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SOME SPECIAL SURFACES IN THE PSEUDO-GALILEAN SPACE

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Abstract. We describe some special surfaces in pseudo-Galilean spaces such as helical surfaces, ruled screw surfaces, surfaces of revolution and in particular tori of revolution. We define special surfaces and find their main properties.

1. Introduction

There has been a long history of studying special classes of surfaces, as surfaces with particularly interesting properties, in Euclidean geometry. For example ruled surfaces, surfaces of revolution, sphere, helical surfaces etc.

Even today there are open problems in the theory of surfaces in the Euclidean geometry. Besides Euclidean geometry, a range of new types of geometries have been invented and developed in the last two centuries. They can be introduced in a variety of manners. One possible way is through projective manner, where one can express metric properties through projective relations. For this purpose a fixed conic (called absolute) in infinity is taken and all metric relations may be considered as projective relations with respect to the absolute. This approach is due to A. Cayley and F. Klein.

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F. Klein noticed that due to the nature of the absolute, various geometries are possible. Details on this development can be found in [8]. Among these geometries there are also Galilean and pseudo-Galilean geometries. These are the ambient geometries in which we investigate some special surfaces.

The main purpose of this article is to define special surfaces, find their main properties and compare them with the properties of corresponding surfaces in Euclidean and some other geometries and to open the field for further investigation.

This paper is written from the pseudo-Galilean point of view. The results can be easily transferred to the Galilean geometry.

We start with the helical surfaces as the most general case where a surface is obtained by rigid screw motion of a given curve. According to the position of a curve we have two classes of screw surfaces.

If a curve which is displaced in a rigid screw motion is a straight line we get a ruled screw surface with constant surface invariants. There are open and closed ruled screw surfaces according to the position of the displaced line with respect to the axis of the screw motion. An interesting case of this class is a helicoid, which happens to be a minimal surface (as in the Euclidean space), but with constant Gaussian curvature (unlike the Euclidean space).

If a pseudo-Euclidean circle undergoes a screw motion, the obtained surface has vanishing Gaussian curvature and constant mean curvature. Therefore it can be related to a cylinder in the Euclidean case.

Separately we treat surfaces of revolution in the pseudo-Galilean space, which are obtained by pseudo-Euclidean or isotropic rotations and also find surfaces of revolution of constant Gaussian curvature.

Finally we describe a torus of revolution as a surface of revolution obtained by revolving a circle around a coplanar axis. There are two different types of tori with respect to the performed rotations and three different types with respect to the existence of intersections of a rotating curve and the rotation axes.

Ruled surfaces and surfaces of constant slope in the pseudo-Galilean space are described in [2] and [3].

We furnish the theory presented here by pictures of some of the investigated surfaces for which we are grateful to Damir Horvat, assistant at the University of Zagreb.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley–Klein geometries (of projective signature $(0,0,+,-)$, explained in [4]). The absolute of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$ where ω is the ideal (absolute) plane, f is a line in ω and I is the fixed hyperbolic involution of points of f.

In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i)$, $i = 1, 2$, is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{\left|(y_2 - y_1)^2 - (z_2 - z_1)^2\right|}, & \text{if } x_1 = x_2. \end{cases} \tag{2.1}$$

The group of motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\bar{x} = a + x, \quad \bar{y} = b + cx + y\cosh\varphi + z\sinh\varphi,$$
$$\bar{z} = d + ex + y\sinh\varphi + z\cosh\varphi.$$

It leaves invariant the absolute figure as well as the pseudo-Galilean distance (2.1) of points.

A vector $\mathbf{x}(x, y, z)$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For isotropic vectors $x = 0$ holds. There are four types of isotropic vectors: spacelike $(y^2 - z^2 > 0)$, time-like $(y^2 - z^2 < 0)$ and two types of lightlike $(y = \pm z)$ vectors. A non-lightlike isotropic vector is a unit vector if $y^2 - z^2 = \pm 1$.

A plane of the form $x = \text{const.}$ is called a pseudo-Euclidean plane (since its induced geometry is pseudo-Euclidean), otherwise it is called isotropic (since its induced geometry is isotropic). An isotropic plane $Ax + By + Cz + D = 0$ is called light-like if $B^2 - C^2 = 0$.

We shall treat a C^r-surface, $r \geqq 1$, as a subset $\Phi \subset G_3^1$ for which there exists an open subset D of $\mathbf{R}^2$ and a C^r-mapping $\mathbf{x} : D \to G_3^1$ satisfying $\Phi = \mathbf{x}(D)$. A C^r-surface $\Phi \subset G_3^1$ is called regular if $\mathbf{x}$ is an immersion, and simple if $\mathbf{x}$ is an embedding. It is admissible if it does not have either pseudo-Euclidean or isotropic light-like tangent planes.

In a tangent plane of a surface parametrized by

$$\mathbf{x}(u_1, u_2) = \big(x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)\big)$$

in a point P_0, there is a unique isotropic direction defined by the condition $x_1\,\mathrm{d}u_1 + x_2\,\mathrm{d}u_2 = 0$, where $x_i = \frac{\partial x}{\partial u_i}$, $i = 1, 2$. A side tangential vector $\boldsymbol{\sigma} = \frac{1}{W}(x_1\mathbf{x}_2 - x_2\mathbf{x}_1)$ is a unit isotropic vector in a tangent plane. The function W, $W > 0$, defined by

$$W = \sqrt{\left|(x_1y_2 - x_2y_1)^2 - (x_1z_2 - x_2z_1)^2\right|} \tag{2.2}$$

is equal to the pseudo-Galilean norm of the side tangential vector $\boldsymbol{\sigma}$.

The isotropic line in the tangent plane meets the absolute line f in a point S, and we denote by $S^{\perp}$ the point on f obtained from S by the hyperbolic involution I. The line connecting P_0 and $S^{\perp}$ is perpendicular to the tangent plane. A unit surface normal field is defined by

$$\mathbf{N} = \frac{1}{W}(0, x_1 z_2 - x_2 z_1, x_1 y_2 - x_2 y_1),$$

where $y_i = \frac{\partial y}{\partial u_i}$, $z_i = \frac{\partial z}{\partial u_i}$, $i = 1, 2$.

Since $\mathbf{N} \cdot \mathbf{N} = \pm 1 = \varepsilon$, we distinguish between two types of admissible surfaces: space-like surfaces having time-like surface normals and time-like surfaces having space-like normals. The pseudo-Euclidean scalar product in the yz-plane is denoted by $\cdot$. A surface is space-like if $(x_1 y_2 - x_2 y_1)^2 - (x_1 z_2 - x_2 z_1)^2 > 0$ in all of its points, time-like otherwise.

The first fundamental form of a surface is introduced in the following way

$$\mathrm{d}s^2 = (x_1 \,\mathrm{d}u_1 + x_2 \,\mathrm{d}u_2)^2 + \delta(\tilde{\mathbf{x}}_1 \,\mathrm{d}u_1 + \tilde{\mathbf{x}}_2 \,\mathrm{d}u_2)^2,$$

where

$$\delta = \begin{cases} 0, & \text{if direction } \mathrm{d}u : \mathrm{d}v \text{ is non-isotropic,} \\ 1, & \text{if direction } \mathrm{d}u : \mathrm{d}v \text{ is isotropic.} \end{cases}$$

By ˜ above, the projection of a vector on the pseudo-Euclidean yz-plane is denoted.

The Gaussian curvature of a surface is defined by means of the coefficients of the second fundamental form

$$K = -\varepsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2}, \tag{2.3}$$

where $\mathbf{N}^2 = \varepsilon = -1$ for space-like surfaces and $\mathbf{N}^2 = \varepsilon = 1$ for time-like surfaces. The second fundamental form II is given by $II = L_{11}\,\mathrm{d}u_1^2 + 2L_{12}\,\mathrm{d}u_1\,\mathrm{d}u_2 + L_{22}\,\mathrm{d}u_2^2$, where L_{ij}, $i, j = 1, 2$, are the normal components of $\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{22}$, respectively. It holds

$$L_{ij} = \varepsilon \frac{1}{x_1}(x_1 \tilde{\mathbf{x}}_{ij} - x_{ij} \tilde{\mathbf{x}}_1) \cdot \widetilde{\mathbf{N}} = \varepsilon \frac{1}{x_2}(x_2 \tilde{\mathbf{x}}_{ij} - x_{ij} \tilde{\mathbf{x}}_2) \cdot \widetilde{\mathbf{N}}.$$

More about the notion of Gaussian curvature in the pseudo-Galilean space can be found in [11].

The mean curvature of a surface is defined by

$$H = -\varepsilon \frac{x_2^2 L_{11} - 2x_1 x_2 L_{12} + x_1^2 L_{22}}{2W^2}.$$

More about curves and surfaces in G_3^1 can be found in [1].

3. Helical surfaces

In Euclidean geometry helical (twisted, screw) surface is a natural generalization of both surfaces of revolution and helicoids. Here we distinguish between two types of rotations and there is only one type of screw (helical) motion.

Let c be a plane curve and e a given line. It is convenient, but not necessary, to start with a plane curve c. A helical surface (or generalized helicoid) H is a surface obtained when c is displaced in a rigid screw motion about e. The normal form of the 1-parameter group of motions called the screw motions is given by

$$(3.1)\qquad \bar{x} = pt + x, \quad \bar{y} = y\cosh t + z\sinh t, \quad \bar{z} = y\sinh t + z\cosh t,$$

where $p \in \mathbf{R}$. By substituting $p = 0$, the pseudo-Euclidean rotations about the x-axis are described, and the obtained surface is a surface of revolution. Surfaces of revolution are presented in one of the following sections.

Since a plane curve c can lie in a pseudo-Euclidean or in an isotropic plane, we treat these two cases separately.

Let a plane curve c, given by $\mathbf{r}(v) = \big(0, f(v), g(v)\big)$, where $f, g \in C^2$, be an admissible curve in a pseudo-Euclidean plane. If c undergoes the screw motion, the helical surface H_p with parametrization is obtained by

$$(3.2)\qquad \mathbf{x}(u,v) = \big(pu, f(v)\cosh u + g(v)\sinh u, f(v)\sinh u + g(v)\cosh u\big).$$

Its first fundamental form is given by

$$(3.3)\quad \mathrm{d}s^2 = (p\,\mathrm{d}u)^2 + \delta\big[(g^2 - f^2)\,\mathrm{d}u^2 + 2(f'g - fg')\,\mathrm{d}u\,\mathrm{d}v + (f'^2 - g'^2)\,\mathrm{d}v^2\big].$$

It is convenient to assume that the curve c is parametrized by the pseudo-Euclidean arc length, that is, $f'^2 - g'^2 = \varepsilon$, where $\varepsilon = \pm 1$. Then the surface normal vector field is

$$\mathbf{N} = \operatorname{sgn}(p)(0, f'\sinh u + g'\cosh u, f'\cosh u + g'\sinh u)$$

and also the first fundamental form (3.3) can be simplified.

From the above expression and the definition of a space-like (time-like) curve as a curve with space-like (time-like) tangent field, the following proposition can easily be obtained.

PROPOSITION 1. *If c is a space-like curve in a pseudo-Euclidean plane then the surface obtained by a rigid screw motion is a space-like surface. On the other hand, if a time-like curve from a pseudo-Euclidean plane is displaced by a screw motion, then a time-like surface is obtained.*

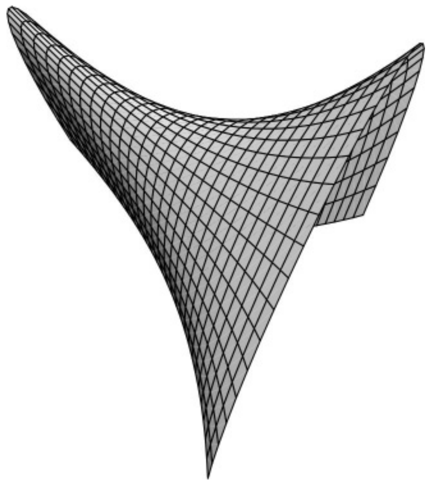

Fig. 1: The helical surface H_p, $p = 1$, $f(v) = \sin v$, $g(v) = \cos v$

For the Gaussian and the mean curvature we get

$$K = -\varepsilon \frac{(fg' - gf')(f''g' - g''f') - 1}{p^2}, \quad H = -\frac{1}{2}\varepsilon \operatorname{sgn}(p)(f''g' - g''f').$$

Since c is parametrized by the arc length (which implies $f'f'' = g'g''$), the expression for K and H can be written also as

$$K = \frac{(fg' - gf')g'' + \varepsilon f'}{p^2 f'}, \quad H = -\frac{1}{2}\varepsilon \operatorname{sgn}(p)\frac{g''}{f'}$$

or equivalently

$$K = \frac{(fg' - gf')f'' + \varepsilon g'}{p^2 g'}, \quad H = -\frac{1}{2}\varepsilon \operatorname{sgn}(p)\frac{f''}{g'}. \tag{3.4}$$

Now we can find minimal surfaces among helical surfaces of the first type. Minimal surfaces are characterized by $H = 0$. According to (3.4) the starting curve has to be a straight line which can be parametrized by $\mathbf{r}(v) = \big(0, av + b, \pm v\sqrt{a^2 - \varepsilon} + c\big)$, where a, b, c are real constants. For $\varepsilon = 1$ we get a minimal time-like ruled surface and for $\varepsilon = -1$ a minimal space-like ruled surface.

THEOREM 1. *There are two classes of minimal surfaces among helical surfaces* H_p : *space-like and time-like surfaces. These surfaces are ruled surfaces with isotropic generator field.*

Now, let a plane curve c be an admissible curve in an isotropic plane. We can treat two cases: the first one when the curve c is parametrized by $\mathbf{r}(v) = \big(g(v), f(v), 0\big)$ and the second when the curve c is parametrized by $\mathbf{r}(v) = \big(g(v), 0, f(v)\big)$. Then the helical surface H_i is given by

$$\mathbf{x}(u,v) = \big(pu + g(v), f(v)\cosh u, f(v)\sinh u\big) \tag{3.5}$$

and

$$\mathbf{x}(u,v) = \big(pu + g(v), f(v)\sinh u, f(v)\cosh u\big) \tag{3.6}$$

respectively. When the expression $f^2 g'^2 - p^2 f'^2 > 0$, the first surface is time-like and the second is space-like and vice versa.

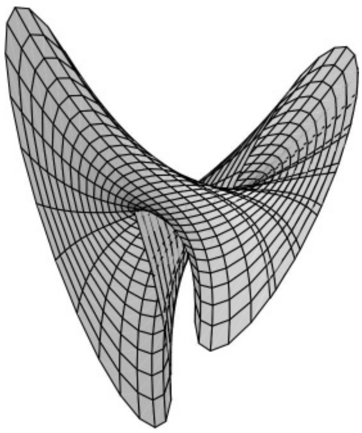

Fig. 2: The helical surface H_i, $p = 1$, $f(v) = \sin v$, $g(v) = \cos v$

The first fundamental form of a helical surface H_i obtained in this way is given by

$$\mathrm{d}s^2 = \big(p\,\mathrm{d}u + g'(v)\,\mathrm{d}v\big)^2 + \lambda\delta\big(f'^2(v)\,\mathrm{d}v^2 - f^2(v)\,\mathrm{d}u^2\big), \tag{3.7}$$

where $\lambda = 1$ for the surface (3.5) and $\lambda = -1$ for the surface (3.6).

If the curve c is parametrized by the isotropic arc length, that is if $g(v) = v$, then the above form can be simplified and the Gaussian and mean curvature are given by

$$K = -\varepsilon\frac{f^3 f'' - f'^4 p^2}{\big(f^2 - f'^2 p^2\big)^2}, \quad H = -\varepsilon\frac{f^2 - 2p^2 f'^2 + p^2 f f''}{2\sqrt{\big(f^2 - f'^2 p^2\big)^3}}. \tag{3.8}$$

Two interesting consequences of the above formulae will be presented in the following subsections.

3.1. Ruled screw surfaces in G_3^1. Ruled screw surfaces are special cases of helical surfaces. The starting curve c is a straight line. We remind the reader that in the pseudo-Galilean space, similar to the Euclidean space, there are three invariants defined for the most general type of ruled surfaces of type I. These invariants are the curvature κ, the torsion τ and the striction σ of the surface. Details on their definition and properties can be found in [3]. It turns out that ruled screw surfaces are ruled surfaces with constant invariants. Our main goal in this section is to determine the invariants.

We distinguish between open and closed ruled screw surfaces according to the position of the line c with respect to the axis e. If a line c intersects the axis e, a closed ruled screw surface is obtained, otherwise we have an open ruled screw surface.

1. *Open ruled screw surfaces.* The straight line $v \mapsto (bv, v, a)$, $a \neq 0$, $b \neq 0$, which does not intersect the x-axis is displaced under the screw motion and the surface

$$\mathbf{x}(t,v) = (pt, a\sinh t, a\cosh t) + v(b, \cosh t, \sinh t) \tag{3.9}$$

is obtained. This is a ruled surface with the striction curve $\mathbf{s}(t) = (pt, a\sinh t, a\cosh t)$. The striction curve lies in a pseudo-Euclidean plane and the obtained surface is a ruled surface of the type I. The arc length parametrization of the striction curve is

$$\mathbf{s}(u) = \left(u, a\sinh\frac{u}{p}, a\cosh\frac{u}{p}\right).$$

The unit generator vector field is given by $\mathbf{e}(u) = \left(1, \frac{1}{b}\cosh\frac{u}{p}, \frac{1}{b}\sinh\frac{u}{p}\right)$. The invariants of this ruled surfaces in G_3^1 are

$$\kappa = \frac{1}{p} = \text{const.}, \quad \tau = -\frac{b}{p} = \text{const.}, \quad \sigma = \frac{ab-p}{pb} = \text{const.}$$

As in the Euclidean and isotropic case ([5], [7]), notice that this surface is a tangent ruled surface of a helix if and only if $p = ab$.

2. *Closed ruled screw surfaces.* If $a = 0, b \neq 0$ in the expressions above, the straight line $v \mapsto (bv, v, 0)$ meets the screw-axis (x-axis) and the obtained surface is a closed ruled screw surface

$$\mathbf{x}(u,v) = (u,0,0) + v\left(1, \frac{1}{b}\cosh\frac{u}{p}, \frac{1}{b}\sinh\frac{u}{p}\right). \tag{3.10}$$

It is a ruled surface of type I with the invariants

$$\kappa = \frac{1}{p} = \text{const.}, \quad \tau = -\frac{b}{p} = \text{const.}, \quad \sigma = -\frac{1}{b} = \text{const.}$$

All space-like ruled surfaces in the pseudo-Galilean space have negative Gaussian curvature. Specially for closed ruled screw surfaces which are space-like, we deduce from (3.8) that $K = -\frac{p^2}{(p^2-b^2v^2)^2}$, where $p = -db$ and d is the parameter of distribution of this surface.

3. *Helicoids.* If c is a straight line orthogonal to the axis of the screw motion (x-axis), the obtained surface is a helicoid. A helicoid is therefore obtained by putting $a = 0$, $b = 0$ in the parametrization of the straight line. It is a ruled surface of type III (according to [3]) since it is a conoidal surface whose direction straight line in infinity is the absolute line.

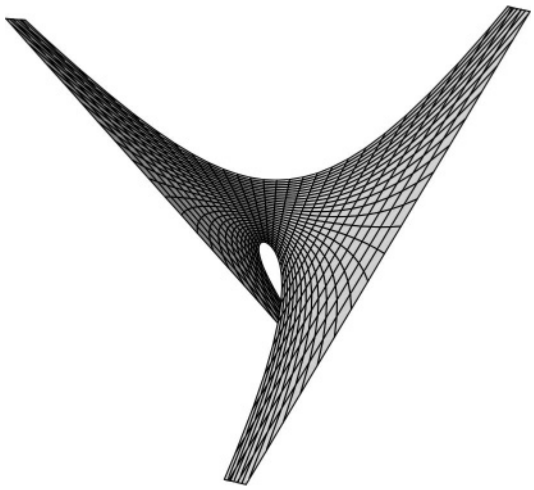

Fig. 3: The helicoid

As a consequence of the formulas for curvature and torsion, from the previous section we can derive that helicoid is minimal surface ($H = 0$) and that its Gaussian curvature $K = \varepsilon \frac{1}{p^2}$ is constant and different from zero. One class of helicoids consists of space-like surfaces

$$\mathbf{x}(u, v) = (pu, v \cosh u, v \sinh u)$$

and another of time-like surfaces

$$\mathbf{x}(u, v) = (pu, v \sinh u, v \cosh u).$$

Notice that a general characteristic of ruled surfaces of type III in Galilean [6] and pseudo-Galilean space [1] is that they are all minimal.

3.2. Circular screw surfaces in G_3^1. Circular screw surfaces are obtained by displacing a circle under a screw motion. We distinguish between two types of circles: pseudo-Euclidean circles and isotropic circles. If a time-like pseudo-Euclidean circle

$$\mathbf{r}(v) = (0, m + R\cosh v, R\sinh v)$$

undergoes the screw motion, the obtained surface is

$$\mathbf{x}(v,t) = \big(pv, (m + R\cosh v)\cosh t + R\sinh v \sinh t, \\ (m + R\cosh v)\sinh t + R\sinh v\cosh t\big).$$

It is a time-like surface with curvatures

$$K = -\frac{m\cosh v}{p^2 R}, \quad H = -\frac{\operatorname{sgn}(p)}{2R^2}(m\cosh v + R).$$

Analogously, if a space-like pseudo-Euclidean circle

$$\mathbf{r}(v) = (0, m + R\sinh v, R\cosh v)$$

undergoes the screw motion, the obtained surface is a space-like surface having

$$K = \frac{m\cosh v}{p^2 R}, \quad H = \frac{\operatorname{sgn}(p)}{2R^2}(m\cosh v + R).$$

We get a specially interesting case if we put $m = 0$ (it means that the center of the starting circle is positioned on the screw axis). Then the obtained time-like surface can be written as

$$\mathbf{x}(u,v) = (pu, R\cosh v, R\sinh v) \tag{3.11}$$

if we put $v = u - t$. Similarly, the obtained space-like surface is

$$\mathbf{x}(u,v) = (pu, R\sinh v, R\cosh v). \tag{3.12}$$

Their first fundamental forms are given by

$$\mathrm{d}s^2 = (p\,\mathrm{d}u)^2 - \varepsilon\delta R^2\,\mathrm{d}v^2.$$

The Gaussian and the mean curvature of the surfaces (3.11), (3.12) are given respectively by

$$K = 0, \quad H = -\varepsilon\frac{\operatorname{sgn}(p)}{2R}.$$

Furthermore, from the equation (3.11) we can notice that this kind of circular screw surface is a special case of ruled surfaces of type II in G_3^1. Types of ruled surfaces are described in [3].

COROLLARY 1. *A circular screw surface obtained from a pseudo-Euclidean circle, whose center is on the screw axis, is a surface of vanishing Gaussian curvature and non-vanishing constant mean curvature.*

These surfaces are analogous to cylinders in Euclidean space since they also have vanishing Gaussian curvature and non-vanishing constant mean curvature.

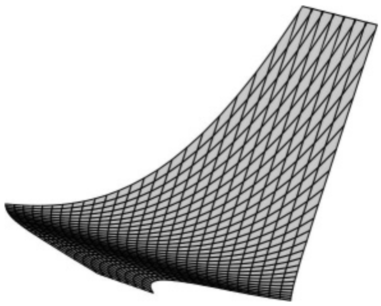

Fig. 4: A circular screw surface obtained from a pseudo-Euclidean circle

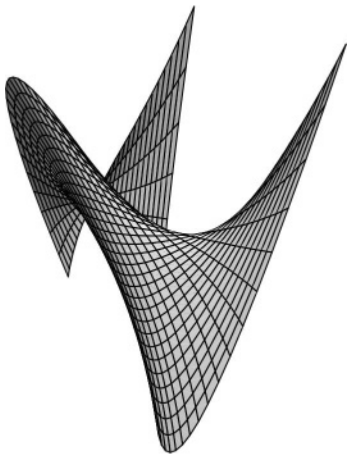

Fig. 5: A circular screw surface obtained from an isotropic circle

If an isotropic circle $\mathbf{r}(v) = (v, 2bv^2 - A, 0)$, $b \neq 0$, is displaced, the obtained surface is

$$\mathbf{x}(u,v) = \big(pu + v, (2bv^2 - A)\cosh u, (2bv^2 - A)\sinh u\big).$$

Its first fundamental form is

$$\mathrm{d}s^2 = (p\,\mathrm{d}u + \mathrm{d}v)^2 + \delta\big(-(2bv^2 - A)\,\mathrm{d}u^2 + 16b^2v^2\,\mathrm{d}v^2\big).$$

These surfaces are a special case of general screw surfaces H_i and from (3.8) it follows that the Gaussian curvature never vanishes. We get analogous results when an isotropic circle $\mathbf{r}(v) = (v, 0, 2bv^2 - A)$, $b \neq 0$, is displaced.

4. Surfaces of revolution in G_3^1

4.1. Rotations in G_3^1. In the pseudo-Galilean space we distinguish between two types of circles and between two types of surfaces of revolution. The first type occurs as the result of a pseudo-Euclidean rotation and the second as the result of the isotropic rotation. We already mentioned that pseudo-Euclidean rotations in G_3^1 are obtained by substituting $p = 0$ in (3.1). Besides them there are isotropic rotations. Their normal form is the following:

$$\bar{x}(t) = x + bt, \quad \bar{y}(t) = y + xt + b\frac{t^2}{2}, \quad \bar{z}(t) = z, \tag{4.1}$$

where $t \in \mathbf{R}$ and $b > 0$. The trajectory of a single point is an isotropic circle, whose normal form is

$$z = \text{const.}, \quad y = \frac{x^2}{2b}. \tag{4.2}$$

The invariant b is the radius of the circle. The fixed line of the isotropic rotations (4.1) is the absolute line f. The trajectory of a point under a pseudo-Euclidean rotation is a pseudo-Euclidean circle with the normal form

$$x = \text{const.}, \quad y^2 - z^2 = R^2, \quad R \in \mathbf{R}. \tag{4.3}$$

The invariant R is the radius of the circle. Pseudo-Euclidean circles intersect the absolute line f in the fixed points of the hyperbolic involution (F_1, F_2), $F_1(0:0:1:1)$, $F_2(0:0:1:-1)$. There are three kinds of pseudo-Euclidean circles: circles of real radius, of imaginary radius and of radius zero. Circles of real radius are time-like curves (having time-like tangent field) and of imaginary radius space-like curves (having space-like tangent field).

4.2. Surfaces of revolution in G_3^1. Let c be an admissible plane curve in G_3^1 and let a be a coplanar line that does not meet c. When the profile curve c is revolved around the axis a, it sweeps out a surface of revolution.

Let c be an admissible plane curve with the parametrization

$$\mathbf{r}(v) = \big(g(v), h(v), 0\big) \quad \text{or} \quad \mathbf{r}(v) = \big(g(v), 0, h(v)\big),$$

where $h(v) > 0$.

The curve c is rotated under the pseudo-Euclidean rotation about the x-axis. Two types of surfaces of revolution S_p are obtained parametrized by (3.5) and (3.6) with $p = 0$.

If we take $h(v) = R$ we get circular screw surface obtained from a pseudo-Euclidean circle.

The Gaussian curvature of S_p is $K = -\varepsilon\left(\frac{h'}{g'}\right)'\frac{1}{g'h}$ and the mean curvature is $H = \varepsilon \operatorname{sgn}(g')\frac{1}{h}$.

Let us take $K = \text{const.} \neq 0$ and parametrize the curve c by the arc length i.e. $g(v) = v$. We can prove (see [11]) that by a pseudo-Euclidean rotation of a curve of the form

$$\mathbf{r}(v) = \left(v, \left(A\cosh\left(\sqrt{|K|}v\right) + B\sinh\left(\sqrt{|K|}v\right)\right), 0\right)$$

or of the form

$$\mathbf{r}(v) = \left(v, \left(A\cos\left(\sqrt{|K|}v\right) + B\sin\left(\sqrt{|K|}v\right)\right), 0\right)$$

we get the surface which has constant Gaussian curvature equal to K. Note that if $K = 0$, then the profile curve is a line $h(v) = Av + B$, $A, B \in \mathbf{R}$, and among these surfaces obtained by rotation of a line there are also the hyperbolic spheres $y^2 - z^2 = \pm R^2$.

Now let c be an admissible plane curve with the parametrization

$$\mathbf{r}(u) = \left(0, g(u), h(u)\right), \tag{4.4}$$

where $g(u) > 0$. The curve c is rotated under the isotropic rotation (4.1) around the z-axis. The surface of revolution S_i with parametrization

$$\mathbf{x}(u, v) = \left(bv, g(u) + b\frac{v^2}{2}, h(u)\right) \tag{4.5}$$

is obtained.

The first fundamental form of a surface of revolution S_i is given by

$$\mathrm{d}s^2 = b^2\,\mathrm{d}v^2 + \delta\left((g'^2 - h'^2)\,\mathrm{d}u^2 + 2g'bv\,\mathrm{d}u\,\mathrm{d}v + b^2v^2\,\mathrm{d}v^2\right). \tag{4.6}$$

When c is parametrized by the arc length e.i. $g'^2 - h'^2 = \varepsilon$, where $\varepsilon = \pm 1$, we get the Gaussian curvature $K = \varepsilon\frac{1}{b}(h')^3\left(\frac{g'}{h'}\right)'$ and the mean curvature

$$H = \varepsilon \operatorname{sgn}(b)\left(\frac{g'}{h'}\right)'(h'^2). \tag{4.7}$$

Then we can easily find surfaces with constant Gaussian curvature (see [11]), flat and minimal surfaces of revolution. From the expressions for the Gaussian and mean curvatures of S_i the next theorem follows.

THEOREM 2. *Surfaces of revolution S_i in G_3^1 are flat* ($K \equiv 0$) *if and only if they are minimal* ($H \equiv 0$).

In the Euclidean space there are no flat minimal surfaces of revolution.

Notice finally, that a parabolic sphere in the pseudo-Galilean space (with the normal equation $x^2 = 2z$) can be obtained by the isotropic rotation (4.1) of an isotropic circle $\mathbf{r}(u) = (u, 0, 2pu^2 - A)$. The obtained surface admits the parametrization

$$\mathbf{x}(u,t) = \left(u + bt, ut + \frac{bt^2}{2}, 2pu^2 - A\right).$$

Its Gaussian, as well as mean curvature is equal to 0, and it satisfies the equation $x^2 = \frac{z+A}{2p} + 2by$.

5. Torus of revolution in G_3^1

The torus of revolution is a surface of revolution obtained by revolving a pseudo-Euclidean circle around a coplanar axis. Tori in the Galilean and the double isotropic case have been described in [12] and [13]. Similarly to the Galilean case, in the pseudo-Galilean case there are two types of tori according to the performed rotation.

I. An isotropic circle is rotated under pseudo-Euclidean rotations. We can suppose that the isotropic circle is of the form

$$\mathbf{r}(v) = (v, 2pv^2 - A, 0), \quad v \in \mathbf{R} \tag{5.1}$$

where $p \neq 0$ and A are real constants. Under a pseudo-Euclidean rotation around the x-axis we obtain the torus T_i parametrized by

$$\mathbf{x}(v,t) = \big(v, (2pv^2 - A)\cosh t, (2pv^2 - A)\sinh t\big). \tag{5.2}$$

It is a time-like surface and a special case of surfaces of revolution S_p. It satisfies the algebraic equation

$$y^2 - z^2 = \left(2px^2 - A\right)^2 \tag{5.3}$$

and therefore it is an algebraic surface of order 4.

Analogously, by rotating (under pseudo-Euclidean rotations around the x-axis) the isotropic circle of the form

$$\mathbf{r}(v) = (v, 0, 2pv^2 - A), \quad v \in \mathbf{R}, \tag{5.4}$$

the obtained torus

$$\mathbf{x}(v,t) = \big(v, (2pv^2 - A)\sinh t, (2pv^2 - A)\cosh t\big) \tag{5.5}$$

is a space-like surface. It satisfies the algebraic equation

$$z^2 - y^2 = (2px^2 - A)^2 \tag{5.6}$$

of order 4.

With respect to the position of the meridian m (i.e. various positions of the rotating curve) of T_i and the rotation axis we distinguish among three types of torus surfaces T_i.

1. $\frac{A}{p} > 0$. The meridian m intersects the rotation axis in two real points and the spindle torus is obtained.

2. $A = 0$. The meridian m intersects the rotation axis in one real point. We call such a surface the horn torus.

3. $\frac{A}{p} < 0$. The meridian m intersects the rotation axis in two imaginary points. We call such a surface the ring torus.

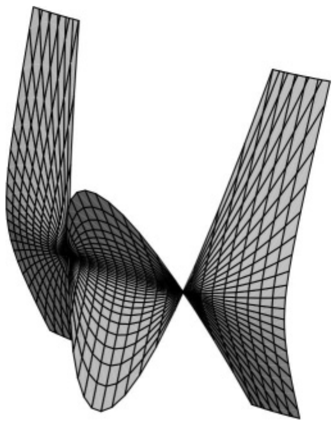

Fig. 6: The spindle torus

A surface T_i contains the absolute line f and the horn torus contains all lightlike lines in the absolute (pseudo-Euclidean) plane.

As in the Euclidean case the intersection of torus with a plane is a curve of order 4. The curves which are intersections of the torus surface and a plane parallel to the rotation axis are called spiral curves. Let us suppose that the plane is an isotropic plane $z = k = \text{const}$. The equation of the spiral curve is

$$y^2 = (2px^2 - A)^2 + k^2. \tag{5.7}$$

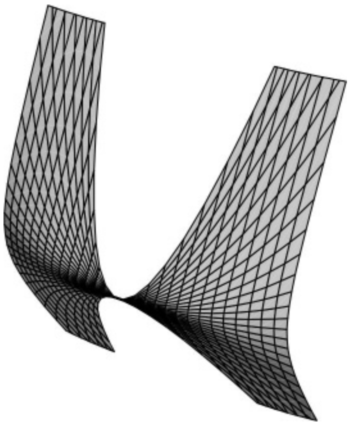

Fig. 7: The horn torus

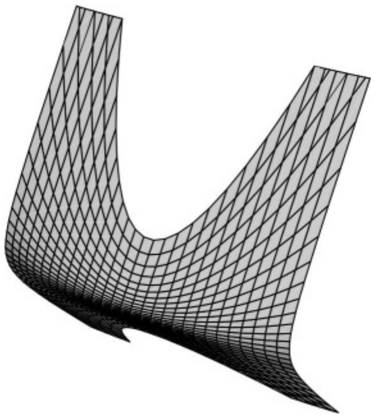

Fig. 8: The ring torus

II. A pseudo-Euclidean circle is rotated under isotropic rotations. Let us suppose that the rotating pseudo-Euclidean circle has real or imaginary radius, so it is given by

$$\mathbf{r}(u) = (0, R\cosh u, R\sinh u) \quad \text{or} \quad \mathbf{r}(u) = (0, R\sinh u, R\cosh u),$$

where $u \in \mathbf{R}$, $R \in \mathbf{R}\backslash\{0\}$. Under the isotropic rotation (4.1) around the z-axis we obtain the torus T_p with the parametrization

$$\mathbf{x}(u,t) = \left(bt, R\cosh u + b\frac{t^2}{2}, R\sinh u\right)$$

or

$$\mathbf{x}(u,t) = \left(bt, R\sinh u + b\frac{t^2}{2}, R\cosh u\right).$$

They can be written in implicit form as

$$\left(y - \frac{x^2}{2b}\right)^2 = z^2 - R^2 \tag{5.8}$$

or

$$\left(y - \frac{x^2}{2b}\right)^2 = z^2 + R^2. \tag{5.9}$$

Both surfaces are algebraic surfaces of order 4.

The surface (5.8) intersects the axis of rotation in two real points and it is called spindle torus of type II. Contrary to that, the surface (5.9) intersects the axis of rotation in two imaginary points and it is called ring torus of type II. Spiral curves are hyperbolas.

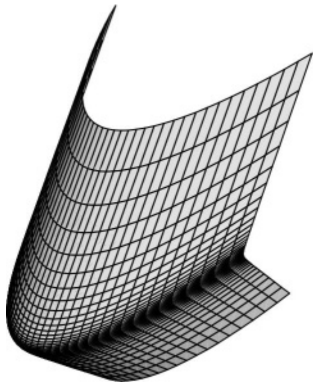

Fig. 9: The torus T_p *of the form* (5.8)

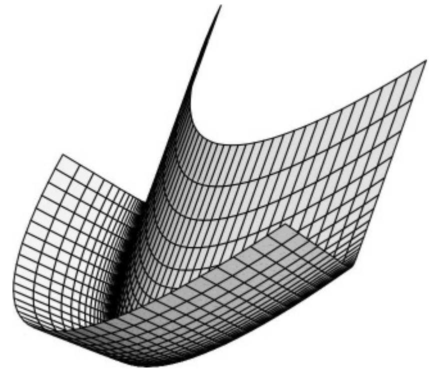

Fig. 10: The torus T_p *of the form* (5.9)

THEOREM 3. *The torus surfaces* T_p *of type II are the surfaces with constant mean curvature.*

PROOF. From (4.7) we have $H = \varepsilon \operatorname{sgn}(b) \frac{1}{R}$.

REMARK 1. In [3] the following theorem was proved: Every helicoid is locally isometric to a torus T_p obtained by rotating a unit pseudo-Euclidean circle by isotropic rotation.

References

[1] B. Divjak, *Geometrija pseudogalilejevih prostora*, PhD thesis, University of Zagreb (Zagreb, 1997).

[2] B. Divjak and Ž. Milin Šipuš, Special curves on ruled surfaces in Galilean and pseudo-Galilean space, *Acta Math. Hungar.,* **98** (2003), 203–215.

[3] B. Divjak and Ž. Milin Šipuš, Minding isometries of ruled surfaces in pseudo-Galilean space, *J. Geom.,* **77** (2003), 35–47.

[4] E. Molnár, The projective interpretation of the eight 3-dimensional homogeneous geometries, *Beitr. Algebra Geom.,* **38** (1997), 261–288.

[5] E. Kruppa, Zur Differentialgeometrie der Strahlflächen und Raumkurven, *Sb. Akad. Wiss. Wien IIa,* **157** (1949), 143–176.

[6] O. Röschel, *Die Geometrie des Galileischen Raumes,* Habilitationsschrift (Leoben, 1984).

[7] H. Sachs, *Isotrope Geometrie des Raumes,* Friedr. Vieweg & Sohn (Braunschweig/ Wiesbaden, 1990).

[8] J. Pierport, The history of mathematics in the nineteenth century, *Bull. Amer. Math. Soc.,* **37** (1999), 9–24.

[9] F. Dillen and W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscipta Math.,* **98** (1999), 307–320.

[10] E. Kreyszig, *Differential geometry,* Dover Publication, Inc (New York, 1991).

[11] Ž. Milin Šipuš and B. Divjak, Surfaces of constant curvature in the pseudo-Galilean space, submitted.

[12] O. Röschel, Torusflächen des Galileischen Raumes, *Studia Sci. Math. Hungar.,* **23** (1988), 401–410.

[13] O. Röschel, Bemerkung zu den Torusflächen des Flagenraumes, *Rad JAZU,* (1987), 428–435.

[14] Y. H. Kim and D. W. Yoon, Classification of ruled surfaces in Minkowski 3-spaces, *J. Geom. Phys.,* **49** (2004), 89–100.