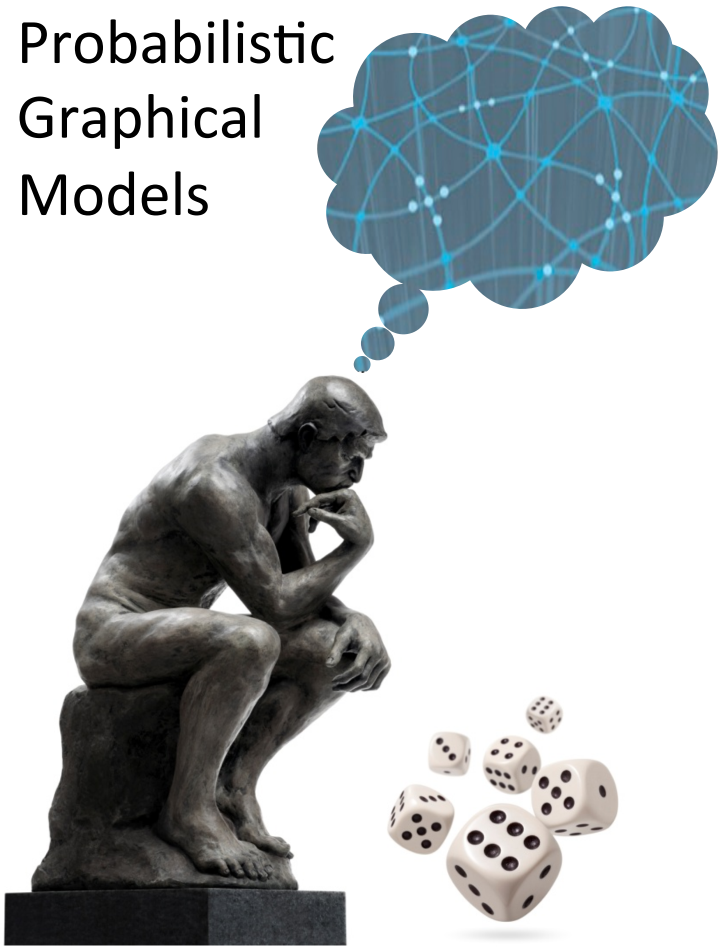
Probabilistic
Graphical
Models

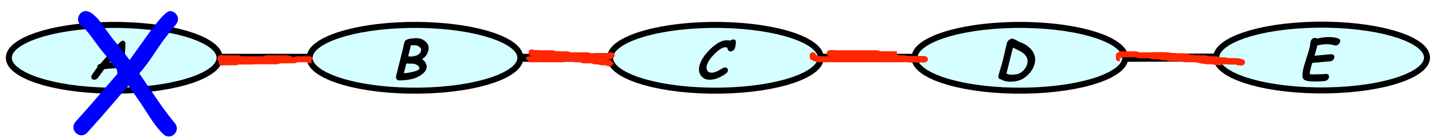


Inference

Variable Elimination

Variable Elimination Algorithm

Elimination in Chains



$$P(E) \propto \sum_D \sum_C \sum_B \sum_A \widetilde{P}(A,B,C,D,E)$$

$$= \sum_D \sum_C \sum_B \sum_A \phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(D,E)$$

$$= \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E) \underbrace{\sum_A \phi_1(A,B)}_{\tau_1(B)}$$

$$= \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E)\tau_1(B)$$

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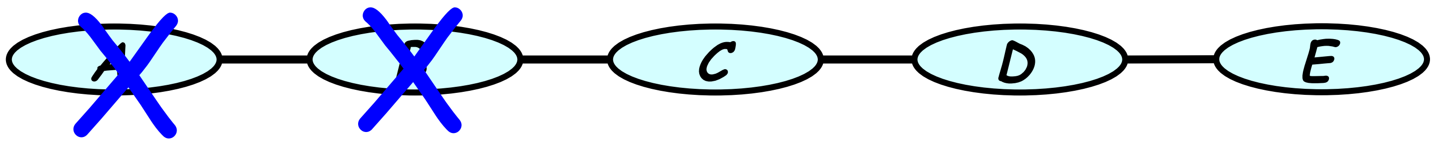
Elimination in Chains



$$P(E) \propto \sum_D \sum_C \sum_B \phi_2(B,C)\phi_3(C,D)\phi_4(D,E)\tau_1(B)$$

$$= \sum_D \sum_C \phi_3(C,D)\phi_4(D,E)\left(\overbrace{\sum_B \phi_2(B,C)\tau_1(B)}^{\tau_2(c)}\right)$$

$$= \sum_D \sum_C \phi_3(C,D)\phi_4(D,E)\tau_2(C)$$

Variable Elimination

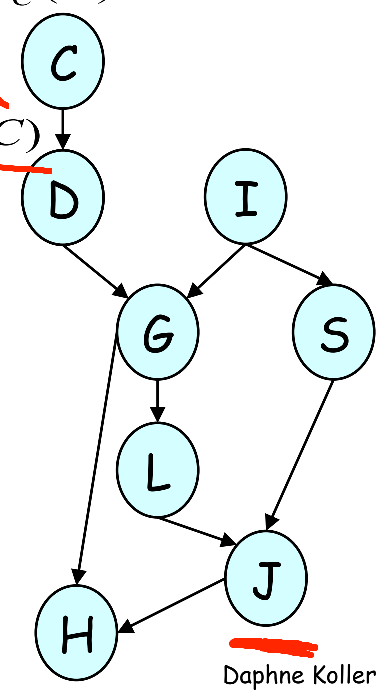
- Goal: P(J)
- Eliminate: C,D,I,H,G,S,L

$$\sum_{L,S,G,H,I,D,C} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\phi_D(C,D)\phi_C(C)$$

$$\sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\overbrace{\sum_C \phi_D(C,D)\phi_C(C)}^{\tau_1(D)}$$

Compute $\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$

$$= \sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_G(G,I,D)\phi_H(H,G,J)\phi_I(I)\tau_1(D)$$



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Variable Elimination

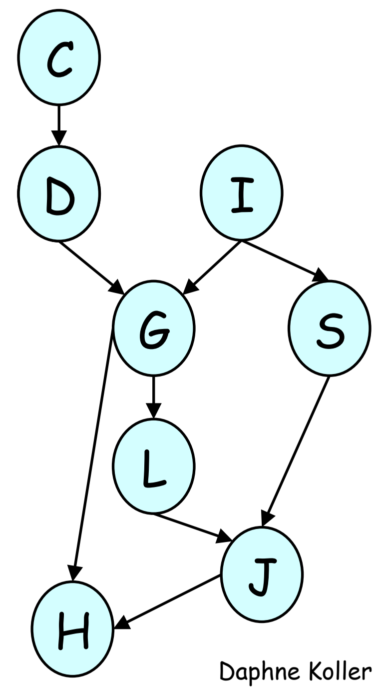
- Goal: P(J)
- Eliminate: $\underline{D}$,I,H,G,S,L

$$\sum_{L,S,G,H,I,D} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\underline{\phi_G(G,I,D)}\phi_H(H,G,J)\phi_I(I)\underline{\tau_1(D)}$$

$$= \sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\sum_D \phi_G(G,I,D)\tau_1(D)$$

Compute $\underline{\tau_2(G,I)} = \sum_{\underline{D}} \overbrace{\phi_G(G,I,D)\tau_1(D)}^{G,I,D}$

$$= \sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\tau_2(G,I)$$



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Variable Elimination

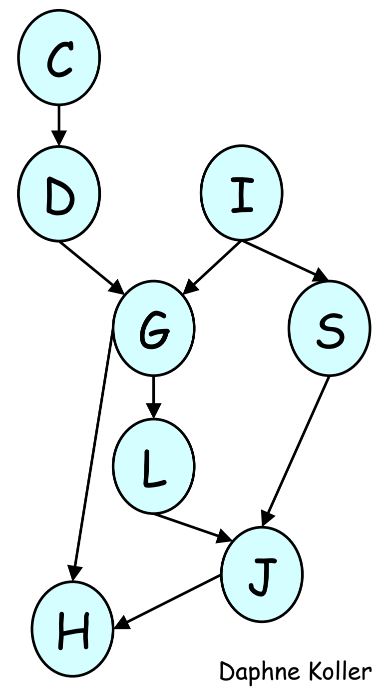
- Goal: P(J)
- Eliminate: I,H,G,S,L

$$\sum_{L,S,G,H,I} \phi_J(J,L,S)\phi_L(L,G)\phi_S(S,I)\phi_H(H,G,J)\phi_I(I)\tau_2(G,I)$$

$$= \sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\overbrace{\sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)}^{G,I,S}$$

Compute $\tau_3(S,G) = \sum_I \phi_S(S,I)\phi_I(I)\tau_2(G,I)$

$$= \sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\tau_3(S,G)$$



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Variable Elimination

- Goal: P(J)
- Eliminate: H,G,S,L

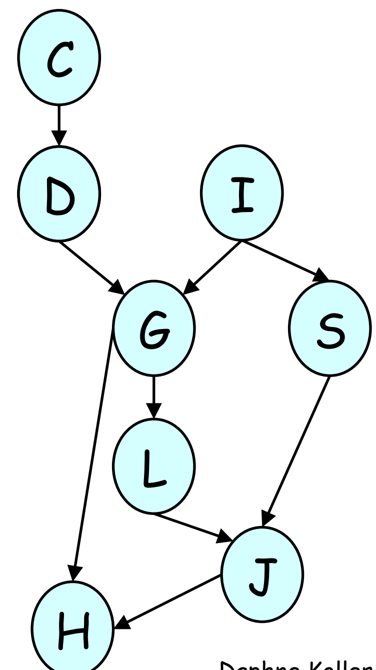
$$\sum_{L,S,G,H} \phi_J(J,L,S)\phi_L(L,G)\phi_H(H,G,J)\tau_3(S,G)$$

$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\underbrace{\sum_H \phi_H(H,G,J)}_{\tau_4(G,J)}$$

$\sum P(H|G,J) = 1$

Compute $\tau_4(G,J) = \sum_H \phi_H(H,G,J)$

$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$



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Variable Elimination

- Goal: P(J)
- Eliminate: G,S,L

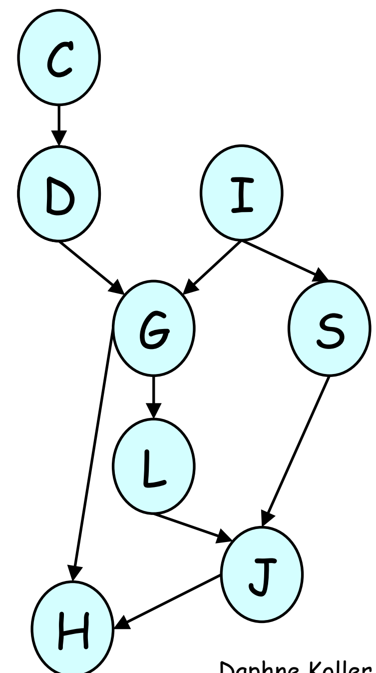
$$\sum_{L,S,G} \phi_J(J,L,S)\phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$$

L, G, S, J

$$\sum_{L,S} \phi_J(J,L,S)\sum_{G} \phi_L(L,G)\tau_4(G,J)\tau_3(S,G)$$

Compute $\tau_5(L,J,S) = \sum_{G} \phi_L(L,G)\tau_3(S,G)\tau_4(G,J)$

$$\sum_{L,S} \phi_J(J,L,S)\tau_5(L,J,S)$$



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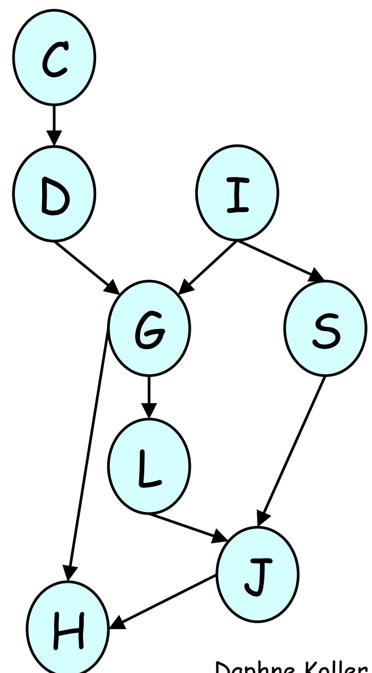
Variable Elimination

- Goal: P(J)
- Eliminate: S,L

$$\sum_{L,S} \phi_J(J,L,S)\tau_5(L,J,S)$$



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Variable Elimination with evidence

- Goal: P(J,I=i,H=h)
- Eliminate: C,D,G,S,L (no H, I)

P(G|i,D) $\Phi_I(i)$

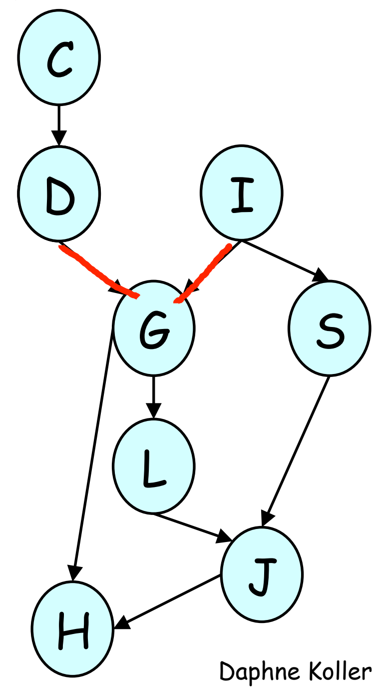
$$\sum_{L,S,G,,D,C} \phi_J(J,L,S)\phi_L(L,G)\phi_S'(S)\phi_G'(G,D)\phi_H'(G,J)\phi_I'()\phi_D(C,D)\phi_C(C)$$

elimination as before

How do we get P(J | I=i,H=h)?

renormalize

P(I=i, H=h) is the renormalizing constant



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Variable Elimination in MNs

- Goal: P(D)
- Eliminate: A,B,C

$$\sum_{A,B,C} \phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(A,D)$$

A,B,D

$$\sum_{B,C} \phi_2(B,C)\phi_3(C,D)\underbrace{\sum_{A} \phi_1(A,B)\phi_4(A,D)}_{\tau_1(B,D)}$$

$$\sum_{B,C} \phi_2(B,C)\phi_3(C,D)\tau_1(B,D)$$

At the end of elimination get $\tau_3(D) = \tilde{P}(D) \propto P(D)$

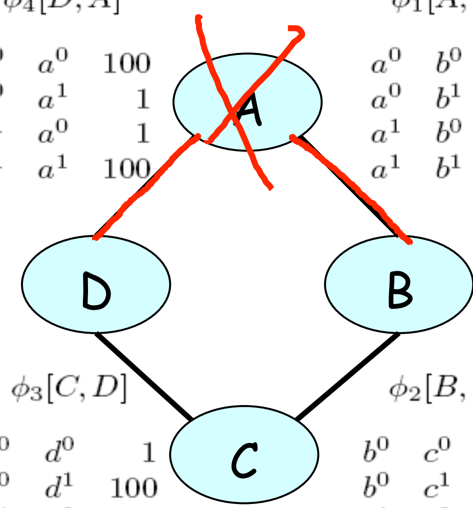
renormalize

$\phi_4[D,A]$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

$\phi_1[A,B]$

a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10



$\phi_3[C,D]$

c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\phi_2[B,C]$

b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

Eliminate-Var Z from Φ

all factors that involve z

$$\Phi' = \{\phi_i \in \Phi \;:\; Z \in Scope[\phi_i]\}$$

$$\psi = \prod_{\phi_i \in \Phi'} \phi_i$$

multiply them

$$\tau = \sum_Z \psi$$

used sum out z

$$\Phi := \Phi - \Phi' \cup \{\tau\}$$

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VE Algorithm Summary

- Reduce all factors by evidence
  - Get a set of factors Φ
- For each non-query variable Z
  - Eliminate-Var Z from Φ

    ~~adds~~ removes Φ'
    adds τ
- Multiply all remaining factors
- Renormalize to get distribution

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Summary

- Simple algorithm
- Works for both BNs and MNs
- Factor product and summation steps can be done in any order, subject to:
  - when Z is eliminated, all factors involving Z have been multiplied in