Ruled Weingarten surfaces in SL(2,R)

Ambient space

■ Coordinates in ambient space - INPUT

In[1]:=
```
u1 := x;
u2 := y;
u3 := z;
```

■ Metric of ambient space - INPUT

In[4]:=
$$g[x\_, y\_, z\_] := \left\{\left\{\frac{1}{2*y^2}, 0, \frac{1}{2*y}\right\}, \left\{0, \frac{1}{4*y^2}, 0\right\}, \left\{\frac{1}{2*y}, 0, 1\right\}\right\}$$

In[5]:=
```
g[u1, u2, u3] // MatrixForm
```

Out[5]//MatrixForm=
$$\begin{pmatrix} \frac{1}{2 y^2} & 0 & \frac{1}{2 y} \\ 0 & \frac{1}{4 y^2} & 0 \\ \frac{1}{2 y} & 0 & 1 \end{pmatrix}$$

In[6]:=
```
INVg := Inverse[g[u1, u2, u3]]
```

In[7]:=
```
MatrixForm[INVg]
```

Out[7]//MatrixForm=
$$\begin{pmatrix} 4 y^2 & 0 & -2 y \\ 0 & 4 y^2 & 0 \\ -2 y & 0 & 2 \end{pmatrix}$$

■ Orthonormal frame field in ambient space - INPUT

In[8]:=
```
e1[x_, y_, z_] := {2 * y, 0, -1}
e2[x_, y_, z_] := {0, 2 * y, 0}
e3[x_, y_, z_] := {0, 0, 1}
```

■ Scalar product in ambient space

In[11]:=
```
SP[v1_, v2_] := Sum[g[u1, u2, u3][[i, j]] * v1[[i]] * v2[[j]], {i, 1, 3}, {j, 1, 3}]
```

■ Check of orthonormality of frame fields

In[12]:=
```
Table[SP[ei[x, y, z][[1 ;; 3]], ej[x, y, z][[1 ;; 3]]], {i, 1, 3}, {j, 1, 3}] // TableForm //
 Simplify
```

Out[12]//TableForm=
```
1 0 0
0 1 0
0 0 1
```

- **Christoffel symbols** $\Gamma_{i,j}^{\{k\}} := \frac{1}{2}\sum_{l=1}^{3}\left(\partial_{u_i} g_{j,l} + \partial_{u_j} g_{l,i} - \partial_{u_l} g_{i,j}\right) h_{l,k}$

In[13]:=
$$\Gamma_1[u_1, u_2, u_3] = \text{Table}\Big[\frac{1}{2}\sum_{l=1}^{3}\left(\partial_{u_i} g[u_1, u_2, u_3][[j, l]] + \partial_{u_j} g[u_1, u_2, u_3][[l, i]] - \partial_{u_l} g[u_1, u_2, u_3][[i, j]]\right) \text{INVg}[[l, 1]], \{i, 1, 3\}, \{j, 1, 3\}\Big] \,//\, \text{Simplify}$$

Out[13]=
$$\left\{\left\{0, -\frac{3}{2\,y}, 0\right\}, \left\{-\frac{3}{2\,y}, 0, -1\right\}, \{0, -1, 0\}\right\}$$

In[14]:=
$$\Gamma_2[u_1, u_2, u_3] = \text{Table}\Big[\frac{1}{2}\sum_{l=1}^{3}\left(\partial_{u_i} g[u_1, u_2, u_3][[j, l]] + \partial_{u_j} g[u_1, u_2, u_3][[l, i]] - \partial_{u_l} g[u_1, u_2, u_3][[i, j]]\right) \text{INVg}[[l, 2]], \{i, 1, 3\}, \{j, 1, 3\}\Big] \,//\, \text{Simplify}$$

Out[14]=
$$\left\{\left\{\frac{2}{y}, 0, 1\right\}, \left\{0, -\frac{1}{y}, 0\right\}, \{1, 0, 0\}\right\}$$

In[15]:=
$$\Gamma_3[u_1, u_2, u_3] = \text{Table}\Big[\frac{1}{2}\sum_{l=1}^{3}\left(\partial_{u_i} g[u_1, u_2, u_3][[j, l]] + \partial_{u_j} g[u_1, u_2, u_3][[l, i]] - \partial_{u_l} g[u_1, u_2, u_3][[i, j]]\right) \text{INVg}[[l, 3]], \{i, 1, 3\}, \{j, 1, 3\}\Big] \,//\, \text{Simplify}$$

Out[15]=
$$\left\{\left\{0, \frac{1}{2\,y^2}, 0\right\}, \left\{\frac{1}{2\,y^2}, 0, \frac{1}{2\,y}\right\}, \left\{0, \frac{1}{2\,y}, 0\right\}\right\}$$

Riemannian connection on ambient space

In[16]:=
```
w1 := {1, 0, 0}
w2 := {0, 1, 0}
w3 := {0, 0, 1}
```

- **Transformation matrices - canonical base $\{\partial_i\}$ vs orthonormal frame $\{e_i\}$**

In[19]:=
$$AC := \begin{pmatrix} 2\,y & 0 & -1 \\ 0 & 2\,y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[20]:= CA = Inverse[AC]

Out[20]=
$$\left\{\left\{\frac{1}{2\,y}, 0, \frac{1}{2\,y}\right\}, \left\{0, \frac{1}{2\,y}, 0\right\}, \{0, 0, 1\}\right\}$$

- **covariant derivations of orthonorml fields ---- CD[i, j] =** $\nabla_{e_i} e_j$

In[21]:=
$$CD[i\_, j\_] := \sum_{l=1}^{3}\sum_{m=1}^{3}\left(e_i[x, y, z][[l]]\left(D[e_j[x, y, z][[m]], u_l]\, w_m + e_j[x, y, z][[m]]\left(\sum_{n=1}^{3}\Gamma_n[u_1, u_2, u_3][[l, m]]\, w_n\right)\right)\right)$$

- **covarian derivations in ambient base**

In[22]:=
```
Table[CD[i, j].CA, {i, 1, 3}, {j, 1, 3}] // MatrixForm // Simplify
```

Out[22]//MatrixForm=
$$\begin{pmatrix} \begin{pmatrix}0\\2\\0\end{pmatrix} & \begin{pmatrix}-2\\0\\-1\end{pmatrix} & \begin{pmatrix}0\\1\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}-1\\0\\0\end{pmatrix} \\ \begin{pmatrix}0\\1\\0\end{pmatrix} & \begin{pmatrix}-1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} \end{pmatrix}$$

- **Levi-Civita connection in SL(2,R) space**

In[23]:=
$$CovDer[w1\_, w2\_] := \sum_{i=1}^{3}\sum_{j=1}^{3} w1[[i]] * (e_i[x, y, z][[i]] * D[w2[[j]], u_i]\, e_j[x, y, z].CA + w2[[j]]\, CD[i, j].CA)$$

Surface r = $\{r_1, r_2, r_3\}$

Parametric equations of surface - INPUT

In[24]:=
```
r1[x_, y_, z_] := f[x]
r2[x_, y_, z_] := h[x]
r3[x_, y_, z_] := z
```

- **Tangent vectors**

In[27]:= $r_u = \{\partial_x r1[x, y, z], \partial_x r2[x, y, z], \partial_x r3[x, y, z]\}$

Out[27]= $\{f'[x], h'[x], 0\}$

In[28]:= $r_v = \{\partial_z r1[x, y, z], \partial_z r2[x, y, z], \partial_z r3[x, y, z]\}$

Out[28]= $\{0, 0, 1\}$

■ Tangent vectors in base {e_i} – INPUT

In[29]:=
```
t1[x_, y_, z_] := {f′[x], h′[x], 0}.CA
t2[x_, y_, z_] := {0, 0, 1}.CA
```

In[31]:=
```
t1[x, y, z]
```

Out[31]= $\left\{\frac{f'[x]}{2\,y}, \frac{h'[x]}{2\,y}, \frac{f'[x]}{2\,y}\right\}$

In[32]:=
```
t2[x, y, z]
```

Out[32]= {0, 0, 1}

■ I. fundamental form

In[33]:=
```
EE[x, y] = SP[r_u, r_u] // FullSimplify // Expand
FF[x, y] = SP[r_u, r_v] // Simplify
GG[x, y] = SP[r_v, r_v] // Simplify
W[x, y] = Sqrt[EE[x, y] GG[x, y] - (FF[x, y])^2] // Simplify // Expand
```

Out[33]= $\frac{f'[x]^2}{2\,y^2} + \frac{h'[x]^2}{4\,y^2}$

Out[34]= $\frac{f'[x]}{2\,y}$

Out[35]= 1

Out[36]= $\frac{1}{2}\sqrt{\frac{f'[x]^2 + h'[x]^2}{y^2}}$

■ II. fundamental form

■ normal field on surface (perpendicular on tangent plane and normalized) - INPUT

In[37]:= $\text{Cross}\left[\left\{\frac{f'[x]}{2\,y}, \frac{h'[x]}{2\,y}, \frac{f'[x]}{2\,y}\right\}, \{0, 0, 1\}\right]$ // Simplify

Out[37]= $\left\{\frac{h'[x]}{2\,y}, -\frac{f'[x]}{2\,y}, 0\right\}$

In[38]:= $\left\{\frac{h'[x]}{2\,y}, -\frac{f'[x]}{2\,y}, 0\right\}.\left\{\frac{h'[x]}{2\,y}, -\frac{f'[x]}{2\,y}, 0\right\}$ // Simplify // Expand

Out[38]= $\frac{f'[x]^2}{4\,y^2} + \frac{h'[x]^2}{4\,y^2}$

In[39]:= $n := \left(\frac{h'[x]}{2\,y} * e_1[x, y, z] - \frac{f'[x]}{2\,y} * e_2[x, y, z]\right) \Big/ \text{Sqrt}\left[\left(\frac{f'[x]^2}{4\,y^2} + \frac{h'[x]^2}{4\,y^2}\right)\right]$

- **Checking of the normal field**

In[40]:= $\mathrm{SP}\left[\left\{\frac{f'[x]}{2\,y}, \frac{h'[x]}{2\,y}, \frac{f'[x]}{2\,y}\right\}, n\right]$ `// Simplify`

```
SP[e3[x, y, z], n] // Simplify
SP[n, n] // Simplify
```

Out[40]= 0

Out[41]= 0

Out[42]= 1

In[43]:= $\mathrm{CovD}[w1\_, w2\_] := \sum_{i=1}^{3}\sum_{j=1}^{3} w1[x, y, z][[i]] * (e_i[x, y, z][[i]] * D[w2[x, y, z][[j]], u_i]\, e_j[x, y, z] + w2[x, y, z][[j]]\, \mathrm{CD}[i, j])$

- ∇_{t1}**t1 – representation in base** $\{e_i\}$

In[44]:= `CovD[t1, t1].CA // FullSimplify`

Out[44]= $\left\{\frac{f'[x]\,(-3\,h'[x] + y\,f''[x])}{2\,y^2}, \frac{2\,f'[x]^2 - h'[x]^2 + y\,f'[x]\,h''[x]}{2\,y^2}, \frac{f'[x]\,(-h'[x] + y\,f''[x])}{2\,y^2}\right\}$

- **Dt1t1 – computed by hand – representation in base** $\{e_i\}$

In[58]:= $\mathrm{Dt1t1} := \left\{\frac{y\,f''[x] - 3\,f'[x]\,h'[x]}{2\,y^2}, \frac{2\,f'[x]^2 - h'[x]^2 + y\,h''[x]}{2\,y^2}, \frac{y\,f''[x] - f'[x]\,h'[x]}{2\,y^2}\right\}$

- ∇_{t1}**t2**

In[45]:= `CovD[t1, t2].CA // Simplify`

Out[45]= $\left\{-\frac{h'[x]}{2\,y}, \frac{f'[x]}{2\,y}, 0\right\}$

- ∇_{t2}**t1**

In[46]:= `CovD[t2, t1].CA // Simplify`

Out[46]= $\left\{-\frac{h'[x]}{2\,y}, \frac{f'[x]}{2\,y}, 0\right\}$

- ∇_{t2}**t2**

In[47]:= `CovD[t2, t2].CA // Simplify`

Out[47]= {0, 0, 0}

Coefficients of the second fundamental form

In[59]:=
```
L[x, y] = SP[Dt1t1, n] // Simplify
```

Out[59]=
$$\frac{-2 f'[x]^3 + y\, h'[x]\, f''[x] - f'[x]\left(2 h'[x]^2 + y\, h''[x]\right)}{4 y^4 \sqrt{\frac{f'[x]^2+h'[x]^2}{y^2}}}$$

In[49]:=
```
M[x, y] = SP[CovD[t1, t2], n] // Simplify
```

Out[49]=
$$-\frac{1}{2}\sqrt{\frac{f'[x]^2 + h'[x]^2}{y^2}}$$

In[50]:=
```
NN[x, y] = SP[CovD[t2, t2], n] // Simplify
```

Out[50]= 0

Curvatures

Gauss curvature

In[60]:=
```
K[x, y, z] = (L[x, y] NN[x, y] - M[x, y]^2) / (W[x, y])^2 // FullSimplify
```

Out[60]= -1

Mean curvature

In[63]:=
```
H[x, y, z] =
  (GG[x, y] L[x, y] - 2 FF[x, y] M[x, y] + EE[x, y] NN[x, y]) / (2 (W[x, y])^2) // FullSimplify
```

Out[63]=
$$\frac{2(-1+y)\, f'[x]^3 + y\, h'[x]\, f''[x] + f'[x]\left(2(-1+y)\, h'[x]^2 - y\, h''[x]\right)}{2 y^4 \left(\frac{f'[x]^2+h'[x]^2}{y^2}\right)^{3/2}}$$

Weingarten surfaces

In[64]:=
```
D[K[x, y, z], x] // FullSimplify
```

Out[64]= 0

In[65]:=
```
D[K[x, y, z], z] // FullSimplify
```

Out[65]= 0

In[66]:= `D[H[x, y, z], x] // FullSimplify`

Out[66]=
$$\frac{1}{2\left(f'[x]^2+h'[x]^2\right)^3\sqrt{\frac{f'[x]^2+h'[x]^2}{y^2}}}\Big(h'[x]^2\left(f''[x]\left(2(-1+y)h'[x]^2-3y\,h''[x]\right)+y\,h'[x]\,f^{(3)}[x]\right)+ f'[x]^2\left(f''[x]\left(2(-1+y)h'[x]^2+3y\,h''[x]\right)+y\,h'[x]\,f^{(3)}[x]\right)+ f'[x]^3\left(-2(-1+y)h'[x]\,h''[x]-y\,h^{(3)}[x]\right)- f'[x]\,h'[x]\left(3y\,f''[x]^2+h''[x]\left(2(-1+y)h'[x]^2-3y\,h''[x]\right)+y\,h'[x]\,h^{(3)}[x]\right)\Big)$$

In[67]:= `D[H[x, y, z], z] // FullSimplify`

Out[67]= 0

In[68]:= `D[K[x, y, z], x] * D[H[x, y, z], z] - D[K[x, y, z], z] * D[H[x, y, z], x] // FullSimplify`

Out[68]= 0