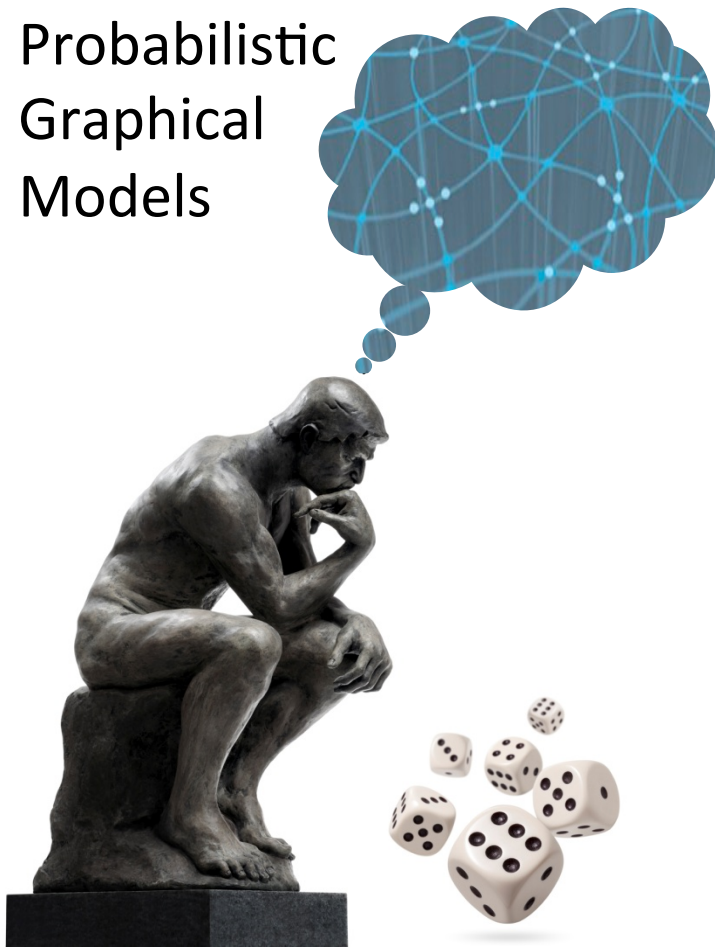Probabilistic Graphical Models

Inference

Message Passing

Clique Tree and VE

Variable Elimination & Clique Trees

- ## Variable elimination
 - Each step creates a factor λ_i through factor product
  - A variable is eliminated in λ_i to generate new factor τ_i
  - τ_i is used in computing other factors λ_j
- ## Clique tree view
  - Intermediate factors λ_i are cliques
  - τ_i are "messages" generated by clique λ_i and transmitted to another clique λ_j
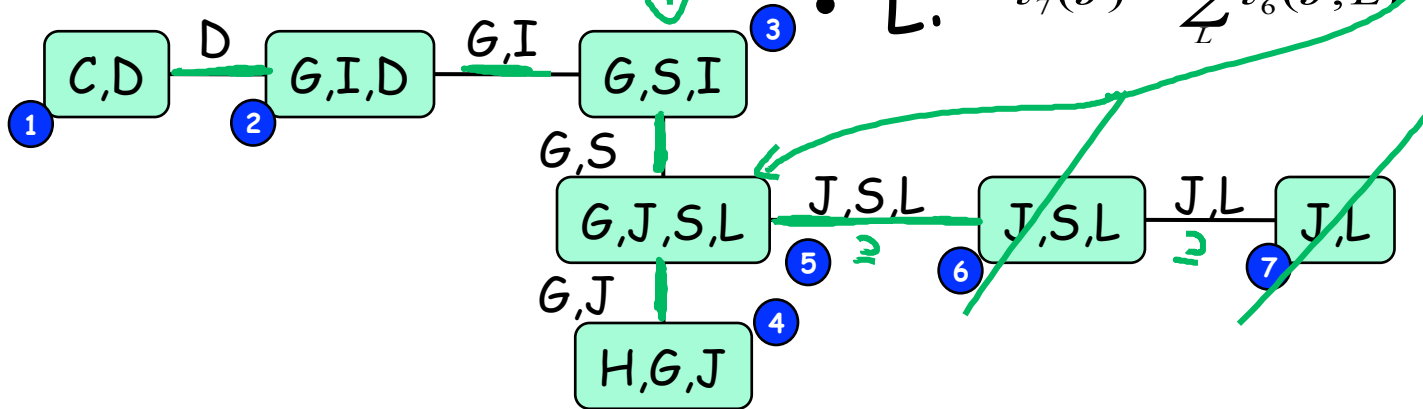
Clique Tree from VE

- ## VE defines a graph
  - Cluster C_i for each factor λ_i used in the computation
  - Draw edge C_i-C_j if the factor generated from λ_i is used in the computation of λ_j
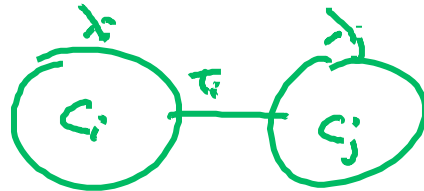
Example

- C: $\tau_1(D) = \sum_C \phi_C(C)\phi_D(C,D)$

- D: $\tau_2(G,I) = \sum_D \phi_G(G,I,D)\tau_1(D)$

- I: $\tau_3(G,S) = \sum_I \phi_I(I)\phi_S(S,I)\tau_2(G,I)$

- H: $\tau_4(G,J) = \sum_H \phi_H(H,G,J)$

- G: $\tau_5(J,L,S) = \sum_G \phi_L(L,G)\tau_3(G,S)\tau_4(G,J)$

- S: $\tau_6(J,L) = \sum_S \phi(J,L,S)\tau_5(J,L,S)$

- L: $\tau_7(J) = \sum_L \tau_6(J,L)$

```
 ┌─────┐  D  ┌───────┐  G,I  ┌───────┐ ③
 │ C,D │─────│ G,I,D │───────│ G,S,I │
 └─────┘     └───────┘       └───────┘
   ①           ②
                          G,S │
                         ┌─────────┐  J,S,L  ┌───────┐  J,L  ┌─────┐
                         │ G,J,S,L │─────────│ J,S,L │───────│ J,L │
                         └─────────┘  ⑤   ⑥ └───────┘   └─────┘ ⑦
                          G,J │  ④
                         ┌───────┐
                         │ H,G,J │
                         └───────┘
```

Remove redundant cliques:
   those whose scope is a subset of adjacent clique's scope

Daphne Koller

Properties of Tree

- **VE process induces a tree**
  - In VE, each intermediate factor is used only once τ_i
  - Hence, each cluster "passes" a factor (message) to exactly one other cluster *(every cluster has at most one parent)*

- **Tree is family preserving:** $\phi \in \Phi$
  - Each of the original factors must be used in some elimination step
  - And therefore contained in scope of associated ψ_i *scope that contains scope(ϕ)*
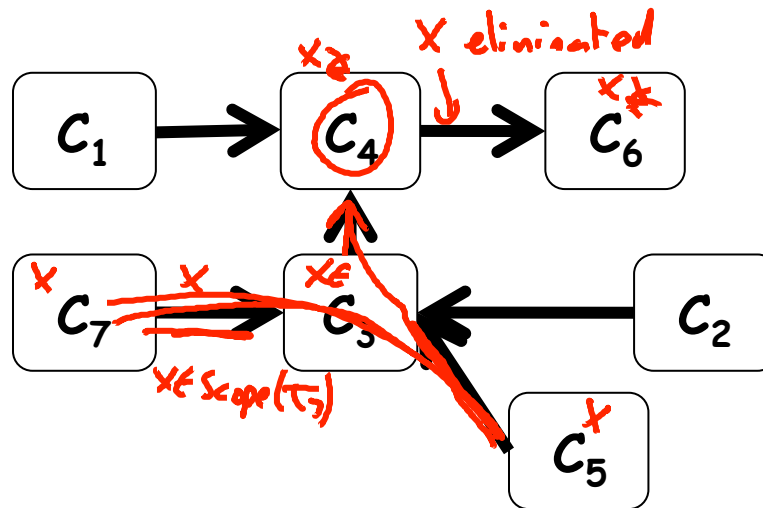
Daphne Koller

Properties of Tree

- ## Tree obeys running intersection property
  - If $X \in C_i$ and $X \in C_j$ then X is in each cluster in the (unique) path between C_i and C_j
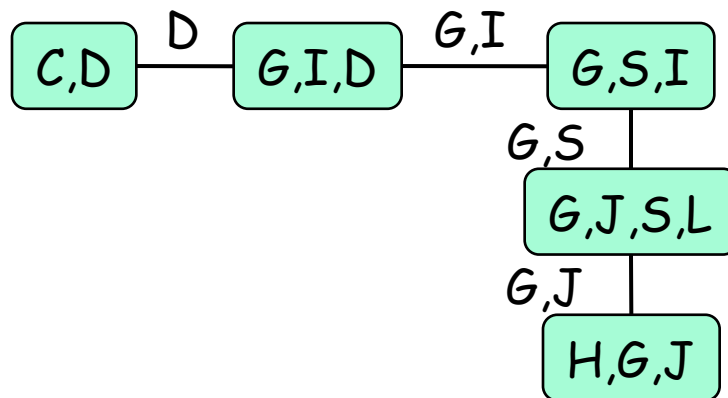
Running Intersection Property

- **Theorem:** If T is a tree of clusters induced by VE, then T obeys RIP

Summary

- A run of variable elimination implicitly defines a correct clique tree
  - We can "simulate" a run of VE to define cliques and connections between them
- Cost of variable elimination is ~ the same as passing messages in one direction in tree
- Clique trees use dynamic programming (storing messages) to compute marginals over all variables at only twice the cost of VE