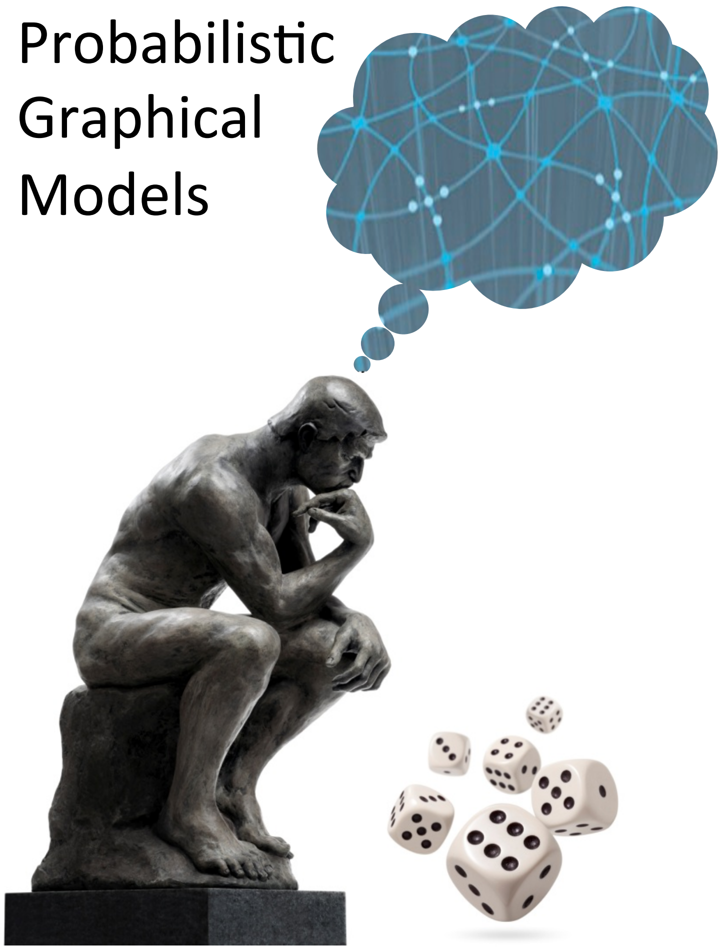
Probabilistic
Graphical
Models



Representation

Independencies

Bayesian Networks

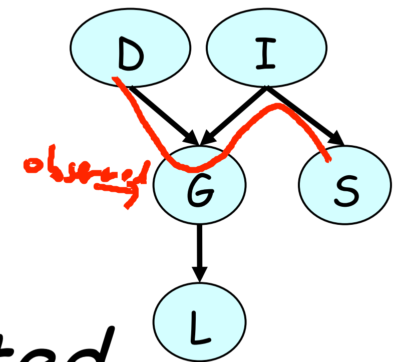
Independence & Factorization

$P(X,Y) = P(X)\,P(Y)$ — X,Y independent

$P(\mathbf{X},\mathbf{Y},\mathbf{Z}) \propto \phi_1(\mathbf{X},\mathbf{Z})\,\phi_2(\mathbf{Y},\mathbf{Z})$ — $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$

- Factorization of a distribution P implies independencies that hold in P
- If P factorizes over G, can we read these independencies from the structure of G?

Flow of influence & d-separation



Definition: **X** and **Y** are *d-separated* in G given **Z** if there is no active trail in G between **X** and **Y** given **Z**

Notation: d-sep$_G$(**X**, **Y** | **Z**)

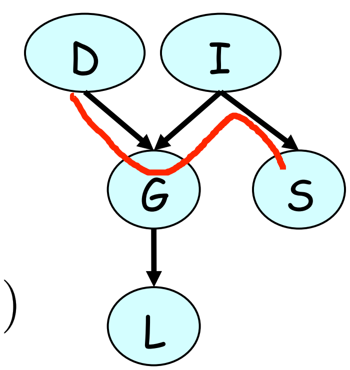
Factorization ⇨ Independence: BNs

Theorem: If P factorizes over G, and d-sep$_G$(**X**, **Y** | **Z**) then P satisfies (**X** ⊥ **Y** | **Z**)

chain rule

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

P ⊨ D ⊥ S



$$P(D,S) = \sum_{G,\cancel{L},I} P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

$\sum_L$

$$= \sum_I P(D)P(I)P(S \mid I) \underbrace{\sum_G \Big(\overbrace{\cancel{P(G \mid D,I)}}^{1} \underbrace{\sum_L \overbrace{\cancel{P(L \mid G)}}^{1}}\Big)}$$

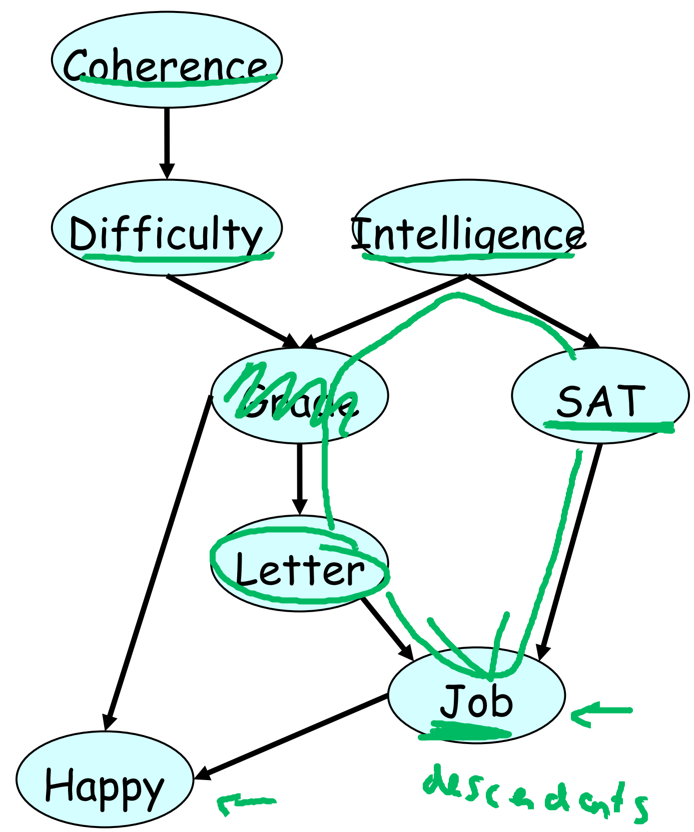
$$= \underbrace{P(D)}_{\phi_1(D)}\Big(\overbrace{\sum_I P(I)P(S \mid I)}^{\phi_2(S)}\Big)$$

Daphne Koller

Any node is d-separated from its non-descendants given its parents

Grade

If P factorizes over G, then in P, any variable is independent of its non-descendants given its parents



Daphne Koller

I-maps

- d-separation in $G \Rightarrow P$ satisfies corresponding independence statement

$$I(G) = \{(X \perp Y \mid Z) \;:\; \text{d-sep}_G(X, Y \mid Z)\}$$

- Definition: If P satisfies $I(G)$, we say that G is an I-map (independency map) of P

I-maps

$P_1 \models D \perp I$

I	D	Prob
i^0	d^0	0.42
i^0	d^1	0.18
i^1	d^0	0.28
i^1	d^1	0.12

$P_2 \not\models D \perp I$

I	D	Prob.
i^0	d^0	0.282
i^0	d^1	0.02
i^1	d^0	0.564
i^1	d^1	0.134

I-map

$I(G_1) = \{D \perp I\}$

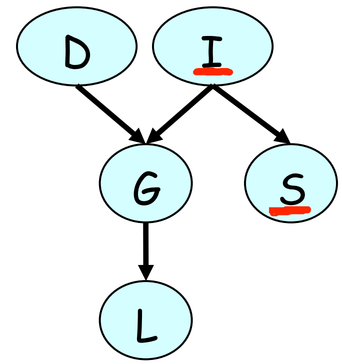
$I(G_2) = \emptyset$

Factorization ⇨ Independence: BNs

Theorem: If P factorizes over G, then G is an I-map for P

Can read from G independencies in P
regardless of parameters

Independence ⇨ Factorization



Theorem: If G is an I-map for P, then P factorizes over G

$I \perp D$

P(I,D) chain rule for probabilities

$$P(D,I,G,S,L) = P(D)P(I \mid \cancel{D})P(G \mid D,I)P(S \mid \cancel{D},I,\cancel{G})P(L \mid \cancel{D},\cancel{I},G,\cancel{S})$$

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

Daphne Koller

Summary

Two equivalent views of graph structure:

- Factorization: G allows P to be represented
- I-map: Independencies encoded by G hold in P

If P factorizes over a graph G, we can read from the graph independencies that must hold in P (an independency map)