Probabilistic
Graphical
Models

Inference

Message Passing

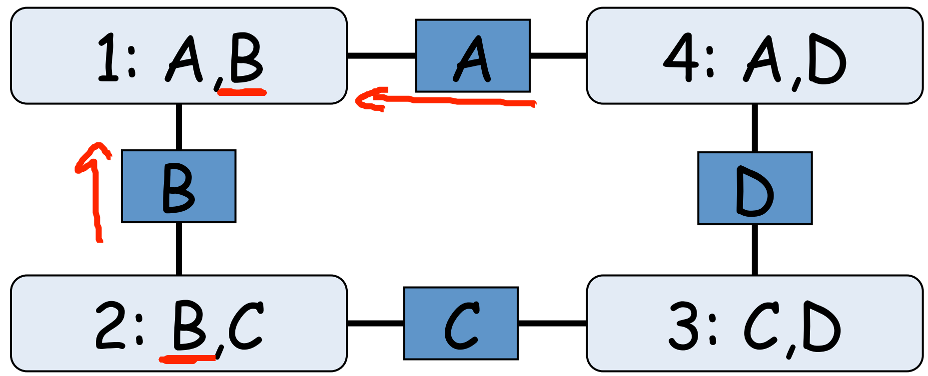
Properties of BP Algorithm

Calibration

$$\beta_1(A, B) = \psi_1(A, B) \times \delta_{4\to 1}(A) \times \delta_{2\to 1}(B)$$

- Cluster beliefs:

$$\beta_i(\boldsymbol{C}_i) = \psi_i \times \prod_{k \in \mathcal{N}_i} \delta_{k\to i}$$



- A cluster graph is calibrated if every pair of adjacent clusters $\boldsymbol{C}_i,\boldsymbol{C}_j$ agree on their sepset $\boldsymbol{S}_{i,j}$

$$\sum_{\boldsymbol{C}_i - \boldsymbol{S}_{i,j}} \beta_i(\boldsymbol{C}_i) = \sum_{\boldsymbol{C}_j - \boldsymbol{S}_{i,j}} \beta_j(\boldsymbol{C}_j)$$

sepset $S_{i,j}$

Convergence ⇨ Calibration

$$\beta_i(\boldsymbol{C}_i) = \psi_i \times \prod_{k \in \mathcal{N}_i} \delta_{k \to i}$$

all msgs

- Convergence: $\delta_{i \to j}(\boldsymbol{S}_{i,j}) = \delta'_{i \to j}(\boldsymbol{S}_{i,j})$

$\delta_{i \to j}$

msgs except from j

$$\delta'_{i \to j}(\boldsymbol{S}_{i,j}) = \sum_{\boldsymbol{C}_i - \boldsymbol{S}_{i,j}} \left(\psi_i \times \prod_{k \in (\mathcal{N}_i - \{j\})} \delta_{k \to i} \right) = \sum_{\boldsymbol{C}_i - \boldsymbol{S}_{i,j}} \frac{\beta_i(\boldsymbol{C}_i)}{\delta_{j \to i}(\boldsymbol{S}_{i,j})} =$$

$$\delta_{j \to i}(\boldsymbol{S}_{i,j}) \delta_{i \to j}(\boldsymbol{S}_{i,j}) = \sum_{\boldsymbol{C}_i - \boldsymbol{S}_{i,j}} \beta_i(\boldsymbol{C}_i)$$

calibration

$$\delta_{j \to i}(\boldsymbol{S}_{i,j}) \delta_{i \to j}(\boldsymbol{S}_{i,j}) = \sum_{\boldsymbol{C}_j - \boldsymbol{S}_{i,j}} \beta_j(\boldsymbol{C}_j)$$

$$\Rightarrow \sum_{\boldsymbol{C}_i - \boldsymbol{S}_{i,j}} \beta_i(\boldsymbol{C}_i) = \sum_{\boldsymbol{C}_j - \boldsymbol{S}_{i,j}} \beta_j(\boldsymbol{C}_j)$$

$$\mu_{i,j}(\boldsymbol{S}_{i,j}) = \delta_{j \to i} \delta_{i \to j} = \sum_{\boldsymbol{C}_j - \boldsymbol{S}_{i,j}} \beta_j(\boldsymbol{C}_j)$$

sepset beliefs

Reparameterization

$$\boxed{\beta_i(\boldsymbol{C}_i) = \psi_i \times \prod_{k \in \mathcal{N}_i} \delta_{k \to i}}$$

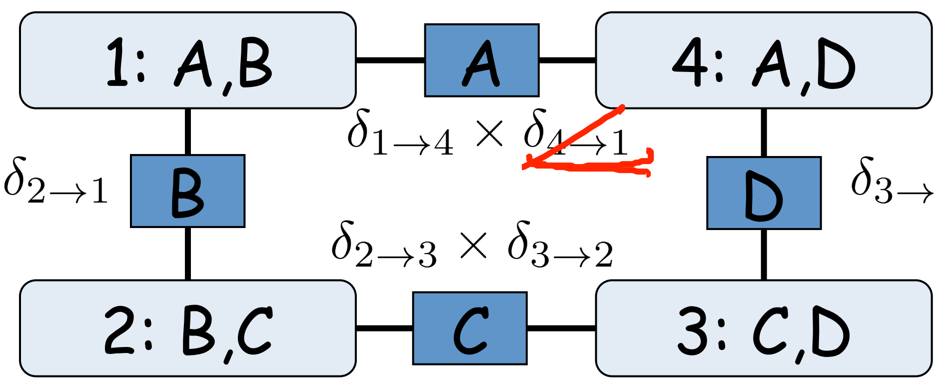
$$\mu_{i,j}(\boldsymbol{S}_{i,j}) = \delta_{j \to i}\delta_{i \to j}$$

sepset beliefs

$$\frac{\prod_i \beta_i}{\prod_{i,j} \mu_{i,j}}$$

$\psi_1 \times \cancel{\delta_{4\to1}} \times \delta_{2\to1}$

$\psi_4 \times \delta_{1\to4} \times \delta_{3\to4}$



$\delta_{1\to4} \times \cancel{\delta_{4\to1}}$

$\underbrace{\cancel{\delta_{1\to2}} \times \delta_{2\to1}}_{\mu_{1,2}}$

$\delta_{3\to4} \times \delta_{4\to3}$

$\delta_{2\to3} \times \delta_{3\to2}$

$\psi_2 \times \cancel{\delta_{1\to2}} \times \delta_{3\to2}$

$\psi_3 \times \delta_{2\to3} \times \delta_{4\to3}$

Daphne Koller

Reparameterization

no information loss

$$\beta_i(\boldsymbol{C}_i) = \psi_i \times \prod_{k \in \mathcal{N}_i} \delta_{k \to i} \qquad \mu_{i,j}(\boldsymbol{S}_{i,j}) = \delta_{j \to i}\delta_{i \to j}$$

$$\boxed{\frac{\prod_i \beta_i}{\prod_{i,j} \mu_{i,j}}} = \frac{\prod_i \left(\psi_i \prod_{j \in \mathcal{N}_i} \cancel{\delta_{j \to i}}\right)}{\cancel{\prod_{i,j} \delta_{i \to j}}}$$

unnormalized measure

$$= \prod_i \psi_i = \tilde{P}_\Phi(X_1, \ldots, X_n)$$

Summary

- At convergence of BP, cluster graph beliefs are calibrated:
  - beliefs at adjacent clusters agree on sepsets
- Cluster graph beliefs are an alternative, calibrated parameterization of the original unnormalized density
  - No information is lost by message passing