Curve acceleration in hyperboloid model of SL(2,R)

Ambient space

Coordinates in ambient space - INPUT

In[•]:=
```
u₁ := r
u₂ := θ
u₃ := ϕ
```

Metric of ambient space - INPUT

In[•]:=
```
g[r_, θ_, ϕ_] :=
  {{1, 0, 0}, {0, Sinh[r]^2 * (Sinh[r]^2 + Cosh[r]^2), Sinh[r]^2}, {0, Sinh[r]^2, 1}}
```

In[•]:=
```
g[u₁, u₂, u₃] // MatrixForm // Simplify
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cosh}[2 r]\,\text{Sinh}[r]^2 & \text{Sinh}[r]^2 \\ 0 & \text{Sinh}[r]^2 & 1 \end{pmatrix}$$

In[•]:=
```
INVg := Inverse[g[u₁, u₂, u₃]]
```

Orthonormal frame field in ambient space - INPUT

In[•]:=
```
e₁[r] := {1, 0, 0}
e₂[r] := {0, 1/(Cosh[r] * Sinh[r]), -Tanh[r]}
e₃[r] := {0, 0, 1}
```

Scalar product in ambient space

In[•]:=
```
SP[v1_, v2_] := Sum[g[u₁, u₂, u₃][[i, j]] * v1[[i]] * v2[[j]], {i, 1, 3}, {j, 1, 3}]
```

Christoffel symbols $\Gamma_{i,j}^{\{k\}}$:=

$$\frac{1}{2}\sum_{l=1}^{3}\left(\partial_{u_i} g_{j,l} + \partial_{u_j} g_{l,i} - \partial_{u_l} g_{i,j}\right) h_{l,k}$$

```
In[•]:= Γ1[u1, u2, u3] :=
  Table[1/2 Sum[(∂_{u_i} g[u1, u2, u3][[j, l]] + ∂_{u_j} g[u1, u2, u3][[l, i]] - ∂_{u_l} g[u1, u2, u3][[i, j]])
      INVg[[l, 1]], {l, 1, 3}], {i, 1, 3}, {j, 1, 3}] // Simplify

In[•]:= Γ2[u1, u2, u3] :=
  Table[1/2 Sum[(∂_{u_i} g[u1, u2, u3][[j, l]] + ∂_{u_j} g[u1, u2, u3][[l, i]] - ∂_{u_l} g[u1, u2, u3][[i, j]])
      INVg[[l, 2]], {l, 1, 3}], {i, 1, 3}, {j, 1, 3}] // Simplify

In[•]:= Γ3[u1, u2, u3] :=
  Table[1/2 Sum[(∂_{u_i} g[u1, u2, u3][[j, l]] + ∂_{u_j} g[u1, u2, u3][[l, i]] - ∂_{u_l} g[u1, u2, u3][[i, j]])
      INVg[[l, 3]], {l, 1, 3}], {i, 1, 3}, {j, 1, 3}] // Simplify
```

Basic fields

```
In[•]:= w1 := {1, 0, 0}
        w2 := {0, 1, 0}
        w3 := {0, 0, 1}
```

Transformation matrices - canonical base $\{\partial_i\}$ vs orthonormal frame $\{e_i\}$

In[•]:= AC :=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\text{Cosh}[r]*\text{Sinh}[r]} & -\text{Tanh}[r] \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[•]:= CA = Inverse[AC]
Out[•]= {{1, 0, 0}, {0, Cosh[r] Sinh[r], Sinh[r]^2}, {0, 0, 1}}
```

covariant derivations of orthonorml fields ---- CD[i, j] = $\nabla_{e_i} e_j$

In[•]:= CD[i_, j_] :=

$$\sum_{l=1}^{3}\sum_{m=1}^{3}\left(e_i[r][[l]]\left(D\left[e_j[r][[m]], u_l\right] w_m + e_j[r][[m]]\left(\sum_{n=1}^{3}\Gamma_n[u_1, u_2, u_3][[l, m]]\, w_n\right)\right)\right)$$

Levi-Civita connection in SL(2,R) ambiant space

In[•]:=
```
Table[CD[i, j].CA, {i, 1, 3}, {j, 1, 3}] // MatrixForm // Simplify
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \text{Coth[r]} + \text{Tanh[r]} \\ 1 \end{pmatrix} & \begin{pmatrix} -\text{Cosh[2 r] Csch[r] Sech[r]} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

covariant derivation of vectors given in base of ambient space $\{e_i\}$

MAIN FORMULA

In[•]:=
```
CovDer[w1_, w2_] :=
```

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \text{w1[[i]]} * \left(e_i[r][[i]] * D[\text{w2[[j]]}, u_i]\, e_j[r].CA + \text{w2[[j]]} \times CD[i, j].CA\right)$$

CURVE ACCELERATION

$$\gamma(t) = r(t)\, e_1 + \theta(t)\, e_2 + \phi(t)\, e_3 \Longrightarrow \gamma'(t) = r'\, e_1 + \theta' \sinh r \cosh r\, e_2 + \left(\theta' \sinh^2 r + \phi'\right) e_3$$

$\nabla_{\gamma'}\gamma' = ?$

Example11 - $\nabla_{r' * e_1}(r' * e_1) = r''\, e_1$

In[•]:=
```
CovDer[{r', 0, 0}, {r', 0, 0}] // FullSimplify
```

Out[•]= {0 & r′, 0, 0}

Example12 - $\nabla_{r' * e_1}(\theta' * \sinh r \cosh r\, e_2) = -r'\, \theta' \sinh r \cosh r\, e_3 + r'\theta' \cosh 2r\, e_2$

In[•]:=
```
CovDer[{r', 0, 0}, {0, θ' Sinh[r] Cosh[r], 0}] // FullSimplify
```

Out[•]= {0, Cosh[2 r] r′ θ′, -Cosh[r] Sinh[r] r′ θ′}

Example13 - $\nabla_{r'*e_1}(\theta' * \sinh^2 r + \phi') e_3$

In[•]:= `CovDer[{r', 0, 0}, {0, 0, θ' (Sinh[r])^2 + ϕ'}] // FullSimplify`

Out[•]= `{0, r′ (Sinh[r]^2 θ′ + ϕ′), Sinh[2 r] r′ θ′}`

Example21 - $\nabla_{\theta' * \sinh r \cosh r\, e_2}(r' * e_1)$

In[•]:= `CovDer[{0, θ' Sinh[r] Cosh[r], 0}, {r', 0, 0}] // FullSimplify`

Out[•]= `{0, Cosh[2 r] r′ θ′, Cosh[r] Sinh[r] r′ θ′}`

Example22 - $\nabla_{\theta' * \sinh r \cosh r\, e_2}(\theta' * \sinh r \cosh r\, e_2)$

In[•]:= `CovDer[{0, θ' Sinh[r] Cosh[r], 0}, {0, θ' Sinh[r] Cosh[r], 0}] // FullSimplify`

Out[•]= $\left\{-\frac{1}{4}\text{Sinh}[4\,r]\,(\theta')^2,\ \text{Cosh}[r]\,(0\,\&)\,\text{Sinh}[r]\,\theta',\ 0\right\}$

Example23 - $\nabla_{\theta' * \sinh r \cosh r\, e_2}(\theta' * \sinh^2 r + \phi') e_3$ - KRIVO!

In[•]:= `CovDer[{0, θ' Sinh[r] Cosh[r], 0}, {0, 0, θ' (Sinh[r])^2 + ϕ'}] // FullSimplify`

Out[•]= `{-Cosh[r] Sinh[r] θ′ (Sinh[r]^2 θ′ + ϕ′), 0, 0 & Sinh[r]^2 θ′}`

Example31 - $\nabla_{(\theta' * \sinh^2 r + \phi') e_3}\, r' * e_1$

In[•]:= `CovDer[{0, 0, θ' (Sinh[r])^2 + ϕ'}, {r', 0, 0}] // FullSimplify`

Out[•]= `{0, r′ (Sinh[r]^2 θ′ + ϕ′), 0}`

Example32 - $\nabla_{(\theta' * \sinh^2 r + \phi') e_3}\, \theta' * \sinh r \cosh r\, e_2$

In[•]:= `CovDer[{0, 0, θ' (Sinh[r])^2 + ϕ'}, {0, θ' Sinh[r] Cosh[r], 0}] // FullSimplify`

Out[•]= `{-Cosh[r] Sinh[r] θ′ (Sinh[r]^2 θ′ + ϕ′), 0, 0}`

Example33 - $\nabla_{(\theta' * \sinh^2 r + \phi') e_3}(\theta' * \sinh^2 r + \phi') e_3$

In[•]:= `CovDer[{0, 0, θ' (Sinh[r])^2 + ϕ'}, {0, 0, θ' (Sinh[r])^2 + ϕ'}] // FullSimplify`

Out[•]= `{0, 0, 0 & (Sinh[r]^2 θ′ + ϕ′)}`

generally ---KRIVO!

$\nabla_{\gamma'}\gamma'$

In[∘]:= CovDer[{r', θ' Sinh[r] Cosh[r], θ' (Sinh[r])^2 + ϕ'},
{r', θ' Sinh[r] Cosh[r], θ' (Sinh[r])^2 + ϕ'}] // FullSimplify

Out[∘]= $\left\{0 \,\&\, r' + \frac{1}{2}\,\mathrm{Sinh}[2\,r]\,\theta'\,((1-2\,\mathrm{Cosh}[2\,r])\,\theta' - 2\,\phi'),\right.$
$\frac{1}{2}\,(0 \,\&\, \mathrm{Sinh}[2\,r] + (-2+6\,\mathrm{Cosh}[2\,r])\,r')\,\theta' + 2\,r'\,\phi',$
$\left.2\,\mathrm{Sinh}[r]\,(0 \,\&\, \mathrm{Sinh}[r] + \mathrm{Cosh}[r]\,r')\,\theta' + 0 \,\&\, \phi'\right\}$

Provjera identiteta

In[∘]:= (1 - 2 Cosh[2 r]) - (-1 - 4 (Sinh[r])^2) // FullSimplify

Out[∘]= 0

In[∘]:= (-1 + 3 Cosh[2 r]) - 2 (1 + 3 (Sinh[r])^2) // FullSimplify

Out[∘]= 0

Usporedba članova akceleracije uz e1

In[∘]:= DM1 := r'' - (θ')^2 Sinh[r] Cosh[r] Cosh[2 r] -
2 (θ')^2 (Sinh[r])^3 Cosh[r] - 2 θ' ϕ' Sinh[r] Cosh[r]
EZ1 := r'' - Sinh[r] Cosh[r] θ' ((1 + 4 (Sinh[r])^2) θ' + 2 ϕ')

In[∘]:= DM1 - EZ1 // FullSimplify

Out[∘]= 0

Usporedba članova akceleracije uz e2

In[∘]:= DM2 := 2 r' θ' Cosh[2 r] + 2 r' θ' (Sinh[r])^2 + 2 r' ϕ' + θ'' Sinh[r] Cosh[r]
EZ2 := Sinh[r] Cosh[r] θ'' + 2 (1 + 3 (Sinh[r])^2) r' θ' + 2 r' ϕ'

In[∘]:= DM2 - EZ2 // FullSimplify

Out[∘]= 0

Usporedba članova akceleracije uz e3

In[∘]:= DM3 := ϕ'' + r' θ' Sinh[2 r] + θ'' (Sinh[r])^2
EZ3 := ϕ'' + (Sinh[r])^2 θ'' + Sinh[2 r] r' θ'

In[∘]:= DM3 - EZ3 // FullSimplify

Out[∘]= 0