Drivers in the case of two assets

INTRODUCTION TO PORTFOLIO ANALYSIS IN R

Kris Boudt
Professor, Free University Brussels & Amsterdam

Future returns are random in nature

Optimizing Portfolio requires expectations:

- about average portfolio return (mean)
- about how far off it may be (variance)

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Why?

Portfolio Return Is A Random Variable

Past performance to predictions

	Mean Portfolio Return
Computed on a sample of T Historical Returns	$\hat{\mu} = \frac{R_1 + R_2 + \ldots + R_T}{T}$
When the return is a random variable	$\mu = E[R]$

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	Portfolio Return Variance
Computed on a sample of T Historical Returns	$\hat{\sigma}^2 = \frac{(R_1 - \hat{\mu})^2 + (R_2 - \hat{\mu})^2 + \ldots + (R_T - \hat{\mu})^2}{T - 1}$
When the return is a random variable	$\sigma^2 = E[(R - \mu)^2]$

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Drivers of mean & variance

- Assume two assets:

Asset 1	Asset 2
Weight: w_1	Weight: w_2
Return: R_1	Return: R_2

- Portfolio Return, $P = w1 \cdot R1 + w2 \cdot R2$
- Thus: $E[P] = w_1 \cdot E[R_1] + w_2 \cdot E[R_2]$

Portfolio return variance

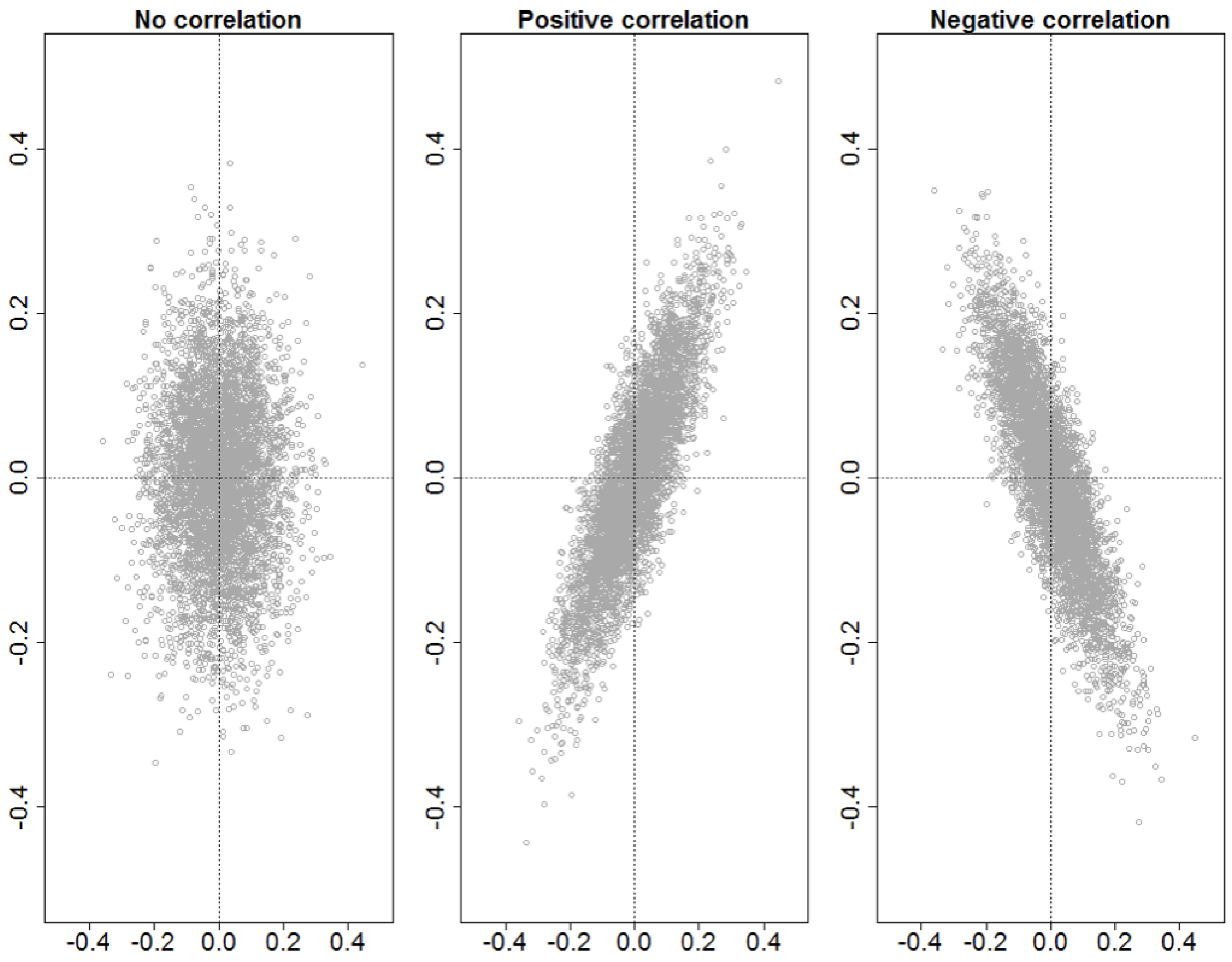
Again, for a portfolio with 2 assets

- $var(P) = w_1^2 \cdot var(R_1) + w_2^2 \cdot var(R_2) + 2 \cdot w_1 \cdot w_2 \cdot cov(R_1, R_2)$

Covariance between return 1 and 2

- $Cov(R_1, R_2)$
  - $= E[(R_1 - E[R_1])(R_2 - E(R_2))]$
  - $= StdDev(R_1) \cdot StdDev(R_2) \cdot corr(R_1, R_2)$

Correlations



Take away formulas

- **E[Portfolio Return]** = $E[P] = w_1 \cdot E[R_1] + w_2 \cdot E[R_2]$
- **var(Portfolio Return)** = $var(P)$ = $w_1^2 \cdot var(R_1) + w_2^2 \cdot var(R_2) + 2 \cdot w_1 \cdot w_2 \cdot cov(R_1, R_2)$

Let's practice!

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Using matrix notation

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Variables at stake for n assets

w: the N x 1 column-matrix of portfolio weights:

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{bmatrix}$$

μ: the N x 1 column-matrix of expected returns:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \dots \\ \mu_N \end{bmatrix}$$

R: the N x 1 column-matrix of asset returns:

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \dots \\ R_N \end{bmatrix}$$

Σ: The N x N covariance matrix of the N asset returns:

$$w = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2N} \\ \dots & \dots & \dots & \dots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_N^2 \end{bmatrix}$$

Generalizing from 2 to n assets

Portfolio Return

$$w_1 * R_1 + w_2 * R_2$$

→

$$w_1 * R_1 + \ldots + w_N * R_N$$

Generalizing from 2 to n assets

Portfolio Return

$$w_1 * R_1 + w_2 * R_2 \Rightarrow w_1 * R_1 + \ldots + w_N * R_N$$

Portfolio Expected Return

$$w_1 * \mu_1 + w_2 * \mu_2 \Rightarrow w_1 * \mu_1 + \ldots + w_N * \mu_N$$

Generalizing from 2 to n assets

Portfolio Return

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Portfolio Expected Return

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Portfolio Variance

$$w_1^2 * var(R_1) + w_2^2 * var(R_2) + 2 * w_1 * w_2 * cov(R_1, R_2)$$

$$\Rightarrow$$

$$w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) + 2 * w_1 * w_2 * cov(R_1, R_2) + \ldots + 2 * w_{N-1} * w_N * cov(R_{N-1}, R_N)$$

Matrices simplify the notation

- Avoid large number of terms by using matrix notation
- We have 4 matrices:
  - weights (w), returns (R), expected returns (μ), and covariance matrix (Σ)

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{bmatrix}$$

$$w' = \begin{bmatrix} w_1 \; w_2 \; \dots \; w_N \end{bmatrix}$$

Simplifying the notation

Portfolio Return

$$w_1 * R_1 + \ldots + w_N * R_N \quad \Rightarrow \quad w'R$$

Simplifying the notation

Portfolio Return

$w_1 * R_1 + \ldots + w_N * R_N$ → $w'R$

Portfolio Expected Return

$w_1 * \mu_1 + \ldots + w_N * \mu_N$ → $w'\mu$

Simplifying the notation

Portfolio Return

$$w_1 * R_1 + \ldots + w_N * R_N \quad \Rightarrow \quad w'R$$

Portfolio Expected Return

$$w_1 * \mu_1 + \ldots + w_N * \mu_N \quad \Rightarrow \quad w'\mu$$

Portfolio Variance

$$\begin{aligned} &w_1^2 + var(R_1) + \ldots + w_N^2 * var(R_N) \\ &+2 * w_1 * w_2 * cov(R_1, R_2) + \ldots \\ &+2 * w_{N-1} * w_N * cov(R_{N-1}, R_N) \end{aligned} \quad \Rightarrow \quad w'\Sigma w$$

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Portfolio risk budget

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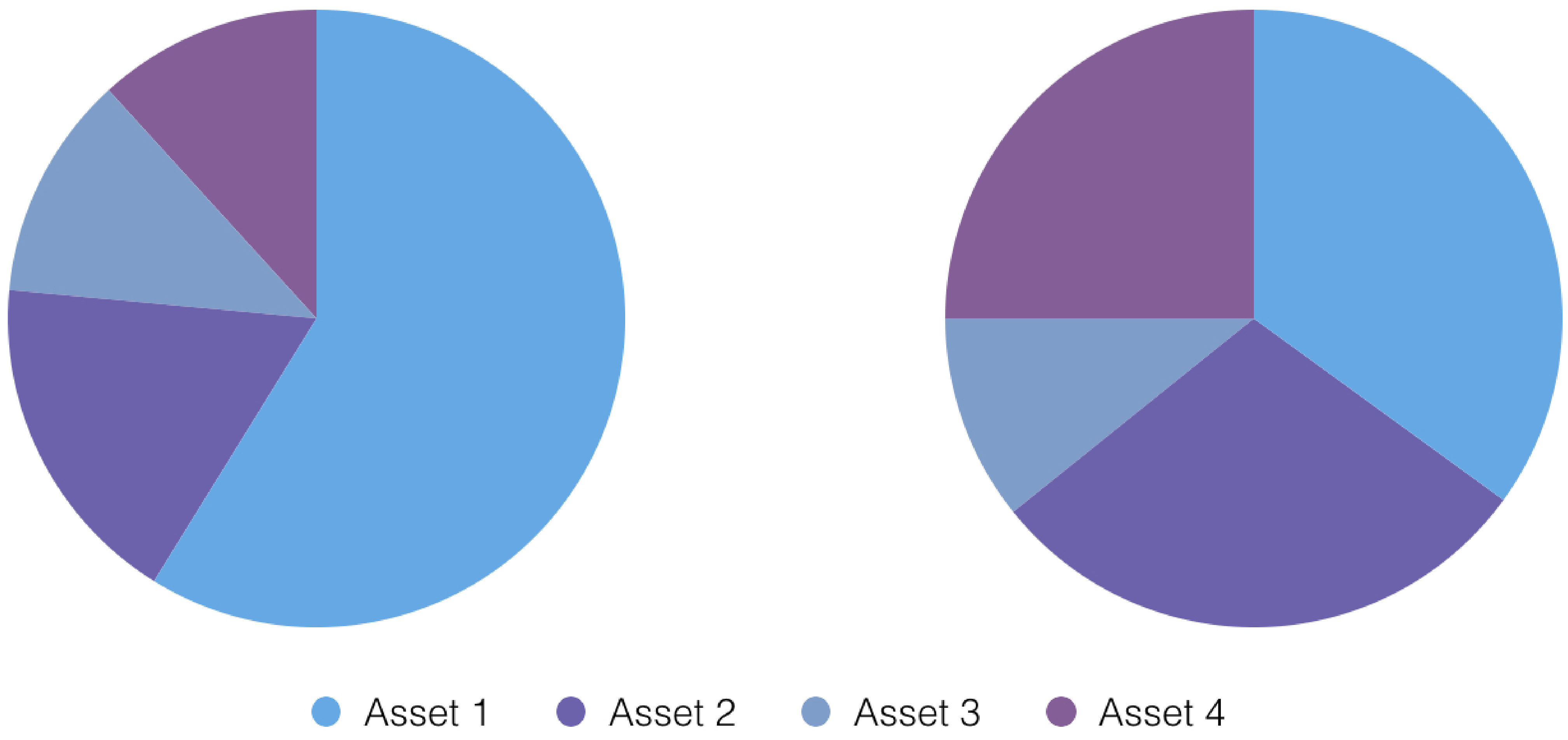
Who did it?



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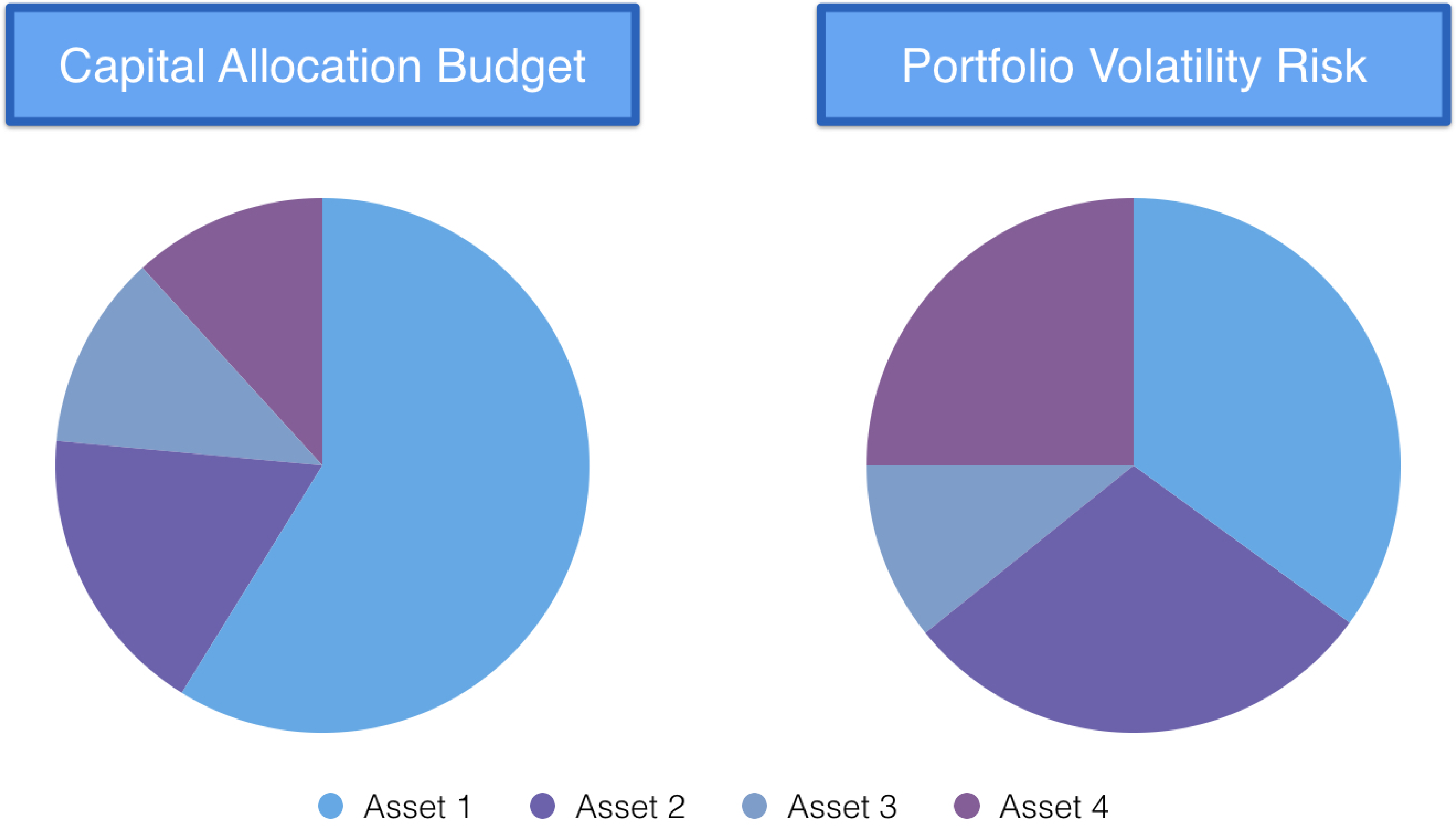
Capital Allocation Budget
Portfolio Volatility Risk
Asset 1
Asset 2
Asset 3
Asset 4

Portfolio volatility in risk contribution

$$\text{Portfolio Volatility} = \sum_{i=1}^{N} RC_i$$

where: $RC_i = \frac{w_i(\Sigma w)_i}{\sqrt{w'\Sigma w}}$

- Risk contribution of asset i depends on
  1. the complete matrix of weights w
  2. the full covariance matrix Σ

Percent risk contribution

$$\%RC_i = \frac{RC_i}{\text{Portfolio volatility}}$$

- where $\sum_{i=1}^{N} \%RC_i = 1$

Relatively less risky assets: $\%RC_i > w_i$

Relatively more risky assets: $\%RC_i < w_i$

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