Random Vectors and the Variance–Covariance Matrix

Definition 1. A *random vector* $\vec{X}$ is a vector $(X_1, X_2, \ldots, X_p)$ of jointly distributed random variables. As is customary in linear algebra, we will write vectors as column matrices whenever convenient.

Expectation

Definition 2. The expectation $E\vec{X}$ of a random vector $\vec{X} = [X_1, X_2, \ldots, X_p]^T$ is given by

$$E\vec{X} = \begin{bmatrix} EX_1 \\ EX_2 \\ \vdots \\ EX_p \end{bmatrix}.$$

This is a definition, but it is chosen to merge well with the linear properties of the expectation, so that, for example:

$$\begin{aligned} E\vec{X} &= E\begin{bmatrix} X_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + E\begin{bmatrix} 0 \\ X_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + E\begin{bmatrix} 0 \\ 0 \\ \vdots \\ X_p \end{bmatrix} \\ &= \begin{bmatrix} EX_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ EX_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ EX_p \end{bmatrix} \\ &= \begin{bmatrix} EX_1 \\ EX_2 \\ \vdots \\ EX_p \end{bmatrix}. \end{aligned}$$

The linearity properties of the expectation can be expressed compactly by stating that for any $k \times p$-matrix A and any $1 \times j$-matrix B,

$$E(A\vec{X}) = AE\vec{X} \quad \text{and} \quad E(\vec{X}B) = (E\vec{X})B.$$

The Variance–Covariance Matrix

Definition 3. The *variance–covariance matrix* (or simply the *covariance matrix*) of a random vector $\vec{X}$ is given by:

$$\operatorname{Cov}(\vec{X}) = E\left[(\vec{X} - E\vec{X})(\vec{X} - E\vec{X})^T\right].$$

Proposition 4.

$$\operatorname{Cov}(\vec{X}) = E[\vec{X}\vec{X}^T] - E\vec{X}(E\vec{X})^T.$$

Proposition 5.

$$\operatorname{Cov}(\vec{X}) = \begin{bmatrix} \operatorname{Var}(X_1) & \operatorname{Cov}(X_1, X_2) & \cdots & \operatorname{Cov}(X_1, X_p) \\ \operatorname{Cov}(X_2, X_1) & \operatorname{Var}(X_2) & \cdots & \operatorname{Cov}(X_2, X_p) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Cov}(X_p, X_1) & \operatorname{Cov}(X_p, X_2) & \cdots & \operatorname{Var}(X_p) \end{bmatrix}.$$

Thus, $\operatorname{Cov}(\vec{X})$ *is a symmetric matrix, since* $\operatorname{Cov}(X, Y) = \operatorname{Cov}(Y, X)$.

Exercise 1. Prove Propositions 4 and 5.

Linear combinations of random variables

Consider random variables $X_1, \ldots, X_p$. We want to find the expectation and variance of a new random variable $L(X_1, \ldots, X_p)$ obtained as a linear combination of $X_1, \ldots, X_p$; that is,

$$L(X_1, \ldots, X_p) = \sum_{i=1}^{p} a_i X_i.$$

Using vector–matrix notation we can write this in a compact way:

$$L(\vec{X}) = \vec{a}^T \vec{X},$$

where $\vec{a}^T = [a_1, \ldots, a_p]$. Then we get:

$$E[L(\vec{X})] = E[\vec{a}^T \vec{X}] = \boxed{\vec{a}^T E\vec{X}},$$

and

$$\begin{aligned}
\mathrm{Var}[L(\vec{X})] &= E[\vec{a}^T \vec{X}\vec{X}^T \vec{a}] - E(\vec{a}^T \vec{X})[E(\vec{a}^T \vec{X})]^T \\
&= \vec{a}^T E[\vec{X}\vec{X}^T]\vec{a} - \vec{a}^T E\vec{X}(E\vec{X})^T \vec{a} \\
&= \vec{a}^T \left(E[\vec{X}\vec{X}^T] - E\vec{X}(E\vec{X})^T \right) \vec{a} \\
&= \boxed{\vec{a}^T \,\mathrm{Cov}(\vec{X})\vec{a}}
\end{aligned}$$

Thus, knowing $E\vec{X}$ and $\mathrm{Cov}(\vec{X})$, we can easily find the expectation and variance of any linear combination of $X_1, \ldots, X_p$.

Corollary 6. *If Σ is the covariance matrix of a random vector, then for any constant vector $\vec{a}$ we have*

$$\vec{a}^T \Sigma \vec{a} \geq 0.$$

That is, Σ satisfies the property of being a positive semi-definite matrix*.*

Proof. $\vec{a}^T \Sigma \vec{a}$ is the variance of a random variable. □

This suggests the question: Given a symmetric, positive semi-definite matrix, is it the covariance matrix of some random vector? The answer is yes.

Exercise 2. Consider a random vector $\vec{X}$ with covariance matrix Σ. Then, for any k dimensional constant vector $\vec{c}$ and any $p \times k$-matrix A, the k-dimensional random vector $\vec{c} + A^T \vec{X}$ has mean $\vec{c} + A^T E\vec{X}$ and has covariance matrix

$$\mathrm{Cov}(\vec{c} + A^T \vec{X}) = A^T \Sigma A.$$

Exercise 3. If $X_1, X_2, \ldots, X_p$ are i.i.d. (independent identically distributed), then $\mathrm{Cov}([X_1, X_2, \ldots, X_p]^T)$ is the $p \times p$ identity matrix, multiplied by a non-negative constant.

Theorem 7 (Classical result in Linear Algebra)**.** *If Σ is a symmetric, positive semi-definite matrix, there exists a matrix $\Sigma^{1/2}$ (not unique) such that*

$$(\Sigma^{1/2})^T \Sigma^{1/2} = \Sigma.$$

Exercise 4. Given a symmetric, positive semi-definite matrix Σ, find a random vector with covariance matrix Σ.

The Multivariate Normal Distribution

A p-dimensional random vector $\vec{X}$ has the *multivariate normal distribution* if it has the density function

$$f(\vec{X}) = (2\pi)^{-p/2}|\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(\vec{X}-\vec{\mu})^T\Sigma^{-1}(\vec{X}-\vec{\mu})\right),$$

where $\vec{\mu}$ is a constant vector of dimension p and Σ is a $p \times p$ positive semi-definite which is invertible (called, in this case, *positive definite*). Then, $E\vec{X} = \vec{\mu}$ and $\mathrm{Cov}(\vec{X}) = \Sigma$.

The *standard multivariate normal distribution* is obtained when $\vec{\mu} = 0$ and $\Sigma = I_p$, the $p \times p$ identity matrix:

$$f(\vec{X}) = (2\pi)^{-p/2} \exp\left(-\frac{1}{2}\vec{X}^T\vec{X}\right).$$

This corresponds to the case where $X_1, \dots, X_p$ are i.i.d. standard normal.

Exercise 5. Let X_1 and X_2 be random variables with standard deviation σ_1 and σ_2, respectively, and with correlation ρ. Find the variance–covariance matrix of the random vector $[X_1, X_2]^T$.

Exercise 6 (The bivariate normal distribution)**.** Consider a 2-dimensional random vector $\vec{X}$ distributed according to the multivariate normal distribution (in this case called, for obvious reasons, the *bivariate normal distribution*). Starting with the formula for the density in matrix notation, derive the formula for the density of $\vec{X}$ depending only on μ_1, μ_2 (the means of X_1 and X_2), σ_1, σ_2 (the standard deviations of X_1 and X_2), and the correlation coefficient ρ, and write it out without using matrix notation.

Exercise 7. Consider a bivariate normal random vector $\vec{X} = [X_1, X_2]^T$, where $E\vec{X} = [5, -4]^T$, the standard deviations are $\mathrm{StDev}(X_1) = 2$ and $\mathrm{StDev}(X_2) = 3$, and the correlation coefficient of X_1 and X_2 is $-4/5$. Use R (or any other software package) to generate 100 independent draws of $\vec{X}$, and plot them as points on the plane.

Hint: To find $\Sigma^{1/2}$, find the eigenvalue decomposition of Σ as:

$$\Sigma = PDP^T,$$

where D is diagonal. Construct $D^{1/2}$ by taking the square root of each diagonal entry, and define

$$\Sigma^{1/2} = PD^{1/2}P^T.$$

In R, *you can find the eigenvalue decomposition of Σ using:*

```
ed <- eigen(sigma)
D <- diag(ed$values)
P <- ed$vectors
```