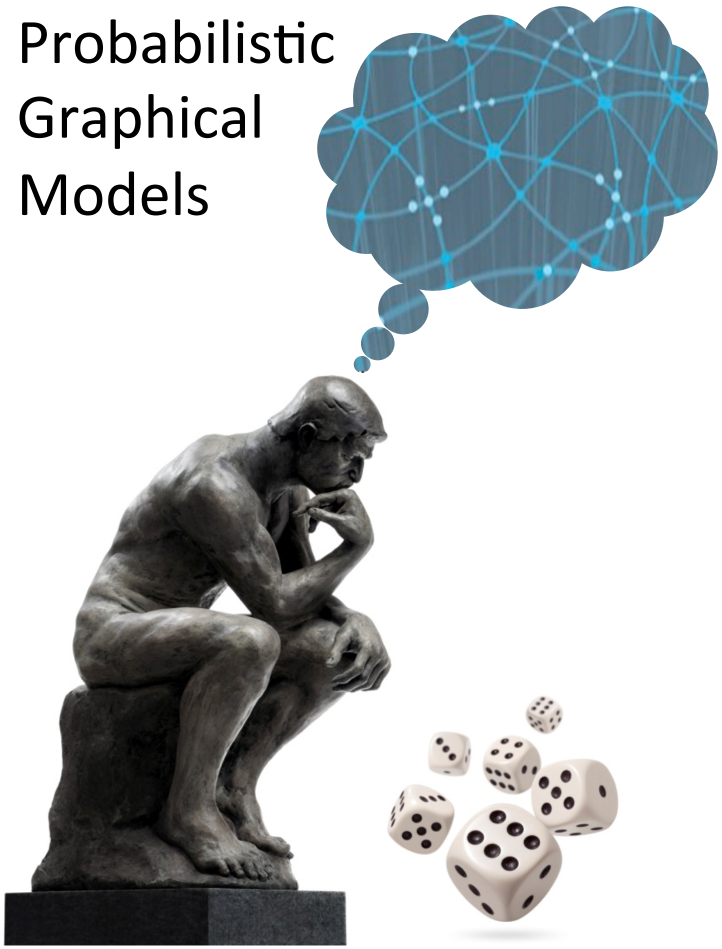
Probabilistic
Graphical
Models



Representation

Independencies

I-maps and Perfect Maps

Capturing Independencies in P

dist   independencies   that hold in P

$$I(P) = \{(\boldsymbol{X} \perp \boldsymbol{Y} \mid \boldsymbol{Z}) \ : \ P \models (\boldsymbol{X} \perp \boldsymbol{Y} \mid \boldsymbol{Z})\}$$

- P factorizes over G ⇨ G is an I-map for P:

d-separation $$I(G) \subseteq I(P)$$

- But not always vice versa: there can be independencies in I(P) that are not in I(G)

Want a Sparse Graph

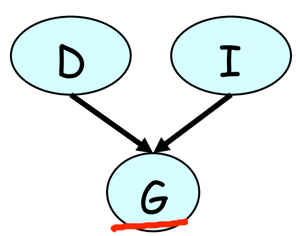
- If the graph encodes more independencies
  - it is sparser (has fewer parameters)
  - and more informative
- Want a graph that captures as much of the structure in P as possible

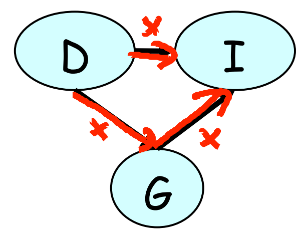
Minimal I-map

- Minimal I-map: I-map without redundant edges

  $X \not\to Y \quad P(Y \mid x^0) = P(Y \mid x^1)$

- Minimal I-map may still not capture I(P)





$D \perp G$

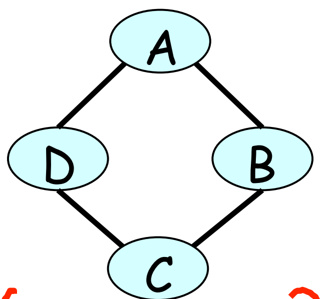
$DG \perp I \mid D$

$D \perp I \mid G$

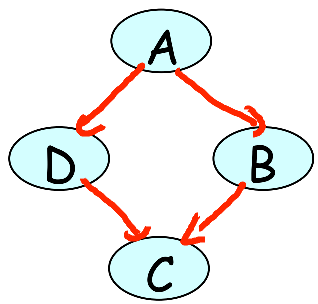
Perfect Map

- Perfect map: I(G) = I(P)
  - G perfectly captures independencies in P

Perfect Map

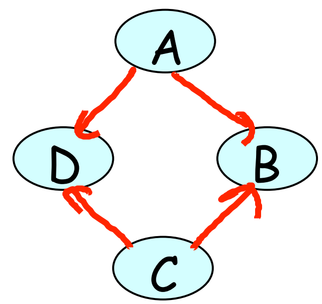


$P \models \left\{ \begin{array}{l} A \perp C \mid B, D \\ B \perp D \mid A, C \end{array} \right\}$



$B \perp D \mid A$

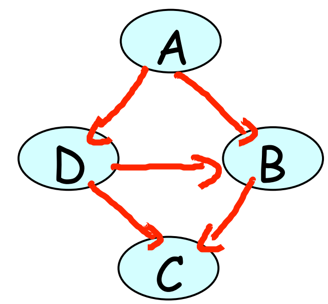
not an I-map



$B \perp D \mid A, C$

$A \perp C$

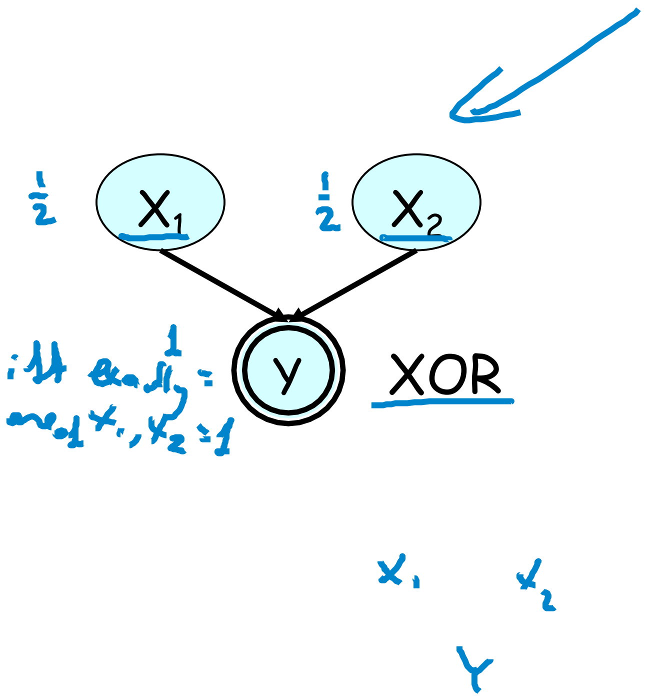
not an I-map



$A \perp C \mid B, D$

$I(G) \subseteq I(P)$

Another imperfect map



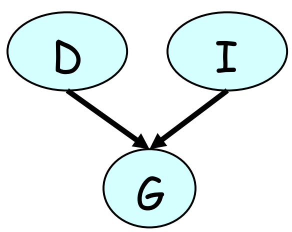
X_1	X_2	Y	Prob
0	0	0	0.25
0	1	1	0.25
1	0	1	0.25
1	1	0	0.25

$X_1 \perp X_2$ $\quad$ $X_1 \perp Y$ $\quad$ $X_2 \perp Y$

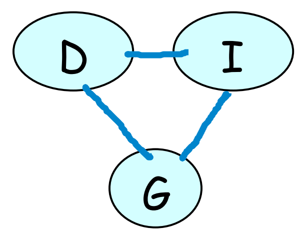
Daphne Koller

MN as a perfect map

- Perfect map: I(H) = I(P)
  - H perfectly captures independencies in P



$D \perp I$



$D \perp I \mid G$

Daphne Koller

Uniqueness of Perfect Map

$G_1 \quad X \rightarrow Y \quad I(G_1) = \emptyset$

$G_2 \quad X \leftarrow Y \quad I(G_2) = \emptyset$

$\Longrightarrow$ can represent exactly the same distribution

I-equivalence

Definition: Two graphs G_1 and G_2 over $X_1, \ldots, X_n$ are *I-equivalent* if $I(G_1)=I(G_2)$

$X \rightarrow Y \rightarrow Z$ $X \leftarrow Y \rightarrow Z$ $X \leftarrow Y \leftarrow Z$

unidentifiable

Most G's have many I-equivalent variants

Summary

- Graphs that capture more of I(P) are more compact and provide more insight
- A minimal I-map may fail to capture a lot of structure even if present *and representable as a PGM*
- A perfect map is great, but may not exist
- Converting BNs ↔ MNs loses independencies
  - BN to MN: loses independencies in v-structures
  - MN to BN: must add triangulating edges to loops