ON KILLING MAGNETIC CURVES IN HYPERBOLOID MODEL OF $\mathrm{SL}(2,\mathbb{R})$ GEOMETRY

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Abstract. Killing magnetic curve is a trajectory of a charged particle on a Riemannian manifold under the action of a Killing magnetic field. In this paper we study Killing magnetic curves in the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ geometry.

1. Introduction

Let F be a closed 2-form on a Riemannian 3-manifold (M,g), called the *magnetic field*. This title comes from the fact that a closed 2-form can be regarded as a generalization of a static magnetic field on 3-dim Euclidean space [15].

A curve $\gamma(t)$ on a Riemannian 3-manifold (M,g) is called a *magnetic curve* if its velocity vector field satisfies the Lorentz equation

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{1.1}$$ maineq

where ∇ is the Levi-Civita connection of g and Φ is $(1,1)$-tensor field on M, called the *Lorentz force*, related to the magnetic field F by

$$g(\Phi(X),Y) = F(X,Y), \quad \forall X,Y \in \mathfrak{X}(M). \tag{1.2}$$ PhiF

For $\Phi = 0$, the differential equation (1.1) coincides with the geodesic equation. Hence we can say that magnetic trajectories are generalizations of geodesics.

The vector field V on M is called a *Killing vector field* if it satisfies the Killing equation

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0, \quad \forall Y,Z \in \mathfrak{X}(M). \tag{1.3}$$ Killeq

The Killing vector field can be interpreted as an infinitesimal generator of isometry on the manifold in the sense that the flow generated by this field is a continuous isometry of the manifold.

In particular, Killing vector fields define an important class of magnetic fields called *Killing magnetic fields* and the trajectories corresponding to the Killing magnetic fields are called the *Killing magnetic curves*.

Killing magnetic curves in Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-spacetime $\mathbb{E}^3_1$ were studied by Druţă-Romaniuc and Munteanu in [5] and [4], respectively. Munteanu and Nistor considered Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ space in [13], Erjavec and Inoguchi in Sol_3 space in [7] and Erjavec in Right half-space model of $SL(2,R)$ geometry in [7].

Key words and phrases. Magnetic curve, Killing vector field, $\mathrm{SL}(2,\mathbb{R})$ geometry.

The goal of this paper is to study the Killing magnetic curves in the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ geometry. Unique features of $\mathrm{SL}(2,\mathbb{R})$ geometry make this task seems more complicated than in other homogeneous geometries. navesti rezultate iz RHS i ocekivanja (nismo mogli rijeiti tamo pa se nadamo da emo ovdje dobiti nove krivulje).

Let us recall that the cross product of two vector fields $X, Y \in \mathfrak{X}(M)$ on Riemannian manifold M is defined as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \mathfrak{X}(M), \tag{1.4}$$ wedge

where dv_g denotes a volume form on M.

It is known that in 3-dim space closed 2-form can be identified with divergence free vector fields via the Hodge operator and the volume form of the oriented manifold. If V is a Killing vector field on M, let $F_V = i_V dv_g$ be the corresponding Killing magnetic field, where i denotes the inner product on M.

Hence, the Lorentz force Φ_V corresponding to the Killing magnetic field F_V is

$$\Phi_V(X) = V \times X, \tag{1.5}$$?Phi?

and then the Lorentz equation (1.1) can be written ([2]) as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{1.6}$$ main2

2. $\mathrm{SL}(2,\mathbb{R})$ geometry

Two models of $\mathrm{SL}(2,\mathbb{R})$ geometry appear in the literature. The first one is the Right half-space model of $\mathrm{SL}(2,\mathbb{R})$ geometry and the second one is the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ geometry. Each of these two models is useful in certain contexts. These two models are isomorphic and the isomorphism between them is constructed in [6].

The Hyperboloid model is introduced in [10] and used in [3, 6, 11]. On the other side, the Right half-space model is in details explained in [14] and used in [1, 8, 9, 12]. As we mentioned, we studied Killing magnetic curves using the Right half-space model of $\mathrm{SL}(2,\mathbb{R})$ geometry in [6].??

Briefly, we recall fundamental properties of the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ geometry.

The idea is to start with the collineation group which acts on projective 3-space $\mathcal{P}^3(\mathbb{R})$ and projective sphere $\mathcal{PS}^3(\mathbb{R})$ and preserves a hyperboloid polarity, i.e. a scalar product of signature $(--++)$. Using the one-sheeted hyperboloid solid

$$\mathcal{H}:\ -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0,$$

with an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of our hyperboloid $\mathcal{H}$ will give us the so-called hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry. Details can be find in [10] and [11].

The Riemannian metric in the Hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ space is given by

$$(ds)^2 = (dr)^2 + \sinh^2 r \cosh^2 r (d\vartheta)^2 + \big(\sinh^2 r (d\vartheta) + (d\varphi)\big)^2, \tag{2.1}$$?metric?

where $r \in [0, \infty)$ and $\vartheta \in [-\pi, \pi)$ are polar coordinates of the intersection point of a fiber and the hyperbolic base plane and $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is a fiber coordinate with extension to $\mathbb{R}$

for the universal covering. One can easily see that the metric is invariant under rotations about a fiber through the origin and translations along fibers.

Therefore, the symmetric metric tensor field g is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r \cosh 2r & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \qquad (2.2)\ \text{?SLg?}$$

The Euclidean coordinates, corresponding to the hyperboloid coordinates (r, ϑ, φ), are given by

$$\begin{aligned} x &= \tan\varphi, \\ y &= \tanh r \cdot \frac{\cos(\vartheta - \varphi)}{\cos\varphi}, \\ z &= \tanh r \cdot \frac{\sin(\vartheta - \varphi)}{\cos\varphi}, \end{aligned} \qquad (2.3)\ \text{?euclideancoord}$$

. This formulas are important for later visualization of surfaces in E^3.

The orthonormal coframe field is given by

$$\theta^1 = dr, \quad \theta^2 = \frac{1}{2}\sinh 2r\ d\vartheta, \quad \theta^3 = \sinh^2 r\ d\vartheta + d\varphi.$$

and the associated orthonormal frame field by

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{2}{\sinh 2r}\frac{\partial}{\partial\vartheta} - \tanh r \frac{\partial}{\partial\varphi}, \quad e_3 = \frac{\partial}{\partial\varphi}. \qquad (2.4)\ \boxed{\text{basis}}$$

Hence,

$$\partial_r = e_1, \quad \partial_\vartheta = \sinh r \cosh r e_2 + \sinh^2 r e_3, \quad \partial_\varphi = e_3. \qquad (2.5)\ \boxed{\text{Cbasis}}$$

In covariant derivative fashion, the Levi-Civita connection ∇ of $\mathrm{SL}(2,\mathbb{R})$ is given by

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = -e_3 & \nabla_{e_1}e_3 = e_2 \\ \nabla_{e_2}e_1 = 2\coth 2r\ e_2 + e_3 & \nabla_{e_2}e_2 = -2\coth 2r\ e_1 & \nabla_{e_2}e_3 = -e_1 \\ \nabla_{e_3}e_1 = e_2 & \nabla_{e_3}e_2 = -e_1 & \nabla_{e_3}e_3 = 0. \end{array} \qquad (2.6)\ \boxed{\text{covariant}}$$

Hence we have the following commutation relations of the basis

$$[e_1, e_2] = -2\coth 2r\ e_2 - 2\ e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

The non-vanishing components of Riemannien curvature tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

up to symmetry properties, are

$$R(e_1,e_2)e_1 = 7e_2 \quad R(e_2,e_3)e_2 = -e_3 \quad R(e_1,e_3)e_1 = -e_3$$

Moreover, if we put $R_{ijkl} = -g(R_{ijk}, e_l)$, where $R_{ijk} = R(e_i, e_j)e_k$, we obtain

$$R_{1212} = -7, \qquad R_{1313} = 1 \qquad \text{and} \qquad R_{2323} = 1.$$

3. Killing vector fields in $\mathrm{SL}(2,\mathbb{R})$ geometry

In this section we recall basic facts on Killing vector fields and determine the Killing vector field in the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ space.

Recall that the vector field V on M is a Killing vector field if it satisfies the Killing equation

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0, \quad \forall Y, Z \in \mathfrak{X}(M). \tag{3.1}$$ Killeq

Let assume that the Killing vector field has a form

$$V = a(r,\vartheta,\varphi)\cdot e_1 + b(r,\vartheta,\varphi)\cdot e_2 + c(r,\vartheta,\varphi)\cdot e_3.$$

Substituting V in the relation (3.1) and taking $Y = e_i,\ Z = e_j$ for all $i,j \in \{1,2,3\}$, we obtain the following system of differential equations for Killing vector fields in $\mathrm{SL}(2,\mathbb{R})$ geometry,

$$\begin{aligned}
\partial_r a &= 0, & \partial_r b + \frac{2}{\sinh 2r}\partial_\vartheta a - \tanh r\,\partial_\varphi a - 2b\coth 2r &= 0,\\
\partial_r c + \partial_\varphi a - 2b &= 0, & \frac{2}{\sinh 2r}\partial_\vartheta b - \tanh r\,\partial_\varphi b + 2a\coth 2r &= 0,\\
\partial_\varphi c &= 0, & \frac{2}{\sinh 2r}\partial_\vartheta c - \tanh r\,\partial_\varphi c + \partial_\varphi b + 2a &= 0.
\end{aligned} \tag{3.2}$$ Killeqsystem

It is easy to check that the solutions of the (3.2) are the following vector fields

$$V_1 = e_3, \qquad V_2 = \sinh 2r\, e_2 + \cosh 2r\, e_3,$$

Usually, instead in the basis of the ambient space, the Killing vector fields are given in canonical basis. Hence, using the relations (2.4) the Killing vector fields in $\mathrm{SL}(2,\mathbb{R})$ are given by

$$\{\partial_\varphi, \quad 2\partial_\vartheta + \partial_\varphi\}. \tag{3.3}$$?KillingVF?

Remark 1. *Komentirati zasto samo dva, a ne 4 KVF. Since* $\mathrm{SL}(2,\mathbb{R})$ *space is a line bundle over the hyperbolic plane, the vector fields V_1, V_3 and V_4 are also Killing vector fields in hyperbolic Poincaré half-plane $\mathbb{H}^2$. Vector field V_2 is a Killing vector field in the fiber direction.??? oprez: KVF su linearno nezavisni u ambijentnoj bazi, ali ne u kanonskoj.*

4. Killing magnetic curves in $\mathrm{SL}(2,\mathbb{R})$ geometry

The goal of this section is to find magnetic curves corresponding to the Killing magnetic fields in the Hyperboloid model of $\mathrm{SL}(2,\mathbb{R})$ space.

4.1. **Case A.**

In this subsection we consider Killing magnetic curves which correspond to the Killing vector field $V = \partial_\varphi$.

First task is to deduce the magnetic curve equation (1.6) for a regular curve $\gamma(t) = (r(t),\vartheta(t),\varphi(t))$ in $\mathrm{SL}(2,\mathbb{R})$. We have

$$\gamma'(t) = r'(t)\frac{\partial}{\partial r} + \vartheta'(t)\frac{\partial}{\partial\vartheta} + \varphi'(t)\frac{\partial}{\partial\varphi},$$

and from (2.5) it follows

$$\gamma' = r'e_1 + \frac{1}{2}\sinh 2r\ \vartheta'\ e_2 + \left(\sinh^2 r\ \vartheta' + \varphi'\right) e_3. \tag{4.1}$$ cprime

Next we compute the covariant derivative $\nabla_{\gamma'}\gamma'$. Taking into account the formulas (2.6), we obtain

$$\begin{aligned}\nabla_{\gamma'}\gamma' &= \left(r'' - \frac{1}{2}\sinh 2r\ \vartheta'\left(\left(1+4\sinh^2 r\right)\vartheta' + 2\varphi'\right)\right)e_1 \\ &\quad+\left(\frac{1}{2}\sinh 2r\ \vartheta'' + 2\left(1+3\sinh^2 r\right)r'\vartheta' + 2r'\varphi'\right)e_2 \\ &\quad+\left(\varphi'' + \sinh^2 r\ \vartheta'' + \sinh 2r\ r'\vartheta'\right)e_3. \end{aligned}\tag{4.2}$$ CD

Using relation (4.1) and the formula (1.4) we have

$$V \times \gamma' = -\frac{1}{2}\sinh 2r\ \vartheta'\ e_1 + r'e_2. \tag{4.3}$$ Veks1

Remark 2. *Remark The relation (4.3) we could obtain in another way. Let* $dv_g = \sinh r \cosh r(dr \wedge d\vartheta \wedge d\varphi)$ *be the volume element of* $\mathrm{SL}(2,\mathbb{R})$. *The Killing vector field* $V = \partial_\varphi$ *by* $F_V = i_V dv_g$, *defines the magnetic field*

$$F_V(X,Y) = dv_g(X,Y,\partial_\varphi) = \frac{1}{2}\sinh 2r(dr\wedge d\vartheta)(X,Y). \tag{4.4}$$ Fphi

From (1.2) and (4.4)

$$\Phi_V(\partial_r) = \frac{2}{\sinh 2r}\partial_\vartheta - \tanh r\partial_\varphi, \quad \Phi_V(\partial_\vartheta) = -\frac{1}{2}\sinh 2r\partial_r, \quad \Phi_V(\partial_\varphi) = 0.$$

Hence, from (2.5) the Lorentz force Φ_V *acts on the basis vectors of* $\mathrm{SL}(2,\mathbb{R})$ *as*

$$\Phi_V(e_1) = e_2, \quad \Phi_V(e_2) = -e_1 \quad \Phi_V(e_3) = 0.$$

Finally we obtain the right hand side of the relation (4.3)

$$\begin{aligned}\Phi_V(\gamma') &= \Phi_V\left(r'e_1 + \frac{1}{2}\sinh 2r\ \vartheta'\ e_2 + \left(\sinh^2 r\ \vartheta' + \varphi'\right)e_3\right) \\ &= r'e_2 - \frac{1}{2}\sinh 2r\ \vartheta'\ e_1.\end{aligned}$$

Further, equalizing the right hand sides of the equations (4.2) and (4.3), we obtain the following system of differential equations

$$\begin{aligned}& r'' - \frac{1}{2}\sinh 2r\ \vartheta'\left(\left(1+4\sinh^2 r\right)\vartheta' + 2\varphi'\right) = -\frac{1}{2}\sinh 2r\ \vartheta', \\ & \frac{1}{2}\sinh 2r\ \vartheta'' + 2\left(1+3\sinh^2 r\right)r'\vartheta' + 2r'\varphi' = r', \\ & \varphi'' + \sinh^2 r\ \vartheta'' + \sinh 2r\ r'\vartheta' = 0.\end{aligned}\tag{4.5}$$ system1

Next, we solve the obtained system (4.5). By homogeneity of SL(2,R), we can extend the solution to limit $r \to 0$, due to given assumption, as follows later on.

The third equation can be written as (total differential?)

$$\frac{d}{dt}\left(\varphi' + \sinh^2 r\ \vartheta'\right) = 0.$$

After integrating we have

$$\varphi'(t) = C - \sinh^2 r\ \vartheta'(t), \quad C \in \mathbb{R}. \tag{4.6}$$ aT1

Substituting (4.6) in the first equation of the system (4.5), we have

$$r'' - \frac{1}{2}\sinh 2r\ \vartheta'\left(\cosh 2r\ \vartheta' + (2C-1)\right) = 0.$$

Hence, for $c_1 = \frac{1}{2}$ we have

$$r'' - \frac{1}{4}\sinh 4r\ \ (\vartheta')^2 = 0. \tag{4.7}$$ ar2

Substituting (4.6) in the second equation of the system (4.5), we have

$$\vartheta'' + 4\coth 2r\ r'\ \vartheta' = 0.$$

After integrating by separating of variables we obtain

$$\vartheta'(t) = \frac{k}{\sinh^2 r(t)}, \quad k \in \mathbb{R}. \tag{4.8}$$ atheta2

Substituting (4.8) in the (4.7) we obtain the following differential equation

$$r'' - \frac{k}{2}\frac{\cosh 2r}{\sinh^3 2r} = 0. \tag{4.9}$$?ar3?

Solving this equation with respect to r, we have

$$r(t) = \int_t \left(\frac{1}{2} - 2cy(\tau) + \lambda y^2(\tau)\right) d\tau. \tag{4.10}$$?aRa?

In consistence with homogeneity we may consider lim $t \to 0$ r(t) = 0. This implies D = E

At the same time we can assume $r(0) = 0; \vartheta(0) = 0; \varphi(0) = 0$, as initial conditions.

Further we consider the arc length

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