On certain classes of Weingarten surfaces in $SL(2,\mathbb{R})$ space

Damjan Klemenčić, Zlatko Erjavec

University of Zagreb, Faculty of Organization and Informatics

Introduction

A **Weingarten surface** or a W surface is a surface satisfying the Jacobi equation

$$\Phi(K,H) = \det\begin{pmatrix} K_u & K_v \\ H_u & H_v \end{pmatrix} = 0,$$

where K is Gaussian curvature and H is mean curvature of the surface. If a surface satisfies a linear equation with respect to K and H $aK+bH=c$, $a,b,c\in\mathbb{R}$, not all zero, then the surface is called **linear Weingarten surface** or LW-surface. It is clear that surface with constant Gauss curvature or constant mean curvature is a Weingarten surface. Therefore, Weingarten surfaces can be regarded as generalization of surfaces of constant Gauss and constant mean curvature.

The study of Weingarten surfaces was initiated by J. Weingarten in 1861. E. Beltrami and U. Dini few years later proved that the only non-developable Weingarten ruled surface in Euclidean 3-space is a helicoidal ruled surface. In the last decade several papers on Weingarten surfaces in different 3-dimensional spaces have appeared. Some results on W-surfaces can be found in [1], [2], [4] and [7].

Motivated by the fact that there are no results about Weingarten surfaces in $SL(2,\mathbb{R})$ geometry, we examine two classes of ruled Weingarten surface in $SL(2,\mathbb{R})$ geometry. The $SL(2,\mathbb{R})$ geometry is one of the eight homogeneous Thurston 3-geometries

$$E^3,\ S^3,\ H^3,\ S^2\times\mathbb{R},\ H^2\times\mathbb{R},\ SL(2,\mathbb{R}),\ \text{Nil},\ \text{Sol}.$$

More about curves and surfaces in $SL(2,\mathbb{R})$ geometry can be found in [5], [6] and [8].

The $SL(2,\mathbb{R})$ geometry

As we mentioned, the $SL(2,\mathbb{R})$ geometry is one of the 3D homogeneous geometries. Generally, the Riemannian manifold (M,g) is called homogeneous if for any $x,y\in M$ there exists an isometry $\Phi: M\to M$ such that $y=\Phi(x)$.

Two models of $SL(2,\mathbb{R})$ geometry appear in the literature. The first one is usually called the Hyperboloid model and the second one, which we will use in this paper, called the Right-half space model of $SL(2,\mathbb{R})$ geometry. The Right half-space model of $SL(2,\mathbb{R})$ geometry is in details explained in [9] and the metric in this model is given by

$$(ds)^2 = \left(\frac{dx}{2y}\right)^2 + \left(\frac{dy}{2y}\right)^2 + \left(\frac{dx}{2y}+d\theta\right)^2$$

A left orthonormal frame $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ in the Right half-space model of $SL(2,\mathbb{R})$ is given by

$$\mathbf{e_1} = 2y\,\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta},\quad \mathbf{e_2} = 2y\,\frac{\partial}{\partial y},\quad \mathbf{e_3} = \frac{\partial}{\partial\theta}.$$

The Levi-Civita connection $\widetilde{\nabla}$ (in terms of the orthonormal frame), is given by

$$\begin{array}{lll} \widetilde{\nabla}_{e_1}e_1 = 2e_2, & \widetilde{\nabla}_{e_1}e_2 = -2e_1 - e_3, & \widetilde{\nabla}_{e_1}e_3 = e_2, \\ \widetilde{\nabla}_{e_2}e_1 = e_3, & \widetilde{\nabla}_{e_2}e_2 = 0, & \widetilde{\nabla}_{e_2}e_3 = -e_1, \\ \widetilde{\nabla}_{e_3}e_1 = e_2, & \widetilde{\nabla}_{e_3}e_2 = -e_1, & \widetilde{\nabla}_{e_3}e_3 = 0. \end{array} \tag{1}$$

The Weingarten surfaces of type $r(u,v) = (x(u), y(u), \theta(v))$

Proposition

The Gauss curvature K and the mean curvature H of the surface $r(u,v) = (x(u),y(u),\theta(v))$ are given by

$$K=-1,\qquad H = \frac{1}{8y^3W^3}\Big(y(x_{uu}y_u - x_uy_{uu}) - x_u(x_u^2+y_u^2)\Big)$$

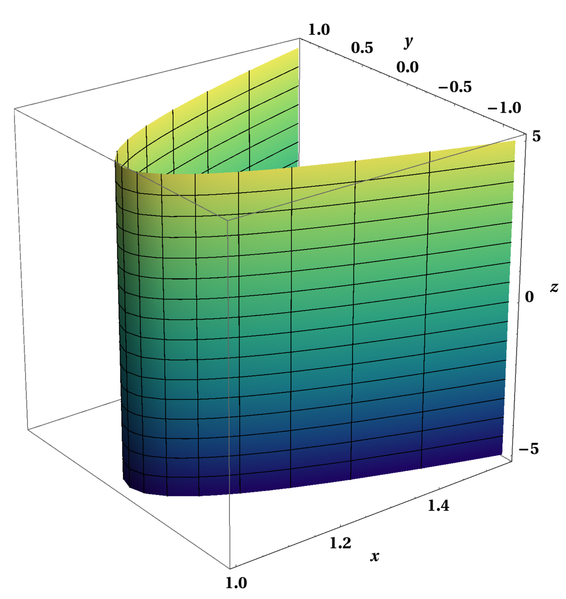
where $x_u = \frac{\partial x}{\partial u}$, $y_u = \frac{\partial y}{\partial u}$, $x_{uu} = \frac{\partial^2 x}{\partial u^2}$, $y_{uu} = \frac{\partial^2 y}{\partial u^2}$ *and* $W = \frac{\theta_v}{2y}\sqrt{x_u^2+y_u^2}$.

Theorem

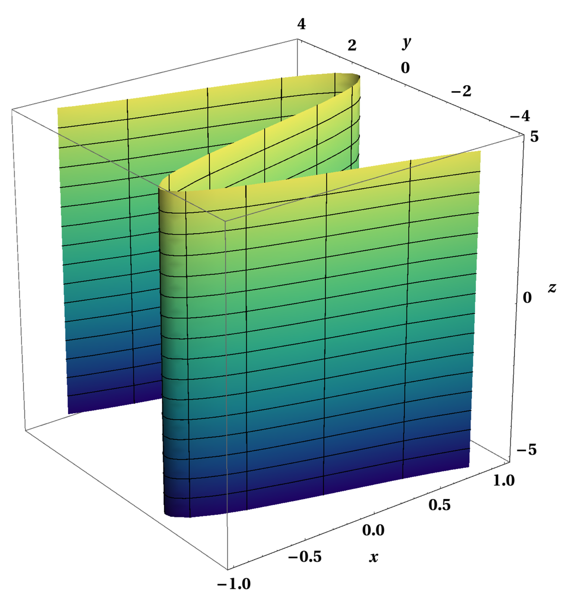
Every surface of the type $r(u,v) = \big(x(u),y(u),\theta(v)\big)$ is a Weingarten surface in $SL(2,R)$ space.

Corollary

Every CMC surface of the type $r(u,v) = \big(x(u),y(u),\theta(v)\big)$ is a linear Weingarten surface in $SL(2,R)$ space.



$r(u,v) = \big(\cosh u,\ \sinh u,\ v\big)$



$r(u,v) = \big(\sin u,\ u,\ v\big)$

The Weingarten surfaces of type $r(u,v) = (u, v, f(u,v))$

Proposition

The Gauss curvature K and the mean curvature H of the level surface $r(u,v) = (u,v,f(u,v))$ are given by

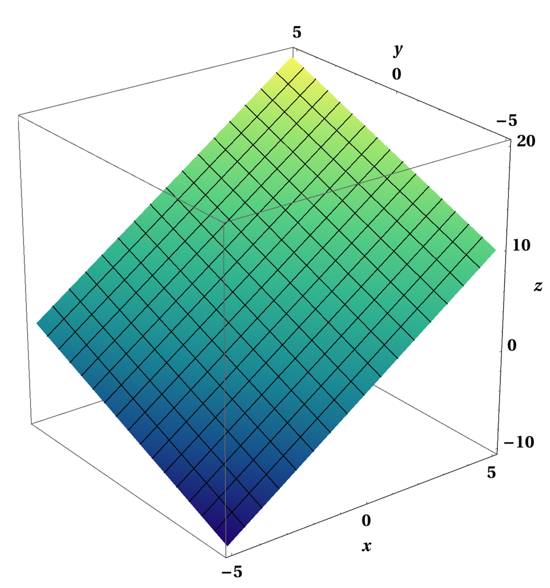
$$K = \frac{1}{16y^4W^4}\left(\left(f_{uu} - 2f_v\left(f_u+\frac{1}{y}\right)\right)\left(f_{vv}+2f_v\left(f_u+\frac{1}{y}\right)\right) - \left(f_u^2 - f_v^2 + f_{uv} + \frac{2f_u}{y} + \frac{1}{2y^2}\right)^2\right),$$

$$H = \frac{1}{8y^2W^3}\left(\left(f_v^2 - \frac{1}{4y^2}\right)\left(f_{uu} - \frac{2f_v}{y} - 2f_uf_v\right) - \left(\frac{f_v}{y} + 2f_uf_v\right)\left(f_u^2 - f_v^2 + f_{uv} + \frac{2f_u}{y} + \frac{1}{2y^2}\right) + \right.$$
$$\left. + \left(f_u^2 + \frac{f_u}{y} + \frac{1}{2y^2}\right)\left(f_{vv} + \frac{2f_v}{y} + 2f_uf_v\right)\right),$$

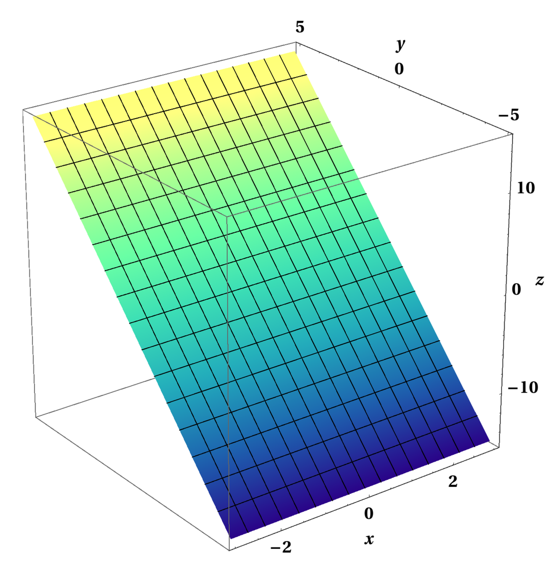
where $W = \frac{1}{2\sqrt{2y^2}}\sqrt{1+2y^2(f_u^2+f_v^2)+2yf_u}$, $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, $f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{vv} = \frac{\partial^2 f}{\partial v^2}$ *and* $f_{uv} = \frac{\partial^2 f}{\partial u\partial v}$.

Proposition

Suppose that $f(u,v) = au+bv+c$ for some $a,b,c\in\mathbb{R}$. Then the surface $r(u,v) = (u,v,f(u,v))$ is a Weingarten surface in $SL(2,R)$ space.



$r(u,v) = \big(u,\ v,\ 2u+v+5\big)$



$r(u,v) = (u,\ v,\ 3v)$

References

[1] F. Dillen, W. Goemans, I. Van de Woestyne, Translation surfaces of Weingarten type in 3-space, *Bull. Transilvania Univ. Brasov*, Ser III, **50** (2008), 109-122.

[2] F. Dillen, W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.* **98** (1999), 307-320.

[3] M.P. Do Carmo, *Riemannian geometry*, Birkhäuser, Boston (1992).

[4] Z. Erjavec, On a certain class of Weingarten surfaces in Sol space, Int. J. Appl. Math., Vol. **28** (2015), 507–514.

[5] J. Inoguchi, Invariant minimal surfaces in the real special linear group of degree 2, Ital. J. Pure Appl. Math. **16** (2004), 61–80.

[6] M. Kokubu, On minimal surfaces in the Real Special Linear Group $SL(2,\mathbb{R})$, Tokyo J Math **20** (1997), 287–297.

[7] Ž. Milin Šipuš, Ruled Weingarten surfaces in the Galilean space, *Periodica mathematica Hungarica* **56** (2008), 213-225.

[8] S. Montaldo, I.I. Onnis, A. Passos Passamani, Helix surfaces in the special linear group, Ann. Math. Pura Appl. **195** (2016), 59–77.

[9] P. Scott, The Geometries of 3-Manifolds, *Bull. London Math. Soc.* **15** (1983), 401-487.

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