt{2}r$ *(*$R > 0$ *radius of the central circle and* r *radius of a meridian circle).*

THANK YOU FOR YOUR ATTENTION!