




Jürgen Richter-Gebert

# Perspectives on Projective Geometry

A Guided Tour  
Through Real  
and Complex Geometry

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# About This Book

*Let no one ignorant of geometry enter here!*

*Entrance to Plato's academy*

*Once or twice she had peeped into the book her sister was reading, but it had no pictures or conversations in it, "and what is the use of a book," thought Alice, "without pictures or conversations?"*

*Lewis Carroll,  
Alice's Adventures in Wonderland*

Geometry is the mathematical discipline that deals with the interrelations of objects in the plane, in space, or even in higher dimensions. Practicing geometry comes in very different flavors. More than any other mathematical discipline, the field of geometry ranges from the very concrete and visual to the very abstract and fundamental. In the one extreme, geometry deals with very concrete objects such as points, lines, circles, and planes and studies the interrelations between them. On the other side, geometry is a benchmark for logical rigor, the elegance of axiom systems, and logical chains of proof. There is a third way of thinking about geometry that stands alongside the visual and the logic-based approaches: the algebraic treatment. Here algebraic structures such as vectors, matrices, and equations are used to form a kind of parallel world, in which each geometric object and relation has an algebraic manifestation. In this parallel world, too, the considerations may be very concrete and algorithmic or very abstract and functorial. This book explains

how to treat the fundamental objects of geometry using appropriate algebraic methods. Many of the techniques presented in this book have their roots in the work of the great geometers of the nineteenth century like Plücker, Grassmann, Möbius, Klein, and Poincaré (to mention only a few).

The algebraic representations are, however, more by far than a way to express geometric objects by numbers. Very often, finding the right algebraic structure unveils the “true” nature of a geometric concept. It may open new perspectives on and deep insights into matters that seemed elementary at first sight and help to generalize, connect, interpret, visualize, and understand. This is what this book is about. Its ultimate aim is to present the beauty that lies in the rich interplay of geometric structures and their algebraic counterparts. A warning should be issued right at the beginning. It is relatively easy to transform geometric objects into algebraic ones. For instance, points in the plane may be easily represented by their  $xy$ -coordinates. However, these “naive” approaches to representing geometric objects are very often not the ones that lead to far-reaching conclusions. Often it is useful to introduce more sophisticated algebraic methods that may seem more abstract at first sight but are ultimately more powerful and elegant. Guided by these more abstract and elegant structures, it may even happen that one is willing to modify the original concept of the geometric first-class citizens (say points or lines) and for instance add some new type of (more abstract) objects. When we talk about *homogeneous coordinates*, one of the most fundamental concepts of this book, exactly this will happen. We will first see that in the plane very elementary operations such as computing the line through two points and computing the intersection of two lines can be very elegantly expressed if lines as well as points are represented by *three-dimensional coordinates* (where nonzero scalar multiples are identified). Taking a closer look at the relation of planar points and their three-dimensional representing vectors, we will observe that certain vectors do not represent points in the real Euclidean plane. This motivates the search for a geometric interpretation of these nonexistent points. It turns out that they may be interpreted as “points that are infinitely far away.” We will then extend the usual two-dimensional plane by these new *points at infinity* and obtain a richer and more elegant geometric system: the system of *projective geometry*.

In a certain sense this way of thinking is quite similar to the work of chemists at the time when the periodic table of the elements was about to be discovered. Based on the elements known so far, they looked for ways to explain their behavior. At some point they spotted a structure and certain symmetry principles into which all the known chemical elements could be fitted (the periodic table of the elements). However, some places in the periodic table did not correspond to known elements. It soon became more reasonable to claim the existence of these undiscovered elements than to give up the inner beauty and explanatory power of the periodic table. Later on, all elements whose existence had been conjectured were indeed discovered. The role

of “discovering elements” in mathematics is played by the “interpretation of concepts.” We will meet such situations quite often in this book.

**The spirit of this book.** In a sense, this book is much more about the “how” than about the “what” of geometry. The reader will recognize that very often we will study very simple objects and their relations. Elementary objects such as points, lines, circles, conics, angles, and distances are the real first-class citizens in our approach. Also, the operations we study will be quite elementary: intersecting two lines, intersecting a line and a conic, calculating tangents, etc. Most of these operations may in principle be performed with some advanced high-school mathematics. Regardless of that, our emphasis will be on structures that at the same time allow us to express the fundamental objects as well as the operations on them in a most elegant way. So the algebraic representation of an object never stands alone; it is always related to the operations that should be performed with the object. As mentioned before, these advanced representations often lead to new insights and broaden our understanding of the seemingly well-known objects. In this respect our philosophy here is very close to Felix Klein’s famous book *“Elementary Mathematics From an Advanced Standpoint.”*

While reading this book, the reader will find that the definitions and concepts are more important than the theorems. Very often the same (sometimes elementary) theorems are re-proved with different approaches. A topic that will show up over and over again is the question of how elegantly and generalizably these proofs can be performed with the various methods. I hope that the reader will find these multiple perspectives on related topics a good way to gain a deeper understanding of what is going on.

**A little history.** As mentioned before, many of the techniques in this book go back to what could be called the golden age of geometry, the hundred years between 1790 and 1890. In this period, starting with Gaspard Monge many fundamental geometric concepts were discovered that went far beyond Euclid’s *Elements* (which until then had dominated geometric thinking). Many of these new concepts were intimately related to the underlying algebraic structures. In that period, algebra and geometry underwent a kind of coevolution, inspiring and enhancing each other. Projective geometry turned out to be one of the most fundamental structures that at the same time had the most elegant algebraic representation. The concepts of linear and multilinear algebra were developed in close connection to their geometric significance. The development culminated in the revolutionary discovery of what now is called “hyperbolic geometry”: a geometric structure that violates the fifth postulate of Euclid and is still logically on an equal footing with his geometry. At its time, this discovery was so revolutionary that C.F. Gauss, who was one of the main protagonists in this discovery and at the same time one of the world’s leading mathematicians, kept it a secret and never published anything on that topic. (We will dedicate several chapters to this topic.) The



key to an elegant treatment of hyperbolic geometry again lies in projective approaches. Nowadays, hyperbolic geometry is a well-established, amazingly rich mathematical subject with flourishing connections to many other fields, such as topology, group theory, number theory, combinatorics, numerics, and many more.

Unfortunately, in the nineteenth century the field of geometry grew perhaps a bit too fast. Many books with many pictures, many theorems, and many proofs of varying mathematical quality were published. Some of the proofs heavily relied on pictorial reasoning. At some time around the turn of the century, a point was reached where it was difficult to say which of these results were to be trusted and which were not. As a kind of antithetical development, this time was the beginning of a school of new and until then unmatched mathematical rigor. David Hilbert was one of the leading figures in the process of rewriting all geometry from scratch in order to place it on a reliable and safe foundation. His book “*Grundlagen der Geometrie*” (Foundations of Geometry) [58] starts with an axiom system that even fixed gaps in Euclid’s axioms and postulates to develop a watertight building of geometry. Hilbert’s famous saying that *one must be able to say “tables, chairs, beer mugs” each time in place of “points, lines, planes”* refers to the demand that an axiom system must be completely formal and not at all depend on the imagination. Following this strict approach, he and several other mathematicians triggered a development in which geometry was treated as a purely formal science. The hardliners of this program claimed that pictures, and in particular pictorial reasoning, had to be abandoned from geometry books.<sup>1</sup>

This development was a kind of catharsis for geometry, and many important and subtle points were revealed in this time (from 1900 to approximately 1970). However, this formal approach also had its disadvantages. There is a famous half-joking quotation from Johann Wolfgang von Goethe about mathematical abstractions:

*Mathematicians are a kind of Frenchmen; whatever you say to them they translate into their own language, and forthwith it is something entirely different.*

Something like this happened to geometry in the time of rigorous abstraction. Abstraction opened mathematicians’ eyes to many far-reaching concepts, such as alternative axiom systems, algebraic geometry, and combinatorial generalizations. At the same time, it changed the concept of what was considered a first-class mathematical citizen. Germs, schemes, matroids, and configuration spaces became more important than (the old-fashioned) points, lines, and planes.

As a sideeffect of this process many important concepts were almost forgotten. Large parts of the still valuable “old geometry” were no longer taught at the universities. The following personal anecdote shall exemplify this. It was around 1993 when I gave a talk at KTH (Kungliga Tekniska Högskolan)

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<sup>1</sup> It is a kind of historical irony that Hilbert, jointly with Cohn-Vossen, wrote a beautiful and highly visual book entitled “*Geometry and the Imagination*” [59].

in Sweden, where I mentioned a certain (and I think really cool) way to construct the foci of an ellipse simply by drawing four specific (complex) tangents and intersecting them (see Figure 19.6). After the talk, a much older colleague came to me and said, “Oh, I am so glad. I thought that today nobody remembered this construction and that I might be one of the last ones who knew it.” In fact, I learned this construction from a book by Blaschke from the 1940s [6], and I hardly know a modern textbook in which it is taught. Perhaps this was one of the points at which I decided to write this book.

**Geometry and computers.** Since the 1970s, the role of geometric reasoning has again undergone a structural change. The reason for this is that *computers*, and in particular computer graphics, have come to play a more and more important role. This has had a twofold effect. On the one hand, in order to obtain good visualizations (also in nonmathematical fields such as CAD, animated movies, games) it is essential to have a good and far-reaching modeling of the objects that are to be visualized, be it the newest automobile design, the dinosaurs in Jurassic Park, or chemical molecules. For such visualizations, even on a very elementary level the elegant treatment of primitives such as points, lines, and circles becomes a key issue. On the other hand, the computer became a tool that allowed mathematicians to visualize abstract concepts and to do precise research on a level that is still quite visual. In particular, computers have made it possible to interact directly with mathematical (and in particular with geometric) structures. All these developments brought a more concrete and more algorithmic treatment of geometry to the mathematical world’s attention once more. In fact, it turned out that many concepts related to nineteenth-century geometry were highly appropriate for dealing with geometric structures in a computational way.

I myself began my research career at a time (around 1985) at which computational methods were seriously entering the everyday work of mathematicians. From then until now I have gone through a chain of thoughts, concepts and problems that definitely shaped the selection of topics in this book. For me, an amazing experience was that this chain went from quite abstract concepts in combinatorial geometry to increasingly elementary (or let us rather say fundamental) concepts and questions. Following these experiences it became more and more clear that the key to an elegant treatment of geometric structures lies in a good algebraic representation and goes straight to the heart of nineteenth century geometry. Since many of these topics I was working on form a kind of “knowledge base” for this book, I will briefly mention this chain. I started working on the structural and computational treatment of so-called realizability questions on combinatorial geometry (we will meet this topic briefly in Section 27.2). In this area it turned out that invariant theoretic and projective methods (see Chapter 6 and Chapter 7) are fundamental. In fact (and this was part of my own doctoral thesis), these methods could be used to implement algorithms that were able to generate “readable algebraic proofs” for many geometric incidence theorems (see Chapter 15)

and by this can form the basis of a kind of geometric expert system. After implementing this prover, I had the desire to have a nice interactive input device for geometric configurations that could be used to feed the prover. What started as a small and seemingly simple project turned out to be a task that is still occupying quite a substantial fraction of my research time. The original demands for this input interface were comparatively simple. The user should be able to use the mouse to construct geometric configurations containing points, lines, circles, conics, etc. After the construction is finished it should be possible to grab basis elements with the mouse, move them, and watch the dependent elements change according to the rules of the construction. If the configuration encodes an incidence theorem, it should be possible to ask the prover for a proof of it. My experience in combinatorial geometry and invariant theory made it immediately clear that such a system, if it was to be elegant, must be based on projective methods, since they have the nice feature of eliminating many special cases. What started at this time (a first prototypical project was undertaken together with Henry Crapo in 1992) for me turned out to be an ongoing search for elegant structures to represent the fundamental objects in geometry. In a sense, this book tells roughly half of this story. In 1996 I started the development of a less prototypical system for dynamic geometry (*Cinderella*), jointly with Ulrich Kortenkamp [112, 113]. In this system we tried to represent the geometric objects in a way that allows for a smooth implementation of geometric primitive operations. One can read the present book as a guide to the representation of these objects and operations. One fundamental breakthrough in the Cinderella project was the discovery that in order to achieve a continuous dynamic behavior in the geometric elements it is necessary to embed the whole situation in an ambient complex space and in a sense navigate on Riemann surfaces (see [72, 73, 74]). This is the other half of the story, on which we will only very briefly touch in the very last section of the very last chapter. To tell it in full length would require another book.

**Applications, beauty, pictures, and formulas:** This book is intended to serve two purposes. On the one hand, it should be very “hands-on” and purposely focuses on elementary objects such as points, lines, circles, conics, and their interrelations. The reader will find many concrete and directly applicable formulas and recipes for performing operations, measurements, and transformations on them. On the other hand, the book is intended to communicate some of the inner beauty of the subject. For me it is one of the most beautiful mathematical topics, with many amazing twists, surprises, and subtleties and still of fundamental importance for many practical applications.

Although this book presents many such explicit algebraic and algorithmic methods for performing primitive operations, the observant reader may recognize that in this book there are comparatively few long algebraic derivations and calculations. This is intimately related to the approach of working on a conceptual level. We will try to derive conceptual setups that make

explicit calculations superfluous whenever possible. In doing so we are close to the philosophy of one of the most important persons in nineteenth century geometry, Julius Plücker. Felix Klein, who was his student, wrote about him:

*In der Plückerschen Geometrie wird die bloße Kombination von Gleichungen in geometrische Auffassung übersetzt und rückwärts durch letztere die analytische Operation geleitet. Rechnung wird nach Möglichkeit vermieden, dabei eine bis zur Virtuosität gesteigerte Beweglichkeit der inneren Anschauung, der geometrischen Ausdeutung vorliegender analytischer Gleichungen ausgebildet und in reichem Maße verwendet.*

Or in the translation by M. Ackermann:

*In Plücker's geometry the bare combination of equations is translated into geometrical terms, and the analytic operations are led back through the geometric. Computation is avoided as far as possible; but by doing this, a mobility, heightened to the point of virtuosity, of inner intuition of the geometric interpretation of given analytic equations, is cultivated and extensively applied.*

Many of the formulas and derivations that are given here are not only used to do a formal derivation that takes one from a statement  $A$  to a statement  $B$ . More importantly, formulas very often have a structural component. Many of them have interesting symmetry properties, a certain rhythm, so to speak. It is perhaps advisable that the reader pause at some point and meditate a bit on this inner structure and symmetry of some of the formulas.

In the book you will also find many pictures, diagrams, and illustrations (so hopefully Alice will find it useful after all). They are intended to illustrate and not to replace the proofs and concepts that are presented. As with the formulas, while reading the book it is highly recommended that one spend a substantial amount of time looking closely at some of the pictures. A picture is worth a thousand words, and not everything that one might see and observe in the pictures is also in the text. So I recommend that the reader take some time for meditation on the pictures, their hidden symmetry structures, their spatial interpretations, their dynamic behavior.

**Why this book?** One might wonder why one should take the effort to write a 570-page book about projective geometry that contains so much “old geometry.” There are several reasons, and I will try to explain a few of them.

*My experience over the past few years:* As already mentioned, much of my own work has been closely related to the representation of geometric objects on the computer. In the area of automated theorem-proving as in the area of dynamic geometry, the classical approaches turned out to be extremely useful. Homogeneous coordinates, invariant theoretic methods, Grassmann-Plücker relations, Cayley-Klein geometries and many other topics that are central in this book were the key to understanding and implementing versatile and flexible tools. This book presents a selection of those topics that I found most helpful either from a structural point of view (how things are related) or from a pragmatic point of view (what is needed for implementations).

Furthermore, many aspects have been added to the purely classical viewpoint that will hopefully reveal some new interrelations between the topics.

*Backing up knowledge:* I had to learn many of these concepts from the old original literature. Mathematical language changes over time, and sometimes it takes quite a bit of decoding to understand what some concept in some original paper really means. Although much of the old mathematics may still be valuable from a modern point of view, it might become increasingly inaccessible. In particular, if (as in the case of classical projective geometry) some concepts are no longer regularly taught at universities, they enter a self-reinforcing loop of fading from commonly available knowledge. Fortunately, the advent of computer visualization has made classical projective geometry an important topic, again. However, many of the deeper concepts are still accessible only to the experts. A few months ago I had a discussion with my colleague Tim Hoffmann on this topic, and in the discussion we found a nice metaphor for what is going on. Writing about classical topics in a modern language is like copying films from videotape to DVDs. The old media still exist; however, it becomes increasingly unlikely that they are used. It needs a refreshing copy procedure that puts the data/knowledge in a format accessible by modern readers (i.e. DVD players). So part of this book project is a kind of backup process. Still I can truly recommend to everyone to read at least once Felix Klein's *Vorlesungen über nicht-euklidische Geometrie* [68] or Plücker's *System der analytischen Geometrie* [100].

*The audience:* This book is intentionally written in a style that should be accessible to students who have basically finished their elementary linear algebra course. It should be accessible to mathematicians as well as computer scientists and physicists. Most of the topics in this book are presented in a relatively self-contained way, allowing even geometry novices to profit from reading it.

**A guided tour:** Here is a brief summary of the topics you will meet in the following chapters. Except for Chapter 1 (which is a bit special, as you will see), this book is divided into three parts. The first part is entitled “Projective Geometry” and deals with the very fundamental objects and concepts. Projective spaces are introduced, first on an axiomatic level (Chapter 2) and then in direct relation to spaces related to real geometry (“real” in the sense of the real numbers  $\mathbb{R}$ ). Homogeneous coordinates are introduced as the main tool for dealing with projective geometry on an algebraic level (Chapter 3). Their transformations are also studied. In particular, it is shown how various transformations can all be handled by a unified framework. Chapter 4 deals with first simple invariants under these transformations. Cross-ratios are prominently introduced. They will form the foundation of many investigations in the later chapters. Chapter 5 is perhaps the theoretically most complicated chapter of the first part. There we show that projective transformations can also be characterized by certain invariant properties (for instance

collinearity). This chapter could be skipped on first reading. Chapters 6 and 7 demonstrate the importance of *determinants* in this context. We outline how one could alternatively build up the framework of projective geometry by taking determinants instead of points as first-class citizens.

The second part is entitled “Working and Playing with Geometry.” In this part a selection of topics is presented that can be handily treated by means of projective concepts. In a way, this part is also largely about the “flexibility of thinking” in Plücker’s sense. Here we try to demonstrate the conceptual power of projective geometry and homogeneous coordinates. Chapter 8 introduces more elaborate invariants. Chapters 9 to 11 deal intensively with conics. These chapters are of fundamental importance for the rest of the book and should not be skipped. Chapter 12 explains how the concepts generalize to higher dimensions. Chapters 13 and 14 are in a sense special again. They introduce a beautiful method of dealing with projective geometry on a diagrammatic level. In this language, each formula can be expressed by a graphical diagram. Algebraic derivations translate to graph manipulations. These two chapters can be skipped at first reading; however, skipping them means missing a wealth of beautiful concepts. Chapter 15 finally tries to present all previously mentioned concepts in a combined way and highlights several interesting geometric incidence theorems and invariant-theoretic proving methods.

The third part is entitled “Measurements.” It deals with a fundamental problem that remains after the first two parts. Over the real numbers, projective geometry and homogeneous coordinates are a powerful system. However, they have one great disadvantage. The only concepts that can be dealt with are those that are stable under projective transformations. This implies that such elementary geometric operations as measuring a distance and measuring an angle have no direct analogue in real projective geometry. Also, such fundamental objects as circles are not objects of real projective geometry. This problem has a beautiful solution. Performing projective geometry over the *complex numbers* allows for the utilization of the geometric properties of this number field. Since multiplication by complex numbers of unit length corresponds to a rotation and rotations implicitly encode distances, this implies that using complex numbers allows one to express measurements in projective geometry. We will see that, for instance, circles can be expressed as special conics that pass through two special complex points  $I$  and  $J$ . Adding these two points to projective geometry will essentially allow us to perform Euclidean operations. The entire third part is about the utilization of complex numbers for performing measurements. Chapter 16 provides a brief introduction to the geometry of complex numbers. Chapter 17 introduces the complex projective line, a first structure in which cocircularity can be expressed in a purely projective framework. Chapter 18 merges the structure of the real projective plane and the complex projective line to arrive at a system that combines the advantages of both spaces. Chapter 19 gives many concrete examples of how this general philosophy applies to various Euclidean concepts.

Chapters 20 to 26 deal with a bold generalization of this approach. It is shown how measurements can be based on projective calculations with respect to a conic. Here all three branches (projective invariants, conics, and complex numbers) are combined to form the very general framework of Cayley-Klein geometries. Chapter 20 introduces the basic concepts, while Chapter 21 introduces the general framework for measurements. Chapters 22 and 23 deal with various special geometric properties and theorems in these spaces. The historically very important topic of *hyperbolic geometry* is a special Cayley-Klein geometry. We dedicate Chapters 24 to 26 to it as the representation of hyperbolic elementary geometry. Hyperbolic geometry turns out to be a so-called nondegenerate Cayley-Klein geometry. This gives it various symmetry properties not shared by general Cayley-Klein geometries.

Finally, in Chapter 27 we briefly mention a few topics that demonstrate how projective geometry influences other parts of mathematics, among them algebraic geometry, combinatorics, quantum information theory, and dynamic geometry.

## Acknowledgements

It has taken what seems to me like an eternity to complete this book, and there were many people involved in reading through the drafts in its various stages. Some of them commented, some of them corrected, some of them protested (at the right places), some of them encouraged. I am sure that I will forget to mention many of them by name here. So first of all a great ‘thank you’ to everyone who gave me any kind of feedback on the manuscript during the last six years.

There are some people who were very active in the final stages of this manuscript. They corrected numerous typos, improved my written English and went through some index battles in the formulas. Among them were Michael Schmid, Thorsten Orendt, Johann Hartl, Susanne Apel, Hermann Vogel, Tim Hoffmann, Peter Lebmeir, Vanessa Krummeck and Martin von Gagern. My special thanks go to Oswald Giering, who went through large parts of the manuscript and made very valuable mathematical comments. Three people deserve a very great *Thank You* regarding the final phase of writing the book. David Kramer and Stephan Lembach took it on themselves to go through the entire (pre-)final version of the manuscript and tried to correct all the spelling<sup>2</sup>, punctuation, formula layout, unidiomatic use of terms and so on. The third person is Jutta Niebauer, who was incredibly patient while entering all these piles of corrections into my original T<sub>E</sub>X files.

Drawings are essential to this book and most of the drawings have been created with suitable software. A great ‘thank you’ goes to Ulrich Kortenkamp, my coauthor of the Cinderella project. Writing the software and the book has

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<sup>2</sup> I wrote this acknowledgement after they finished their work.

been a tightly interwoven process and I am quite convinced that without our mathematical discussions on the software several sections of this book would never have been written. In Section 26.5 you will see beautiful pictures of hyperbolic ornaments. They have been produced with the software project *morenaments* by Martin von Gagern (using hand-drawn sketches of myself as input). Also this project was essential for shaping some of the mathematics presented in this book.

I cannot count the mathematical discussions I have had with colleagues and students on various topics in this book. Many students who attended my classes on Projective Geometry helped to clarify several mathematical and stylistic issues, and many of them definitely helped to clarify the exposition. I am especially grateful to those of them who encouraged me to leave the book in its present, rather explanatory style. They convinced me that even today students are willing to read fat books and that it is worse to approach the same topic from very different directions. Discussions with colleagues were also essential; here I would like to mention a few of the main players who have consciously or unconsciously contributed to the book in its present form (i.e. I learned a lot from them): Günter Ziegler, Jim Blinn, Jürgen Bokowski, Bernd Sturmfels, Henry Crapo, Walter Whiteley, Tim Hoffmann, Ulrich Kortenkamp, AleXX Below, and Martin von Gagern. A special thank you goes to Anders Björner, who in the earliest stages of this project encouraged me to write this book. Perhaps here I should also mention two other mathematicians from whom I learned a lot although I will here on earth unfortunately never have the chance to meet them: Julius Plücker and Felix Klein.

Publishing this book with Springer Verlag means a lot to me. I am very happy to be able to collaborate with Martin Peters and Ruth Allewelt. They were always friendly and remained patient, even though writing this book took much longer than promised.

My greatest, warmest, and foremost thanks go to the one person who has influenced me most throughout my entire life. Without the love, encouragement, faith in me, patience, and deep understanding of my wife, Ingrid, all else would mean nothing at all. Thank you!

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*Jürgen Richter-Gebert*  
Garching, October 2010





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# Pappos's Theorem: Nine Proofs and Three Variations

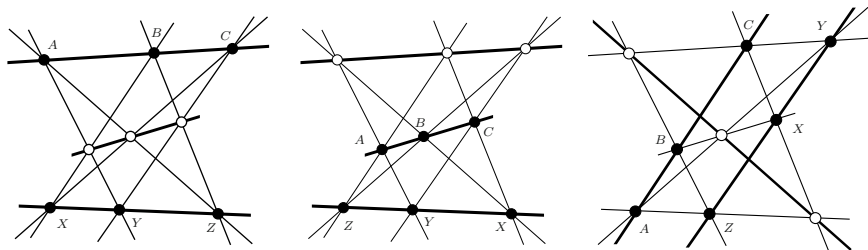
*Bees, then, know just this fact which is of service to themselves, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different figures. We, however, claiming as we do a greater share in wisdom than bees, will investigate a problem of still wider extent, namely, that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest plane figure of all those which have a perimeter equal to that of the polygons is the circle.*

*Pappos of Alexandria, ca. 340 CE*

*Everything in the world is strange and marvelous to well-open eyes.*

*José Ortega y Gasset*

We will begin our journey through *projective geometry* in a slightly uncommon way. We will have a very close look at one particular geometric theorem—namely *The hexagon theorem of Pappos*. Pappos of Alexandria lived around 290–350 CE and was one of the last great Greek geometers of antiquity. He was the author of several books (some of them are unfortunately lost) that covered large parts of the mathematics known at that time. Among other topics, his work addressed questions in mechanics, dealt with the volume/circumference properties of circles, and even gave a solution to the angle trisection problem (with the additional help of a conic). The reader may take this first chapter as a kind of overture to the remainder of the book in which several topics that are important later on are introduced. Without any harm one can also skip this chapter on first reading and come back to it later.



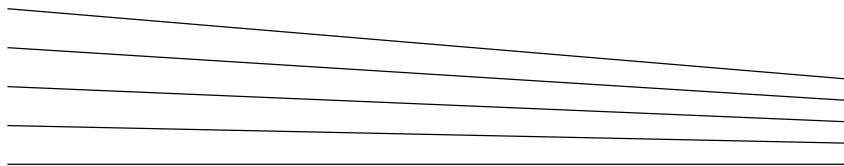
**Fig. 1.1** Three versions of Pappos's theorem.

## 1.1 Pappos's Theorem and Projective Geometry

The theorem that we will investigate here is known as *Pappos's hexagon theorem* and usually attributed to Pappos of Alexandria (though it is not clear whether he was the first mathematician who knew about this theorem). We will later see that this theorem is special in several respects. Perhaps the most important property is that in a certain sense Pappos's theorem is the *smallest* theorem expressible in elementary terms only. The only objects involved in the statement of Pappos's theorem are *points* and *lines*, and the only relation needed in the formulation of the theorem is incidence. Properly stated, the theorem consists only of nine points and nine lines, and there is no such theorem with fewer items. Another remarkable fact is that the incidence configuration underlying Pappos's theorem has beautiful symmetry properties. Some of them are obvious, some of them slightly hidden.

**Theorem 1.1 (Pappos's hexagon theorem).** *Let  $A, B, C$  be three points on a straight line and let  $X, Y, Z$  be three points on another line. If the lines  $\overline{AY}$ ,  $\overline{BZ}$ ,  $\overline{CX}$  intersect the lines  $\overline{BX}$ ,  $\overline{CY}$ ,  $\overline{AZ}$ , respectively, then the three points of intersection are collinear.*

Here *intersecting* means that two lines have exactly one point in common. The nine points of Pappos's theorem are the two triples of points on the initial two lines and the three points of intersection, which finally turn out to be collinear. The nine lines are the two initial lines, the six zigzag lines between the points, and finally the line on which the three intersection points lie. Figure 1.1 shows several instances of Pappos's theorem. The six black points correspond to the initial points, whereas the three white points are the intersections that turn out to be collinear. Observe that in our examples the positions of the nine points and lines (taken as a set) are identical. However, the role of the initial two triples of points is played by different points in each example. The first example shows the picture most often drawn in textbooks, with the final conclusion line between the two initial lines. The second picture shows that the roles of these three lines can be freely interchanged. The



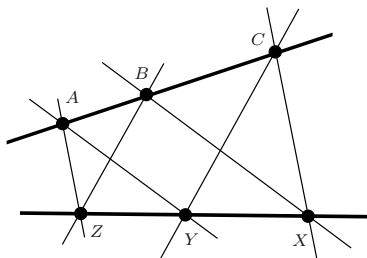
**Fig. 1.2** An almost parallel bundle of lines that meet at a point far on the right.

last picture shows that also one of the inner lines can play the role of the conclusion line (by symmetry of the construction this line can be an arbitrary inner line). In fact, the automorphism group of the combinatorial structure behind Pappos's theorem admits that any pair of lines that do not have a point of the configuration in common can be taken as initial lines for the theorem.

The exact formulation of the theorem already has some subtleties, which we want to mention here. The theorem as stated above requires that the pairs of lines  $(\overline{AY}, \overline{BX})$ ,  $(\overline{BZ}, \overline{CY})$ , and  $(\overline{CX}, \overline{AZ})$  actually intersect, so that we can speak of the collinearity of the *intersection points*. Stated as in Theorem 1.1, Pappos's theorem is perfectly valid in Euclidean geometry. However, if we interpret it in Euclidean geometry it does not exhaust its full generality. There are essentially two different ways in which it can happen that two lines  $a$  and  $b$  may not intersect in Euclidean geometry. Either they are identical (then they have infinitely many points in common) or they are parallel (then they have no point in common). Now, *projective geometry* is an extension of Euclidean geometry in which points are added that are infinitely far away. By this we can properly speak of the intersection of parallel lines (the intersection point lies at infinity) and we get an interpretation of Pappos's theorem in which all instances of parallelism are covered as well.

The essence of real projective geometry may be summarized in the following two sentences: *Bundles of parallel lines meet at an infinite point. All infinite points are incident to a line at infinity.* Thus (real) projective geometry is an extension of Euclidean geometry by certain elements at infinity. In the next two chapters we will elaborate in depth on this extension of Euclidean geometry. In this chapter we will be content with a kind of pre-formal understanding of it.

Imagine a horizontal line  $a$  and a line  $b$  that is almost parallel to it. Both lines meet (since they are not parallel), but the point of intersection will be relatively far out. If the line  $b$  has a small negative slope, the intersection point will be far to the right of the picture. If the slope of  $b$  is small but positive, the intersection point will be far to the left. What happens if we move line  $b$  continuously from the situation with small negative slope via zero slope to the situation with small positive slope? The point of intersection will first



**Fig. 1.3** Euclidean version of Pappos's theorem.

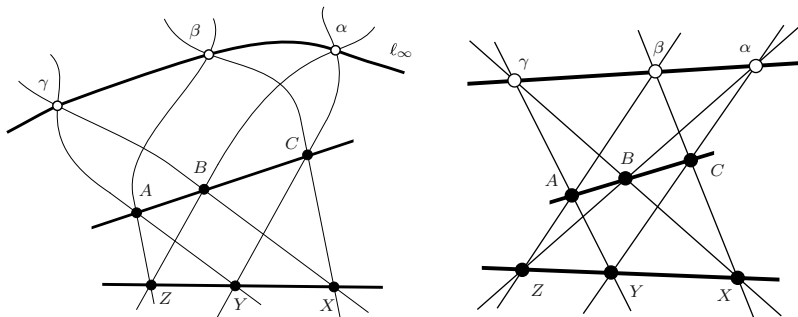
move farther and farther to the right (in fact, it can be arbitrarily far away). In the situation with zero slope both lines are parallel and the intersection point vanishes. After this, the point comes back from a very far position on the left side. Projective geometry now eliminates the special case of parallel lines by postulating an additional point at infinity on the parallels. Figure 1.2 shows a bundle of lines that meet in a point very far out on the right. If this point is moved to infinity, then the lines will eventually become parallel.

It is important to notice that in the concept of projective geometry one assumes the existence of *many* different points at infinity: one for each bundle of parallel lines. All these points together form the line at infinity  $\ell_\infty$ . By introducing these additional elements, special cases get eliminated from geometry. As a matter of fact, these extensions imply that in the projective plane any two distinct points will have a unique line connecting them and any two distinct lines will have a unique point of intersection (it just may be at infinity). Furthermore, from an intrinsic viewpoint of the projective plane the infinite elements are indistinguishable from the finite elements. They have exactly the same incidence properties. (For more details see the next chapter.)

## 1.2 Euclidean Versions of Pappos's Theorem

By passing to a projective framework we get two kinds of benefit. First of all, we extend the scope in which Theorem 1.1 (in exactly the same formulation) is valid. Any point or any line may as well be located at an infinite position—the theorem remains true (we will prove this later). On the other hand, we may get interesting Euclidean specializations of Pappos's theorem by sending elements to infinity. One of them is given by the theorem below:

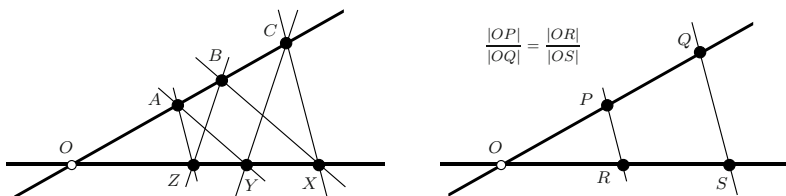
**Theorem 1.2 (A Euclidean version of Pappos's theorem).** *Consider two straight lines  $a$  and  $b$  in Euclidean geometry. Let  $A, B, C$  be three points on  $a$  and let  $X, Y, Z$  be three points on  $b$ . Then the following holds: If  $\overline{AY} \parallel \overline{BX}$  and  $\overline{BZ} \parallel \overline{CY}$  then automatically  $\overline{AZ} \parallel \overline{CX}$ .*



**Fig. 1.4** Euclidean version of Pappos's theorem with points at infinity and line at infinity added (left). The straightened version (right).

For a drawing of this theorem see Figure 1.3. Figure 1.4 illustrates how the parallelism of lines is translated to the projective setup. If  $\overline{AY} \parallel \overline{BX}$  then these two lines intersect (projectively) at a point  $\gamma$  at infinity. Similarly we get an infinite intersection  $\alpha$  for  $\overline{BZ} \parallel \overline{CY}$ . Pappos's theorem (in its projective version) states that  $\gamma$  and  $\alpha$  and the intersection  $\beta$  of  $\overline{AZ}$  with  $\overline{CX}$  are collinear. Since  $\gamma$  and  $\alpha$  span the line  $\ell_\infty$  at infinity,  $\overline{AZ}$  and  $\overline{CX}$  must be parallel as well. In other words, the conclusion line (i.e. the line that encodes the final conclusion of the theorem) has been sent to infinity. The drawing on the right shows a straightened version of the situation with the conclusion line at a finite location. Observe the similarity of the combinatorics. Once we have introduced the concept of *projective transformation*, we will see that by a suitable transformation we can send any instance of Pappos's theorem to the above situation. Thus our Euclidean version is essentially equivalent to the full Pappos's theorem and not just a special case of it.

We will start our collection of proofs with two proofs of Theorem 1.2. It should be remarked in advance that most of our proofs will be algebraic and rely on translations of geometric facts to algebraic identities. There is a general problem with algebraic proofs: *one should never divide by zero!* This seemingly obvious fact leads to many difficulties and misunderstandings when geometric theorems are concerned. Very often, proofs work perfectly in generic situations in which no points or lines coincide or additional collinearities occur, but in certain degenerate cases they may break down. In fact, many algebraic proofs given in geometry textbooks suffer from this (d)effect and a whole branch of current ongoing research deals with the proper treatment of nondegeneracy conditions. The very statement of Theorem 1.1 carries non-degeneracy conditions in stating that the three crucial pairs of lines should actually intersect.



**Fig. 1.5** Euclidean version of Pappos's theorem (left). Relation of parallels and segment ratios (right).

In our investigations we will bypass these degeneracy problems by assuming a few (rather strong) generic nondegeneracy properties. All nine points of the configuration should be distinct and all nine lines of the configuration should be distinct. If for a certain proof additional nondegeneracy assumptions are necessary, we will state them in the context of the proof.

Our first proof is extremely simple but (in its naive version) also of limited scope. It will be based on ratios of segment lengths. We present the proof in a version that works only under the following two additional assumptions: *The two initial lines must intersect in a point O. The triples of points on these lines should not be separated by O.* By introducing oriented lengths the proof can be easily extended to get rid of the second assumption. But we will not do this here.

*Proof one: segment ratios.* By  $|PQ|$  we denote the distance between two points  $P$  and  $Q$ . Our first proof relies on the following fact, which is well known from school lessons on elementary geometry (compare Figure 1.5, right). Let  $a$  and  $b$  be two lines intersecting at  $O$  and let  $P$  and  $Q$  be two points on  $a$  not separated by  $O$ . Similarly, let  $R$  and  $S$  be two points on  $b$  not separated by  $O$ . Then  $\overline{PR}$  and  $\overline{QS}$  are parallel if and only if

$$\frac{|OP|}{|OQ|} = \frac{|OR|}{|OS|}.$$

Using this fact and the hypotheses of the theorem, the parallelism of  $\overline{AY}$  and  $\overline{BX}$  implies that

$$\frac{|OA|}{|OB|} = \frac{|OY|}{|OX|}.$$

Similarly, the parallelism of  $\overline{BZ}$  and  $\overline{CY}$  implies that

$$\frac{|OB|}{|OC|} = \frac{|OZ|}{|OY|}.$$

Since none of the six points are allowed to coincide with  $O$ , none of the denominators in the above expression are zero. Multiplying the two left sides of the equations and the two right sides of the equations and canceling the terms  $|OB|$  and  $|OY|$ , we obtain

$$\frac{|OA|}{|OC|} = \frac{|OZ|}{|OX|}.$$

This in turn is equivalent to the fact that  $\overline{AZ}$  and  $\overline{CX}$  are parallel.  $\square$

At first sight the above proof seems to be very simple and elegant: Multiply two equations, cancel out terms, and get the result. Unfortunately, it has several drawbacks. One of the main problems is that we translated parallelism into ratios of lengths of segments. This translation works correctly only if the decisive points are not separated by the intersection of the lines. One can circumvent this problem by considering *oriented* line segments. The sign of the ratios used in our proof will be negative if the points are separated by  $O$ , and positive otherwise. However, to make this formally correct one should provide a case-by-case analysis that proves that the signs really have the desired behavior. A closer look shows that the proof is problematic, since we introduced the auxiliary point  $O$  and we made the proof dependent on its existence. The complete proof breaks down if the lines  $a$  and  $b$  are parallel and point  $O$  does not exist at all. In fact, the Euclidean version of Pappos's theorem does not at all depend on these special position requirements. The following proof uses only the six points of Theorem 1.2. However, we will need three slightly less trivial facts concerning polynomials and oriented areas of triangles and quadrangles.

*Fact 1: Oriented triangle area.*

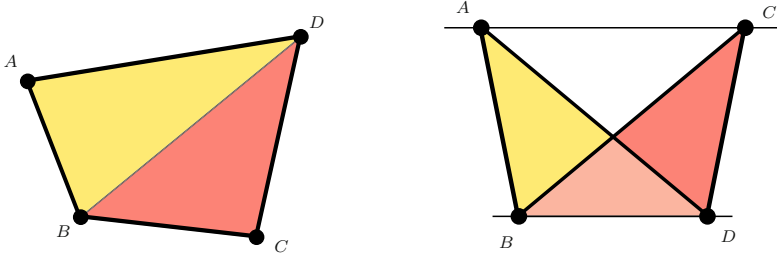
For three points  $A, B, C$  with coordinates  $(a_x, a_y)$ ,  $(b_x, b_y)$ , and  $(c_x, c_y)$  we can express the oriented area of the triangle  $\Delta(A, B, C)$  by a polynomial in the coordinates. To be more specific, the desired polynomial is

$$\frac{1}{2} \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{2} (a_x b_y + b_x c_y + c_x a_y - a_x c_y - b_x a_y - c_x b_y).$$

In fact, the specific shape of this polynomial is not important for our next proof. What is more important is the meaning of *oriented*: If the sequence of points  $(A, B, C)$  is in counterclockwise order, then the area will be calculated with positive sign. If they are in clockwise order, we will get a negative sign. If the three points are collinear, then the triangle vanishes and the area will be zero. We will denote the triangle area by  $\mathbf{area}(A, B, C)$ .

*Fact 2: Oriented quadrangle area.*

The oriented area of a quadrangle  $\square(A, B, C, D)$  can be defined as



**Fig. 1.6** Area of a quadrangle. The convex case (left) and a self-intersecting zero-area case (right).

$$\mathbf{area}(A, B, C, D) = \mathbf{area}(A, B, D) + \mathbf{area}(B, C, D).$$

This function is again a polynomial in the coordinates of the points. If the boundary of this triangle (the polygonal chain from  $A$  to  $B$  to  $C$  to  $D$  and back to  $A$ ) is free of self-intersections, then the usual area is calculated (with sign depending of the orientation). However, if the polygon has self-intersections, then one of the triangles in the sum contributes a positive value and the other a negative value. The area of a self-intersecting quadrangle  $(A, B, C, D)$  is zero if and only if the two triangles involved in the sum have equal areas with opposite signs. Since both triangles share the edge  $(B, D)$ , the zero case implies that  $A$  and  $C$  have the same altitude over this edge. In other words, the line through  $A$  and  $C$  is parallel to the line through  $B$  and  $D$ . Altogether we obtain

$$\overline{AC} \parallel \overline{BD} \text{ if and only if } \mathbf{area}(A, B, C, D) = 0.$$

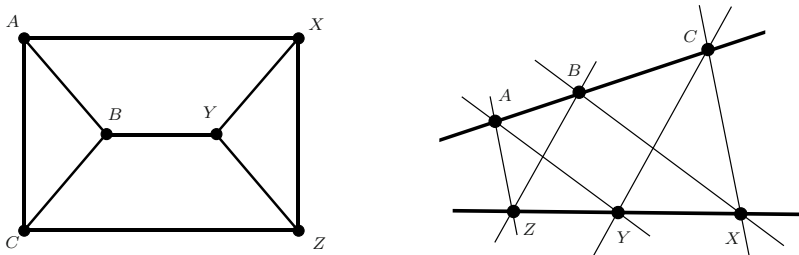
*Fact 3: Zero polynomials.*

If a polynomial in several variables is zero in a full-dimensional region of the space of parameters, then it must be the zero polynomial. In other words, if we have a polynomial that evaluates to zero at a certain point and also for all small perturbations away from that point, then it must be the zero polynomial.

Now we have collected everything to formulate a proof of Pappos's theorem by area arguments. The following proof was given as a motivating example by D. Fearnly Sander in an article on the conceptual power of areas for theorem-proving [39].

*Proof two: area method.* Consider six points  $A, B, C, X, Y, Z$  in the Euclidean plane located at positions that roughly resemble the situation in Figure 1.7 on the left. This figure can be considered as being composed of two triangles  $\Delta(A, C, B)$ ,  $\Delta(X, Y, Z)$ , and two quadrangles  $\square(B, Y, X, A)$





**Fig. 1.7** Pappos proof by the area method.

and  $\square(C, Z, Y, B)$ . The sum of the oriented areas (with counterclockwise vertex labels) of these tiles equals the area of the surrounding quadrangle  $\square(C, Z, X, A)$ . Thus we have

$$\begin{aligned}
 &+ \text{area}(A, C, B) \\
 &+ \text{area}(X, Y, Z) \\
 &+ \text{area}(B, Y, X, A) \\
 &+ \text{area}(C, Z, Y, B) \\
 &- \text{area}(C, Z, X, A) = 0.
 \end{aligned}$$

The expression on the left is obviously a polynomial, and it does not depend on the exact position of the points (since for our argument only the fact that all involved polygons are labeled counterclockwise and the fact that the inner tiles decompose the outer quadrangle were relevant). Hence by Fact 3 this formula must hold for arbitrary positions of the six points—even in degenerate cases. Now let the six points correspond to the points in Pappos's theorem. The hypotheses of Theorem 1.2 state that  $(A, B, C)$  and  $(X, Y, Z)$  are two collinear triples of points. Furthermore, we have  $\overline{AY} \parallel \overline{XB}$  and  $\overline{BZ} \parallel \overline{YC}$ . In terms of areas, this means that

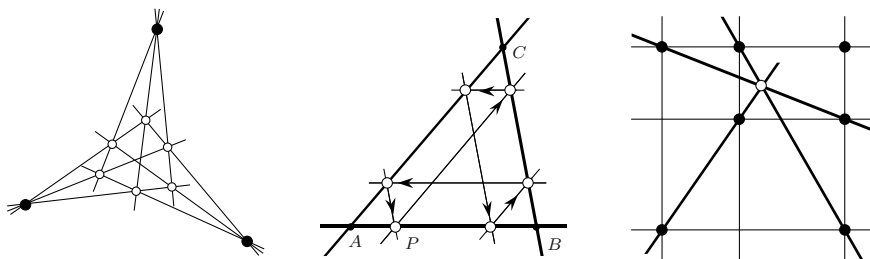
$$\text{area}(A, C, B) = \text{area}(X, Y, Z) = \text{area}(B, Y, X, A) = \text{area}(C, Z, Y, B) = 0.$$

This implies immediately that we also have

$$\text{area}(C, Z, X, A) = 0,$$

since otherwise the above area-sum formula would be violated. Hence we have  $\overline{AZ} \parallel \overline{XC}$  and the theorem is proved.  $\square$

This proof is conceptually far less trivial than our first one, but as a benefit we get several things for free. In essence, the proof says that if four of the areas in the formula above vanish, then the last one has to vanish as well. In this form the theorem holds without any restrictions. It covers even the case of coinciding points.



**Fig. 1.8** Three versions of Pappos's Theorem.

As a second benefit we may observe that this proof is very useful for generalizations. We may consider the drawing in Figure 1.7 as the projection of a three-dimensional prism over a triangle. The five faces of the prism (two triangles and three quadrangles) correspond to the five areas involved in the proof. We can play a similar game with every three-dimensional polyhedron that has only triangles and quadrangles in its boundary. This gives an infinite collection of incidence theorems for which Pappos's theorem is the smallest example. The reader is invited to explore this field on his/her own. For instance, what is the corresponding theorem if we consider a cube as the underlying combinatorial structure?

Before we start to investigate proofs of Pappos's theorem based on concepts of projective geometry we will present some other interesting instances of Pappos's theorem. They are drawn in Figure 1.8. Lines that seem to be parallel in the drawings are really assumed to be parallel. The first picture shows a nice instance that reveals the order-three symmetry that is inherent to Pappos's theorem. The other two pictures show Euclidean specializations in which some of the points are sent to infinity. So the Euclidean instance in the second drawing could be formulated as follows.

**Theorem 1.3 (Another Euclidean version of Pappos's theorem).**

*Start with a triangle  $A, B, C$ . Draw a point  $P$  on the line  $\overline{AB}$ . From there draw a parallel to  $\overline{AC}$  and form the intersection with  $\overline{BC}$ . From this intersection draw a parallel to  $\overline{AB}$  and form the intersection with  $\overline{AC}$  and continue this procedure as indicated in the picture. After six steps you will reach point  $P$  again.*

The patient reader is invited to find out how the drawings in Figure 1.8 correspond to the labeling in our original version of the theorem.

## 1.3 Projective Proofs of Pappos's Theorem

In this section we want to present proofs in which (in contrast to the last section) we make no particular use of parallelism. All proofs in this section will rely on the collinearity properties of points only. In this respect these proofs are projective in nature, since incidence and collinearity are genuine projective concepts, while parallels are not.

The main algebraic tool used in this section is *homogeneous coordinates*, which will be introduced in much detail in later chapters. In contrast to the usual  $(x, y)$ -coordinates in the plane, homogeneous coordinates present points in the plane by three coordinates  $(x, y, z)$ . Coordinate vectors that differ only by a nonzero scalar multiple are considered to be equivalent. The zero vector  $(0, 0, 0)$  is excluded from consideration. Thus the nonzero points in a one-dimensional subspace of  $\mathbb{R}^3$  represent the same point. A usual Euclidean plane  $H$  can be embedded in a homogeneous framework in the following way. Embed  $H$  as an affine subspace of  $\mathbb{R}^3$  that does not contain the origin. Each point  $p$  of  $H$  corresponds to the one-dimensional subspace  $V_p$  spanned by  $p$  and may be represented by any nonzero vector of  $V_p$ . Conversely, each homogeneous vector  $(x, y, z)$  spans a subspace  $V_{(x,y,z)}$ . In general, this subspace intersects the embedded plane  $H$  at some point  $p$ . This is the point that corresponds to  $(x, y, z)$ . It may happen that  $V_{(x,y,z)}$  does not intersect  $H$  (this happens whenever the subspace is parallel to  $H$ ). Then there is no Euclidean point associated to  $(x, y, z)$ . In this case this homogeneous coordinate vector represents an infinite point (see Chapter 3 for details). Thus the finite and the infinite points can be represented by homogeneous coordinates in a completely generalized manner.

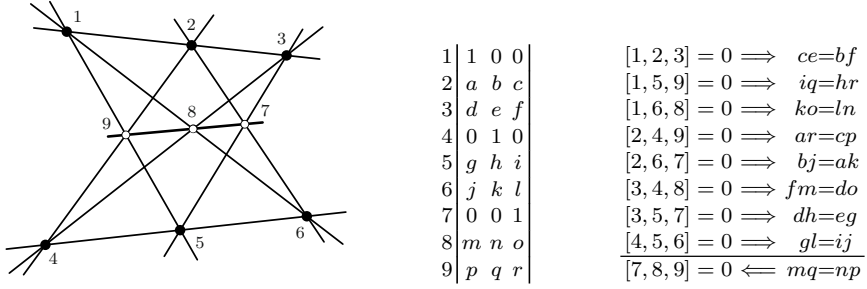
Collinearity of points in  $H$  translates to the fact that the three points in  $\mathbb{R}^3$  lie in a single plane (the plane spanned by the corresponding line and the origin of  $\mathbb{R}^3$ ). Thus if  $A = (x_1, y_1, z_1)$ ,  $B = (x_2, y_2, z_2)$ , and  $C = (x_3, y_3, z_3)$  are homogeneous coordinates of points, then one can test collinearity by checking the condition

$$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0.$$

This condition works for finite as well as for infinite points. The following proof is based on this observation.

*Proof three: determinant cancellations.* For matters of better readability we have exchanged the labels of the points by simple digits from 1 to 9 (see Figure 1.9). For the proof we need the additional nondegeneracy condition that the triple of points  $(1, 4, 7)$  is *not collinear*. The generic nondegeneracy conditions (no identical points and no identical lines) should still be valid.

Assume that  $(1, 4, 7)$  is not collinear. After a suitable affine transformation (which does not affect the incidence relations of points and lines) we may assume without loss of generality that  $(1, 4, 7)$  forms an equilateral triangle.



**Fig. 1.9** Determinant cancellation for Pappos's theorem.

Now we embed the plane in which our configuration resides into three-space in such a way that the points 1, 4, and 7 are at the three-dimensional unit vectors  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ .

Since the configuration is now embedded in  $\mathbb{R}^3$ , each point is represented by three-dimensional (homogeneous) coordinates. Three points  $P, Q, R$  in our picture are collinear if and only if the determinant of the  $3 \times 3$  matrix formed by their coordinates is zero. We abbreviate this determinant by  $[PQR]$ . The matrix in Figure 1.9 represents the coordinates of the configuration.

The letters in the matrix represent the coordinates of the remaining points. The generic nondegeneracy assumptions imply that none of the letters can be 0. This can be seen as follows. The triple of points  $(3, 4, 7)$  cannot be collinear, since otherwise two of the configuration lines would coincide. However, the determinant formed by these points equals exactly  $a$ . Thus we get

$$0 \neq \det \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = a.$$

A similar argument works for each of the other variables.

With our special choice of coordinates, each of the eight collinearities of the hypotheses can be expressed as the vanishing of a certain  $2 \times 2$  sub-determinant of the coordinate matrix. If we write down all these equations (compare Figure 1.9), multiply all left sides, and multiply all right sides, we are left with another equation  $mq = np$ , which translates back to the collinearity of  $(7, 8, 9)$ . By our nondegeneracy assumptions, all variables involved in the proof will be nonzero; therefore the cancellation process is feasible.  $\square$

A proof that is essentially based on this structure first appeared in [14]. This proof carries remarkable symmetric structures concerning the cancellation patterns among the determinants. Structurally, it reduces to the facts

that all collinearities correspond to  $2 \times 2$  determinants and that each letter occurs on the left as well as on the right. The first fact is highly dependent on the choice of our basis, since only the zeros in the unit vectors are allowed to express each of the collinearities as a  $2 \times 2$  determinant.

One can circumvent this problem by an even more abstract approach. Instead of dealing with concrete coordinates of points, we may deal with general properties of determinants. A fundamental role in this context is played by the *Grassmann-Plücker relations*. These relations state that for arbitrary five points  $A, B, C, D, E$  in the projective plane the following relation holds among the determinants of the homogeneous coordinates:

$$[ABC][ADE] - [ABD][ACE] + [ABE][ACD] = 0.$$

This remarkable identity is of fundamental importance for projective geometry, and we will dedicate a large part of Chapter 6 to it. For now we take the identity as an algebraic fact. On it we base our next proof.

*Proof four: Grassmann-Plücker relations.* We again assume that  $(1, 4, 7)$  is not collinear. We consider the fact that  $(1, 2, 3)$  is collinear in our theorem. Taking this Grassmann-Plücker relation

$$[147][123] - [142][173] + [143][172] = 0$$

together with the fact that  $[123] = 0$ , we obtain

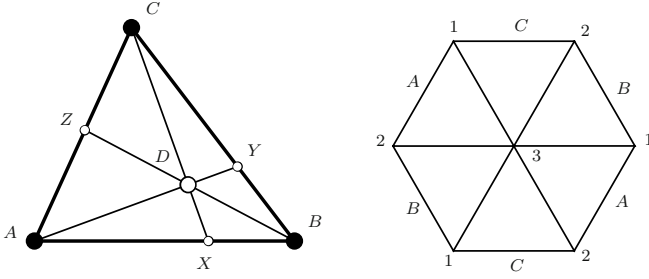
$$[142][173] = [143][172].$$

For each of the eight collinearities of the hypotheses we can get one such equation:

$$\begin{array}{ll} [147][123] - [142][173] + [143][172] = 0 & \implies [142][173] = [143][172] \\ [147][159] - [145][179] + [149][175] = 0 & \implies [145][179] = [149][175] \\ [147][186] - [148][176] + [146][178] = 0 & \implies [148][176] = [146][178] \\ [471][456] - [475][416] + [476][415] = 0 & \implies [475][416] = [476][415] \\ [471][483] - [478][413] + [473][418] = 0 & \implies [478][413] = [473][418] \\ [471][429] - [472][419] + [479][412] = 0 & \implies [472][419] = [479][412] \\ [714][726] - [712][746] + [716][742] = 0 & \implies [712][746] = [716][742] \\ [714][753] - [715][743] + [713][745] = 0 & \implies [715][743] = [713][745] \end{array}$$

Multiplying again all left sides and all right sides of the equations (and taking care of the signs of the determinants) and canceling out terms that occur on both sides, we end up with the equation

$$[718][749] = [719][748].$$



**Fig. 1.10** Ceva's theorem  $\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1$  (left). The pasting scheme for the proof (right).

(The cancellation is feasible since all involved determinants will be nonzero by our nondegeneracy conditions.) By the Grassmann-Plücker relation

$$[714][789] - [718][749] + [719][748] = 0,$$

this implies that  $[714][789] = 0$ . Since  $[147]$  was assumed to be nonzero, this implies that  $[789] = 0$ , which in turn is equivalent to the collinearity of  $(7, 8, 9)$ .  $\square$

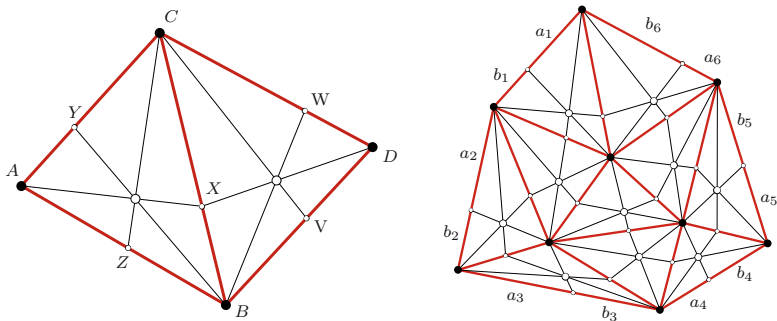
This proof is very similar to the previous one. However, working directly on the level of determinants makes the special choice of the basis no longer necessary. There are amazingly many theorems in projective geometry that can be proved by this generic determinant calculus, and one can even base methods for automatic theorem-proving on them. (For details on this subject see [15, 30, 109].)

Our next proof reveals a topological structure that underlies Pappos's theorem. The proof can be thought of as gluing together several triangular shapes to form a closed oriented surface. The fact that the surface is closed (has no boundary) corresponds to the conclusion of the theorem.

For this proof to work out we need a kind of basic building block: The *theorem of Ceva* (see for instance [28]). Ceva's theorem states that if in a triangle the sides are cut by three concurrent lines that pass through the corresponding opposite vertices, then the product of the three (oriented) length ratios along each side equals 1.

In fact, this theorem is almost trivial if one views the length ratios as ratios of certain triangle areas. For this observe that if the line  $(A, B)$  is cut by the line  $(C, D)$  at a point  $X$ , then we have

$$\frac{|AX|}{|XB|} = -\frac{\text{area}(C, X, A)}{\text{area}(C, X, B)} = -\frac{\text{area}(C, D, A)}{\text{area}(C, D, B)}, \quad (*)$$



**Fig. 1.11** Pasting copies of Ceva's theorem.

where  $\mathbf{area}(A, B, C)$  denotes the oriented area of the triangle  $(A, B, C)$ . In order to prove Ceva's theorem, we consider the obvious identity

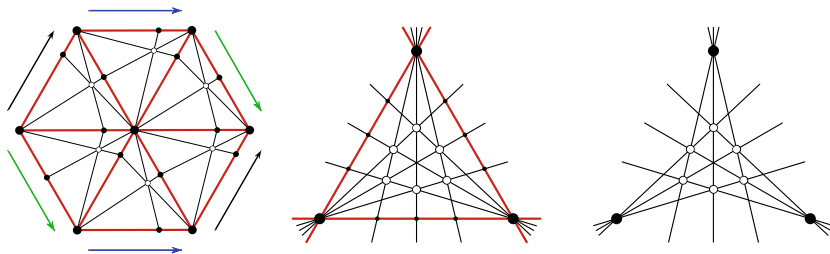
$$\frac{\mathbf{area}(CDA)}{\mathbf{area}(CDB)} \cdot \frac{\mathbf{area}(ADB)}{\mathbf{area}(ADC)} \cdot \frac{\mathbf{area}(BDC)}{\mathbf{area}(BDA)} = -1$$

(note that the oriented triangle area is an alternating function and that each triangle in the denominator occurs as well in the numerator). Applying the above identity (\*), we immediately get Ceva's theorem. The converse of Ceva's theorem holds as well: If the product of the three ratios equals 1, then the three lines in the interior will meet.

Now consider the situation in which two Ceva triangles are glued together along an edge in a way such that they share the point on this edge. Multiplying the two Ceva expressions, we see that the ratio on the inner edge cancels, and we are left only with terms that live on the boundary of the figure (see Figure 1.11 (left)). We obtain

$$\frac{|AZ|}{|ZB|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|BV|}{|VD|} \cdot \frac{|DW|}{|WC|} = 1.$$

We can extend this process to an arbitrary collection of triangles that are glued edge to edge. An edge can be used either by only one triangle (then it is a boundary edge) or by exactly two triangles. The whole collection of patched triangles should be orientable (thus we obtain an orientable triangulated 2-manifold with boundary). All triangles of the collection should be equipped with Ceva configurations that have the additional property that points on interior edges are shared by the Ceva configurations of two adjacent triangles. We consider the product of all corresponding Ceva expressions. After cancellation of the ratios that correspond to inner edges we are left with an expression that contains only oriented length ratios from the boundary. For instance, in the situation of Figure 1.11 (right) we get



**Fig. 1.12** Creating Pappos's theorem from six copies of Ceva's theorem.

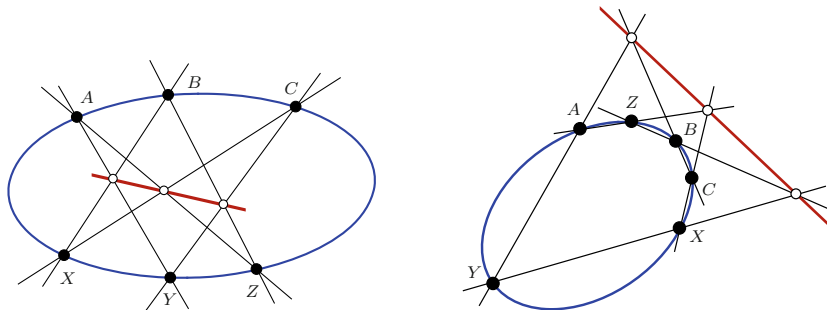
$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} = 1.$$

The inner part of the structure cancels completely and does not contribute to the product on the boundary. Now, if we have a collection of triangles that has nothing more than a triangular boundary (i.e., a 2-manifold with a single triangular hole), then the Ceva condition on the whole is automatically satisfied, and we can paste in a final triangle that carries a Ceva configuration. In other words, if we have an orientable triangulated 2-manifold *without boundary* and we have a Ceva configuration on all triangles but one (such that the edge points are shared), then a Ceva configuration can automatically be put on the final triangle. This is an incidence theorem. We now will show that using the right manifold, Pappos's theorem can be put in exactly this form.

*Proof five: pasting copies of Ceva's theorem.* Consider six triangles that are arranged as in Figure 1.10 on the right. Furthermore, identify opposite edges of the hexagon as indicated in the drawing. This can be done by placing the six triangles one over the other (think of the hexagon as made of paper and fold it appropriately) and gluing together corresponding opposite edges. Now place a Ceva configuration on each of the edges in a way such that whenever two triangles meet at an edge, the corresponding two points on this edge are identified. Our considerations above show that if the edge points are located such that five of the triangles carry proper Ceva configurations, then the last Ceva configuration is satisfied automatically. The figure in the middle shows the situation after all the triangle edges have been identified. Observe that the points on the edges of the outer triangle as well as the edges themselves do not contribute to the incidence theorem. What is left after these elements are deleted is exactly a drawing of Pappos's theorem.  $\square$

A proof very similar to this was given by H.S.M. Coxeter and S.L. Greitzer [28]. Their proof was based on Menelaus configurations instead of Ceva configurations but is essentially similar. In [110] one can find an elaborate treatment of the question of which geometric theorems can be proved by similar manifold arguments.





**Fig. 1.13** Two instances of Pascal's theorem.

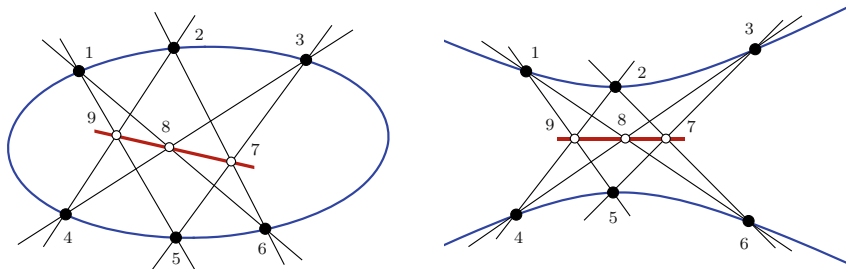
## 1.4 Conics

This section deals with generalizations and variations of Pappos's theorem. In particular, we will study what happens if we consider pairs of lines as degenerate cases of a degree-two curve (an ellipse, hyperbola, or parabola) in the plane. Degree-two curves are often also called *conics*, and they correspond to solutions of (homogeneous) quadratic equations in homogeneous coordinates. More specifically, a conic in the plane is characterized by six homogeneous parameters  $a, b, c, d, e, f$  and consists of the set of all points with homogeneous coordinates  $(x, y, z)$  that satisfy the equation

$$a \cdot x^2 + b \cdot y^2 + c \cdot xy + d \cdot xz + e \cdot yz + f \cdot z^2 = 0.$$

Let  $(x, y, z)$  be a solution of this equation. Since the total degree in  $x, y, z$  of each summand is the same (namely two), every scalar multiple  $\lambda \cdot (x, y, z)$  is also a solution of this equation. Thus we may think of each solution as a point in the real projective plane. The totality of these points forms the conic. The geometric form of the conic depends on the special values of the parameters. Projectively, there is no difference between ellipse, hyperbola, and parabola. These three cases simply reflect different ways in which the line at infinity  $\ell_\infty$  intersects the conic. If there is no intersection, the conic is an ellipse; if there are two intersections, the conic is a hyperbola (it has two infinite points, which correspond to the two asymptotes); if there is just one intersection, the conic is a parabola (which turns out to be a limit case between the two other possibilities).

There is one interesting special case that is also important from a projective point of view: the conic may degenerate into two lines (which may even coincide). This happens whenever the term  $ax^2 + by^2 + cxy + dxz + eyz + fz^2$  factorizes into two linear components:



**Fig. 1.14** Deformations of Pascal's theorem and labeling for the proof.

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = (\alpha_1x + \beta_1y + \gamma_1z) \cdot (\alpha_2x + \beta_2y + \gamma_2z).$$

In this case the conic consists of two lines, each one described by the linear equation in one of the factors.

In general, five points in the projective plane determine a unique conic passing through each of them. Thus it is a truly projective condition whether six points lie on a common conic or not. In Chapter 10 we will prove that six points  $A, B, C, X, Y, Z$  are on a common conic if and only if the following condition among the determinants of the homogeneous coordinates holds:

$$[ABC][AYZ][XBZ][XYC] = [XYZ][XBC][AYC][ABZ].$$

We will use this nice characterization to prove the following well-known variation (or better generalization) of Pappos's theorem:

**Theorem 1.4 (Variation 1: Pascal's Theorem).** *Let  $A, B, C, X, Y, Z$  be six points on a conic. If the lines  $\overline{AY}$ ,  $\overline{BZ}$ ,  $\overline{CX}$  intersect the lines  $\overline{BX}$ ,  $\overline{CY}$ ,  $\overline{AZ}$  respectively, then the three points of intersection are collinear.*

Two instances of the theorem can be found in Figure 1.13. Pascal's theorem is named after the famous Blaise Pascal and was discovered by (the 16-year-old) Pascal in 1640. This is about 1300 years after the discovery of Pappos's theorem. Nevertheless, it is obviously a generalization of Pappos's theorem. If the conic in Pascal's theorem degenerates to consist of two lines, then we immediately obtain Pappos's theorem. We will prove Theorem 1.4 by a determinant cancellation argument similar to the one used in our fourth proof. Figure 1.14 shows two instances of Pascal's theorem one with an ellipse and one with a hyperbola. If we smoothly deform the first into the second, we will pass through the degenerate situation that resembles Pappos's theorem.

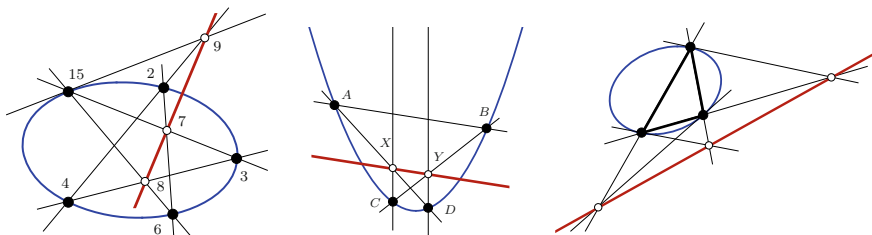
*Proof six: Pascal's theorem.* Again we assume for nondegeneracy reasons that no points and no lines of the theorem coincide. For the labeling in the proof we refer to Figure 1.14. Consider the following determinant equations:

$$\begin{array}{ll}
\text{conic:} & \Rightarrow [125] [136] [246] [345] = + [126] [135] [245] [346] \\
[159] = 0 & \Rightarrow [157] [259] = - [125] [597] \\
[168] = 0 & \Rightarrow [126] [368] = + [136] [268] \\
[249] = 0 & \Rightarrow [245] [297] = - [247] [259] \\
[267] = 0 & \Rightarrow [247] [268] = - [246] [287] \\
[348] = 0 & \Rightarrow [346] [358] = + [345] [368] \\
[357] = 0 & \Rightarrow [135] [587] = - [157] [358] \\
\hline
[987] = 0 & \Leftarrow [287] [597] = + [297] [587]
\end{array}$$

The first line encodes that the points  $1, \dots, 6$  lie on a conic. The next six lines are consequences of Grassmann-Plücker relations and the six collinearity hypotheses of our theorem. If we multiply all expressions on the left and all expressions on the right and cancel determinants that occur on both sides, we end up with the last expression, which (under the nondegeneracy assumption that  $[157] \neq 0$ ) implies the desired collinearity of  $(7, 8, 9)$ .  $\square$

Similar to Pappos's theorem, there is a variety of reformulations and specializations. A nice reformulation is the following: *If a hexagon is inscribed in a conic in the projective plane, then the opposite sides of the hexagon meet in three collinear points.* Or if one prefers a Euclidean variant of this in which the conclusion line is sent to infinity, one could state the following: *If a hexagon is inscribed in a conic and two pairs of opposite edges are parallel, then so is the third pair.* There is another nice way to derive even more incidence theorems from Pascal's theorem. Assume that the conic has a fixed position. If two of the points in Pascal's theorem that are joined by a line continuously approach each other until they meet, their joining line will in the limit case become a tangent to the conic at the position where the two points are located. Thus we obtain as limit cases situations in which also tangents are involved (observe that tangents are proper concepts of projective geometry).

Instances of degenerate versions are given in Figure 1.15. The leftmost picture shows a smallest degenerate situation. The label 15 symbolizes that points 1 and 5 are identified. The labeling is consistent with the labeling in Figure 1.14. The join of 1 and 5 becomes the tangent at the point 15. One can also read the construction in the reverse direction. If a conic  $\mathcal{C}$  and a point 15 on it are given, then one can construct the tangent at 15 by choosing four arbitrary points 2, 3, 4, 6 on  $\mathcal{C}$  and constructing the joins and intersections as given by the picture to arrive finally at point 9 another point on the tangent. This fact was also known to Pascal, and is one of the main applications of his theorem. The second picture shows in essence the same situation as the first one. However, here the point 15 has been sent to infinity and the corresponding tangent is located at the line at infinity. By this the conic becomes a parabola, and the two other lines through 15 become parallel to the symmetry axis of the parabola. Now the theorem reads as follows: *Start*

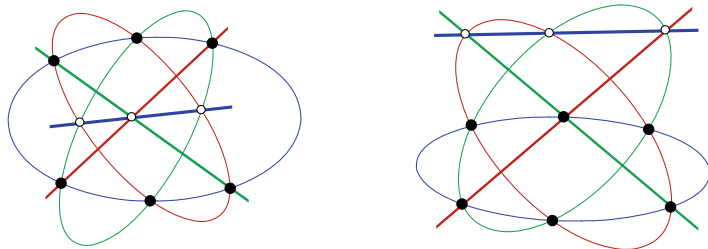


**Fig. 1.15** Degenerate versions of Pascal's theorem.

with four points  $A, B, C, D$  on a parabola. Draw two lines through  $C$  and  $D$  parallel to the symmetry axis of the parabola. Intersect them with  $\overline{AD}$  and  $\overline{BC}$ , respectively. Then the join of the two intersections is parallel to the join of  $A$  and  $B$ . The right figure shows an even more degenerate situation: Inscribe a triangle into a conic. Form the tangents at the vertices. Intersect them with the opposite sides of the triangle. The three intersections are collinear.

## 1.5 More Conics

We can think of Pascal's theorem being derived from Pappos's theorem by considering two lines that do not have a configuration point in common as a (degenerate) conic. Pascal's theorem says that the theorem stays valid even if the conic is not degenerate. The same process can be applied two more times to obtain a theorem with three conics and three lines. For this consider the left part of Figure 1.16. The blue conic arises from merging the upper and the lower lines of the drawing. The red and the green conics arise from merging two other lines. Amazingly, the new configuration still forms a theorem. If all incidences except for the blue line are satisfied as indicated in the picture, then the three white points are automatically collinear (we will prove this in a minute). First we observe that there are two combinatorially different ways of merging three pairs of lines in Pappos's theorem to three conics. The second possibility is shown in Figure 1.16 on the right. Also in this case we get a theorem. To see that they are combinatorially different, observe that in one picture the three lines meet in a point; in the other one they don't. Both theorems are an instance of an even more general fact that is a consequence of *Bézout's theorem* from algebraic geometry (see [19, 40]). An *algebraic curve* of degree  $d$  is the zeroset of a homogeneous polynomial of degree  $d$ . Thus conics are algebraic curves of degree 2. Bézout's theorem can be stated in the following way: *If an algebraic curve of degree  $n$  and an algebraic curve of degree  $m$  intersect, then either the number of intersections is finite and less*



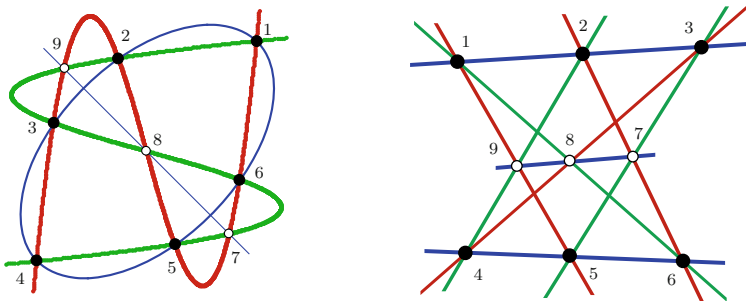
**Fig. 1.16** Generalizations of Pascal's Theorem

than or equal to  $n \cdot m$ , or the curves intersect in infinitely many points and share a component. Now we can prove the following very strong statement:

**Theorem 1.5 (Variation 2: Cayley-Bacharach-Chasles theorem).** *Let  $A$  and  $B$  be two curves of degree three intersecting in nine proper points. If six of these points are on a conic, the remaining three points are collinear.*

*Proof seven: algebraic curves.* Let  $A$  and  $B$  be the curves and let  $p_A(x, y, z)$  and  $p_B(x, y, z)$  be the corresponding homogeneous polynomials of degree three. Bézout's theorem implies that if the two curves  $A$  and  $B$  have only finitely many points in common, then they can have at most nine points of intersection. Call them  $1, \dots, 9$ . And assume that  $1, \dots, 6$  are on a conic  $C$  with polynomial  $p_C$ . We will prove that  $7, 8, 9$  are collinear. Consider a linear combination  $p_\mu = p_A + \mu \cdot p_B$  of the two polynomials for some real parameter  $\mu$ . The polynomial  $p_\mu$  has the following properties. First it is again a degree-three polynomial. Second, it passes through all nine points  $1, \dots, 9$  (each of these points is a zero of both  $p_A$  and  $p_B$ , so it is also a zero of any linear combination of them). Now consider an additional point  $q$  on the conic  $C$  distinct from  $1, \dots, 6$ . There is a  $\mu$  such that  $p_\mu$  also passes through  $q$  (to find  $\mu$  we just have to solve a linear equation  $p_A(q) + \mu p_B(q) = 0$ ). Consider  $p_\mu$  with this specific value  $\mu$ . The curve  $p_\mu$  passes through  $1, \dots, 6$  and through  $q$ . Thus it shares these *seven* points with the conic  $C$ . Bézout's theorem implies that  $p_\mu$  must have  $C$  as one component. Thus we have  $p_\mu = p_C \cdot L$  with a linear equation  $L$  (otherwise  $p_\mu$  cannot have degree three). This implies that the points  $7, 8, 9$  are all contained in the line described by the linear equation  $L$ .  $\square$

The situation of the theorem is sketched in Figure 1.17. This theorem was independently discovered by several people. Most probably Chasles was the first to discover this theorem, in a slightly more general version in 1885. As so often in mathematics, the theorem is usually attributed to others, in this case namely to Cayley and to Bacharach, who published similar results



**Fig. 1.17** If two cubics intersect in nine points six of which are on a conic, then the remaining three points are collinear.

later than Chasles (for a historic account see [38, 63]). The theorems shown in Figure 1.16 are immediate specializations of this theorem. There the two curves of degree three decompose into the product of a quadratic curve (the conic) and a linear curve (the line). So the two red components of the picture form one curve of degree three, and the two green components form the other one. The rest is a literal application of the above theorem. One can even go one step further and consider Pappos's original theorem as a direct consequence of Theorem 1.5. For this one simply has to consider three of the lines as one cubic and another three as the other cubic. The color coding in Figure 1.17 makes the decomposition clear.

## 1.6 Complex Numbers and Circles

We are almost at the end of our journey around Pappos's theorem. In this section we want to take the considerations of the last chapter still a little further and draw a surprising connection to the geometry of circles in the plane. Circles are an intrinsically Euclidean concept. Thus if we do so we have again to talk about the exact position of our line at infinity  $\ell_\infty$ . As already mentioned in Section 1.3, homogeneous coordinates can be considered as embedding the Euclidean plane into  $\mathbb{R}^3$  at some affine hyperplane. This time (and this will be done quite often later in the book) we will choose the affine hyperplane  $\{(x, y, z) \mid z = 1\}$  for this embedding. Thus a point with Euclidean coordinates  $(x, y)$  can be represented by homogeneous coordinates  $(x, y, 1)$  or any nonzero scalar multiple of this vector. The infinite points are those with coordinates  $(x, y, 0)$ .

We now want to study *circles* under this special embedding. A circle is a special conic. Thus we want to find out which quadratic equations will correspond to circles. A circle is usually given by its center  $(c_x, c_y)$  and a

radius  $r$ . In Euclidean geometry the circle equation can be written as

$$(x - c_x)^2 + (y - c_y)^2 - r^2 = 0.$$

Expanding this term and interpreting it in homogeneous coordinates with  $z = 1$ , we can rewrite it as

$$\begin{aligned} (x - c_x \cdot z)^2 + (y - c_y \cdot z)^2 - r^2 \cdot z^2 &= \\ x^2 - 2c_x xz + c_x^2 z^2 + y^2 - 2c_y yz + c_y^2 z^2 - r^2 z^2 &= \\ x^2 + y^2 - 2c_x xz - 2c_y yz + (c_x^2 + c_y^2 - r^2)z^2 &= 0. \end{aligned}$$

The last line gives the interpretation of the circle in parameters of a general conic. The circle is a special conic for which the coefficients of  $x^2$  and  $y^2$  are equal and the coefficient of  $xy$  vanishes.

There is a surprising (and very deep) connection between circles and complex numbers. Let us investigate what happens when we intersect a circle with the line at infinity. In other words, we search for solutions of the above equation with  $z = 0$ . Clearly the solution must be complex, since no circle has real intersections with the line at infinity (this property is possessed only by hyperbolas and parabolas). In the circular case for  $z = 0$  the equation degenerates to

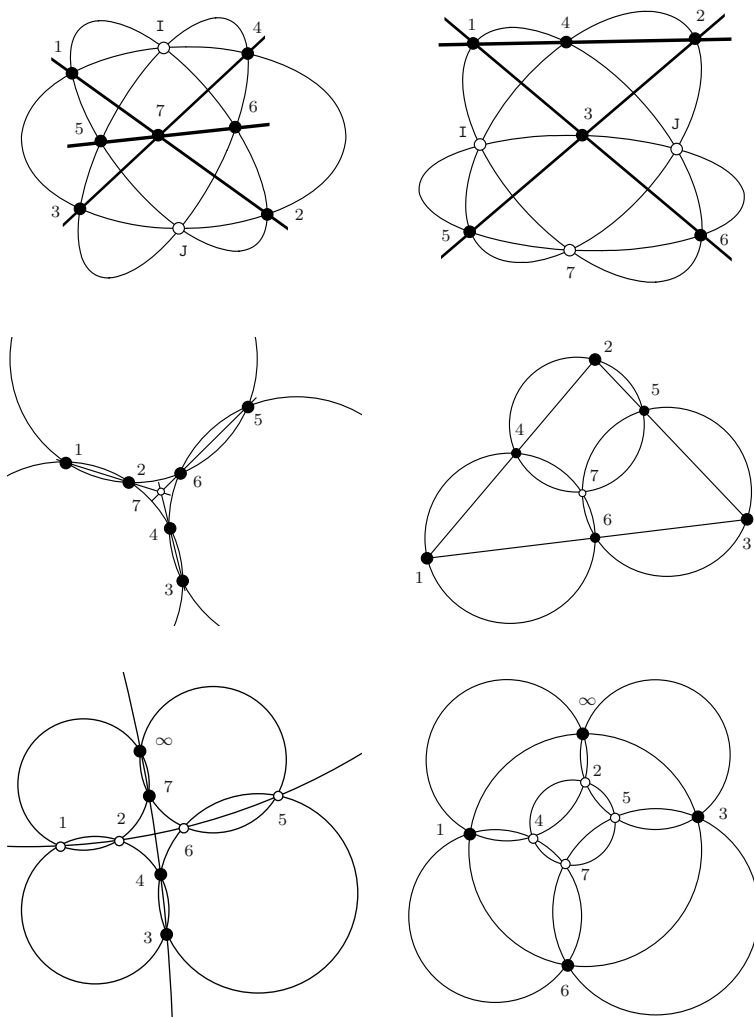
$$x^2 + y^2 = 0.$$

Up to scalar multiples we get the two solutions

$$\mathbf{I} = (1, i, 0) \quad \text{and} \quad \mathbf{J} = (1, -i, 0).$$

These solutions are complex points at the line at infinity. Moreover (and this is an important fact!), they do not depend on the specific choice of the specific circle. Thus we can say *All circles pass through I and J and any conic passing through these points is a circle.*

This fact is perhaps the most important connection of Euclidean and projective geometry. It allows us to express relations about circles as incidence relations of conics that involve the points  $\mathbf{I}$  and  $\mathbf{J}$ . In a very strong sense we could say that *every Euclidean incidence theorem can be expressed as a projective theorem in which two points play the special roles of I and J*. In a sense, Chapters 16 to 26 of this book are dedicated to the elaboration of this fact. Here we will make a small application of it in the context of Pappos's theorem. Consider again the two generalizations given in Figure 1.16. These two pictures are reproduced again in the first row of Figure 1.18. In these pictures the points in which three conics meet are marked by white dots. In the same way as we assumed in Section 1.2 that a certain line is located at the line at infinity we will now assume that in each picture two of these points are located at the points  $\mathbf{I}$  and  $\mathbf{J}$ . All other points should stay at real positions. A conic that passes through  $\mathbf{I}$  and  $\mathbf{J}$  is a circle. Thus the conics in our theorem become circles (this is similar to the effect that two lines become

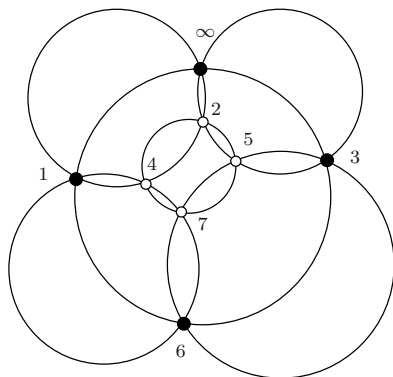


**Fig. 1.18** Metamorphoses of theorems.

parallel if their point of intersection is located at an infinite position). So the two theorems can be interpreted as Euclidean theorems about seven points, three lines, and three circles. The corresponding pictures are shown in Figure 1.18 in the second row. For instance, the first of these two theorems can be stated as follows: *Given three circles that intersect mutually in two points, the three lines spanned by the intersections of each pair of circles meet in a point.* The meeting point corresponds to point 7 in the original theorem.

We can even go one step further. We can interpret straight lines as circles with infinite radius. There is a particular way of extending Euclidean





**Fig. 1.19** Miquel's theorem.

geometry that reflects this way of thinking. For this we introduce *one* point  $\infty$  at infinity and assume that straight lines are circles that in particular contain this point. (A word of caution: one should not confuse this extension of Euclidean geometry by *one* point with the projective plane we introduced earlier. In the projective plane a line at infinity was introduced. The extension by only one point used here has something to do with the *projective complex line* and is called the *one-point compactification of the Euclidean plane* and will be investigated later, in Chapter 17.

In this setup we no longer have to distinguish between lines and circles. Lines are just circles of infinite radius. In this interpretation our two theorems could be stated as theorems on six circles and eight points (we interpret the infinite point  $\infty$  just as an ordinary point). The last row of Figure 1.18 gives a drawing of the situation in which  $\infty$  is located at a finite position. For instance, the second theorem (which is a well-known fact from circle geometry) could be stated as follows.

**Theorem 1.6 (Variation 3: Miquel's theorem).** *Consider four points  $A, B, C, D$  on a circle. Draw four more circles  $C_1, C_2, C_3, C_4$  that pass through the pairs of points  $(A, B)$ ,  $(B, C)$ ,  $(C, D)$ , and  $(D, A)$ , respectively. Now consider the other intersections of  $C_i$  and  $C_{i+1}$  for  $i = 1, \dots, 4$  (indices modulo 4). These four intersections are again cocircular.*

We will give an elementary proof of this theorem by calculations of angle sums. The basic fact that we will need for this proof is illustrated in Figure 1.20. If we consider a secant  $AB$  of a circle and if we look at this secant from two other different points  $C$  and  $D$  of the circle (which are on the same side of  $AB$ ), we will see the secant in the same angle. If the points  $C$  and  $D$  are at opposite sides of the secant we will have complementary angles. Observe that the angles in Figure 1.20 are assumed to be oriented angles. Thus the complementary angle has to be counted with negative sign. If one takes

care of the orientation of the angles one could say that the difference of the two angles at  $C$  and  $D$  will in both cases be a multiple of  $\pi$ . Conversely, four points  $A, B, C, D$  lie on a common circle if the difference of the angles (under which  $AB$  is seen) at  $C$  and  $D$  is a multiple of  $\pi$ . Thus we get a characterization of four points on a circle in terms of angles.

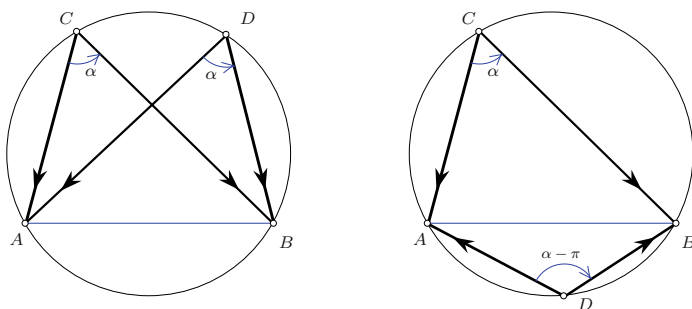
In principle, Miquel's theorem can now easily be proven by considering angle sums among the six involved circles. However, we here will prefer a more algebraic approach that expresses the angle relations in terms of complex numbers. For this assume that all eight points in the picture are finite and consider the picture of Miquel's theorem embedded in the complex number plane  $\mathbb{C}$ . We consider  $A, B, C, D$  from Figure 1.20 as complex numbers. Then, for instance,  $A - C$  forms a complex number that points in the direction from  $C$  to  $A$ . Forming the quotient  $\frac{A-C}{B-C}$ , we get a complex number whose argument (the angle with respect to the real axis) is exactly the angle at point  $C$ . Similarly,  $\frac{A-D}{B-D}$  gives a complex number that describes the angle at point  $D$ . We can compare these two angles by forming again the quotient of these two numbers:  $\frac{A-C}{B-C} / \frac{A-D}{B-D}$ . This number will be real if and only if the two angles differ by a multiple of  $\pi$ .

Taking everything together, we get the following characterization of four points being cocircular (possibly with infinite radius): *Four points  $A, B, C, D$  in the complex plane are cocircular if and only if*

$$\frac{(A - C)(B - D)}{(B - C)(A - D)}$$

*is a real number.*

The above expression is called a *cross-ratio*, and we will later on see that cross-ratios play a fundamental and omnipresent role in projective geometry (see Chapters 4 and 5). With the help of cross-ratios we can easily state a proof of Miquel's theorem.



**Fig. 1.20** Angles in a circle.

*Proof eight: cross-ratio cancellations.* Assume that the quadruples of points  $(A, B, C, D)$ ,  $(A, B, E, F)$ ,  $(B, C, F, G)$ ,  $(C, D, G, H)$ ,  $(D, A, H, E)$  are cocircular. From this we obtain that the following cross-ratios are all real:

$$\frac{(A-B)(C-D)}{(C-B)(A-D)}, \quad \frac{(F-B)(A-E)}{(A-B)(F-E)}, \quad \frac{(C-B)(F-G)}{(F-B)(C-G)}, \quad \frac{(H-D)(C-G)}{(C-D)(H-G)}, \quad \frac{(A-D)(H-E)}{(H-D)(A-E)}.$$

Multiplying all these numbers and canceling terms that occur in the numerator as well as in the denominator, we are left with the expression

$$\frac{(F-G)(H-E)}{(H-G)(F-E)}.$$

Since this expression is the product of real numbers, it must itself be real. By our above observations this expresses exactly the cocircularity of  $(E, F, G, H)$ , which is the conclusion of our theorem.  $\square$

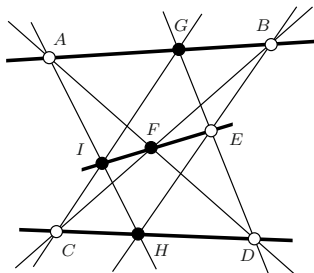
## 1.7 Finally...

We will end this section with an almost trivial proof of Pappos's theorem in its full generality by simply expanding an algebraic term. Still we need a little preparation for this. Again consider the original points of Pappos's theorem expressed in homogeneous coordinates. Thus we assume that the drawing plane  $H$  is again embedded in  $\mathbb{R}^3$  at a position that does not contain the origin of  $\mathbb{R}^3$ . As before, each point  $p$  is represented by a three-dimensional vector  $(x, y, z)$ . This time we will take *all* points of  $\mathbb{R}^3 - \{(0, 0, 0)\}$  into account. For this we identify the vector  $(x, y, z)$  with all of its nonzero scalar multiples  $(\lambda x, \lambda y, \lambda z)$ ,  $\lambda \neq 0$ . By this  $\mathbb{R}^3 - \{(0, 0, 0)\}$  is divided into equivalence classes. Each equivalence class represents a point of the projective plane. A point of the drawing plane  $H$  can be represented by its actual  $(x, y, z)$  position or by any nonzero scalar multiple of it. Conversely, for a point  $(x, y, z)$  of  $\mathbb{R}^3 - \{(0, 0, 0)\}$  we consider the line  $l_{(x, y, z)}$  through it and the origin. The point in  $H$  that is represented by  $(x, y, z)$  is the intersection of  $l_{(x, y, z)}$  and  $H$ . If this intersection does not exist,  $(x, y, z)$  represents an infinite point.

In this setup a straight line  $g$  in  $H$  may be considered a two-dimensional linear space spanned by the elements of  $g$  and the origin of  $\mathbb{R}^3$ . Such a line may be represented by a linear equation

$$\{(x, y, z) \in \mathbb{R}^3 - \{(0, 0, 0)\} \mid ax + by + cz = 0\}$$

given by parameters  $(a, b, c) \in \mathbb{R}^3 - \{(0, 0, 0)\}$ . Thus points as well as lines are represented by nonzero vectors in  $\mathbb{R}^3$ . A line  $g$  is incident to a point  $p$  if and only if the standard scalar product  $\langle p, g \rangle$  is zero.



**Fig. 1.21** A construction sequence for Pappos's theorem.

This observation gives us the key to a very elegant method of calculating the line that connects two points  $p$  and  $q$ . We simply need a vector  $g$  with the property  $\langle p, g \rangle = \langle q, g \rangle = 0$ . Such a vector can simply be calculated by the cross product  $p \times q$ . Similarly, the intersection of two lines  $g$  and  $h$  asks for a vector  $p$  with the property  $\langle p, g \rangle = \langle p, h \rangle = 0$ . Thus the intersection can be calculated by  $g \times h$ . So we can apply the cross product to calculate intersections and joins in projective geometry. (We will learn much more of this in Chapter 3.)

What happens if we try to form the join of two identical points  $p$  and  $q$ ? If  $p$  and  $q$  represent the same point, they must be scalar multiples of each other:  $q = \lambda p$ . Performing the cross product, we obtain  $p \times q = p \times \lambda p = \lambda(p \times p) = (0, 0, 0)$ . Obtaining a zero vector as result is an indication of performing a degenerate operation. A similar effect results when we try to intersect two identical lines.

How can we test collinearity of three points  $p, q, r$ ? The points are collinear if and only if the representing vectors are linearly dependent in  $\mathbb{R}^3$ . Thus we can test collinearity by the condition  $\det(p, q, r) = 0$ .

Now we can express Pappos's theorem as a sequence of nested cross products and a determinant. Expanding the final term and observing that it is zero will prove the theorem.

*Proof nine: brute force.* We give a construction sequence for the Pappos's configuration. We start with five free points  $A, B, C, D, E$  (compare Figure 1.21). The coordinates for the remaining four points in the construction can be calculated by

$$\begin{aligned} F &= (A \times D) \times (B \times C), \\ G &= (A \times B) \times (D \times E), \\ H &= (C \times D) \times (B \times E), \\ I &= (A \times H) \times (C \times G). \end{aligned}$$

Testing the final collinearity boils down to testing whether  $\det(E, F, I) = 0$ . The following session of the computer algebra program *Mathematica* shows

an evaluation of these expressions. All output except for the final result has been suppressed. The final “0” proves Pappos’s theorem.

```

In[43]:= cross[{a_, b_, c_}, {x_, y_, z_}] := {b*z - c*y, -a*z + c*x, a*y - b*x}

In[50]:= a = {a1, a2, a3};
         b = {b1, b2, b3};
         c = {c1, c2, c3};
         d = {d1, d2, d3};
         e = {e1, e2, e3};
         f = cross[cross[a, d], cross[b, e]];
         g = cross[cross[a, b], cross[d, e]];
         h = cross[cross[c, d], cross[b, e]];
         i = cross[cross[a, h], cross[c, g]];
         Det[{e, f, i}]

Out[50]= 0

```

What does this evaluation indeed prove? It shows that when we perform the construction sequence independently of the initial choice of the coordinates of  $A, B, C, D, E$ , the final determinant will be zero. This may happen for two different reasons. Either during the construction sequence we run into a degenerate situation (such as the intersection of identical lines) that introduce a zero vector as an intermediate result. Or all operations were valid (this will be the case for almost every instance) and the final points  $E, F, I$  are indeed collinear.  $\square$

A word of caution: The last proof is very general and seems to be straightforward. Still, the help of a computer is essential here. Performing the calculations by hand would require one to perform all cross products and to evaluate the final determinant. The final term has altogether 15456 summands of degree 15. They can be canceled in pairs, which gives the final result.



# Part I

## Projective Geometry





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# Projective Planes

*Möge dieses Büchlein dazu beitragen den Schatz geometrischer Schönheit [...] über unsere Zeit hinwegzuretten.*

*W. Blaschke, Projektive Geometrie 1949*

The basis of all investigations in this book will be projective geometry. Although projective geometry has a tradition of more than 400 years, it gives a fresh look at many problems, even today. One could even say that the essence of this book is to view many well-known geometric effects/setup/statements/environments from a projective viewpoint.

One of the usual approaches to projective geometry is the axiomatic one (see for instance [3, 25, 44, 58]). There, in the spirit of Euclid, a few axioms are set up and a *projective geometry* is defined as any system that satisfies these axioms. We will very briefly meet this approach in this chapter. The main part of this book will, however, be much more concrete and “down to earth.” We will predominantly study projective geometries that are defined over a specific coordinate field (most prominently the real numbers  $\mathbb{R}$  or the complex numbers  $\mathbb{C}$ ). This gives us the chance to directly investigate the interplay of geometric objects (points, lines, circles, conics, ...) and the algebraic structures (coordinates, polynomials, determinants, ...) that are used to represent them. The largest part of the book will be about surprisingly elegant ways of expressing geometric operations or relations by algebraic formulas (see also [26]). We will in particular focus on understanding the geometry of real and of complex spaces. In the same way as the concept of complex numbers explains many of the seemingly complicated effects for real situations (for instance in calculus, algebra, or function theory), studying the complex projective world will give surprising insights into the geometry over the real numbers (which to a large extent governs our real life).

The usual study of Euclidean geometry leads to a treatment of special cases at a very early stage. Two lines may intersect or not depending on whether they are parallel or not. Two circles may intersect or not depending on their radii and on the position of their midpoints. In fact, these two effects already lead to a variety of special cases in constructions and theorems all over Euclidean geometry. The treatment of these special cases often unnecessarily obscures the beauty of the underlying structures. Our aim in this book is to derive statements and formulas that are elegant and general, and carry as much geometric information as possible. In particular, we will try to reduce the necessity of treating special cases to a minimum. Here we do not strive for complicated formulas but for formulas that carry much structural insight and often simplicity. In a sense, this book is written in the spirit of Julius Plücker (1801–1868), who was, as Felix Klein (1849–1925) expressed it [69], a master of “reading in the equations.”

Starting from the usual Euclidean plane we will see that there are two essential extensions needed to bypass the special situations described in the last paragraph. First, one has to introduce *elements at infinity*. These elements at infinity will nicely unify special cases that come from parallel situations. Second (in the third part of this book), we will study the geometry over complex numbers, since they allow us also to treat intersections of circles that are disjoint from each other in real space.

## 2.1 Drawings and Perspectives

- *In the Garden of Eden, God is giving Adam a geometry lesson: “Two parallel lines intersect at infinity. It can’t be proven but I’ve been there.”*
- *If parallel lines meet at infinity, infinity must be a very noisy place with all those lines crashing together!*

*Two math jokes from a website*

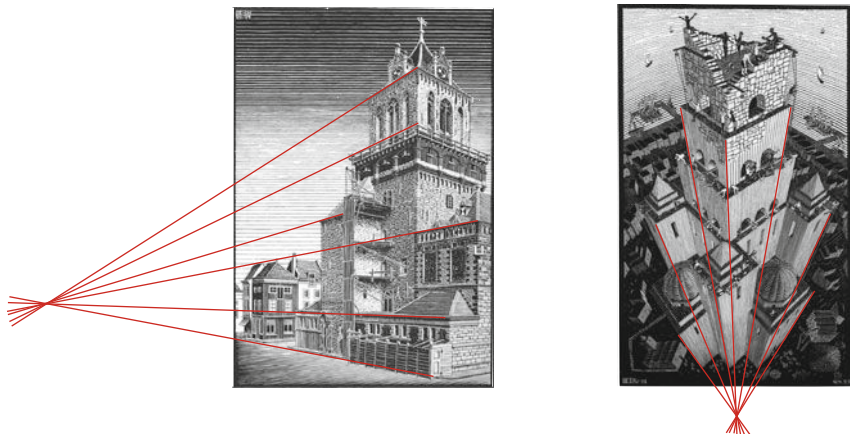
It was one of the major achievements of the Renaissance period of painting to understand the laws of perspective drawing. If one tries to produce a two-dimensional image of a three-dimensional object (say a cube or a pyramid), the lines of the drawing cannot be in arbitrary position. Lines that are parallel in the original scene must either be parallel or meet in a finite point. Lines that meet in a point in the original scene have either to meet in a point in the drawing or they may become parallel in the picture for very special choices of the viewpoint. The artists of that time (among others Dürer, Leonardo da Vinci and Raphael) used these principles to produce (for the standards of that time) stunningly realistic-looking images of buildings, towns, and other



**Fig. 2.1** A page of Dürer's book *Underweysung der Messung, mit dem Zirkel unn Richtscheyt, in Linien, Ebenen unn gantzen corporen*.

scenes. The principles developed at this time still form the basis of most computer-created photorealistic images. The basic idea is simple. To produce a two-dimensional drawing of a three-dimensional scene, fix the position of the canvas and the position of the viewer's eye in space. For each point on the canvas consider a line from the viewer's eye through this point and plot a dot according to the object that your ray meets first (compare Figure 2.1).

By this procedure a line in object space is in general mapped to a line in the picture. One may think of this process in the following way: Any point in object space is connected to the viewpoint by a line. The intersection of this line with the canvas gives the image of the point. For any line in object space we consider the plane spanned by this line and the viewpoint (if the line does not pass through the viewpoint this plane is unique). The intersection of this plane and the canvas plane is the image of the line. This simple construction principle implies that—almost obviously—incidences of points and lines are preserved by the mapping process and that lines are again mapped to lines. Parallelism, orthogonality, distances, and angles, however, are not preserved by this process. So it may happen that lines that were parallel in object space are mapped to concurrent lines in the image space. Two pictures by the Dutch

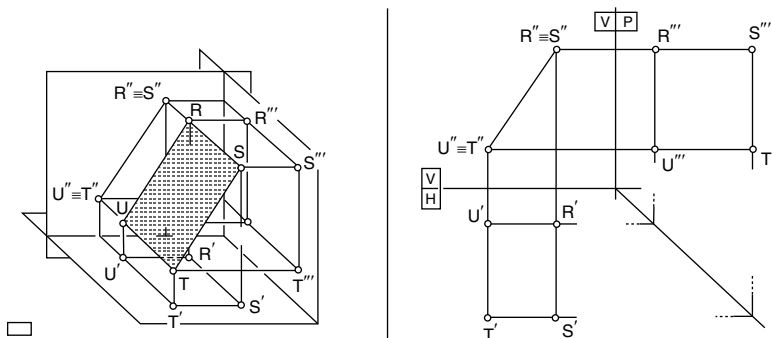


**Fig. 2.2** Two copperplates of the dutch graphic artist M.C. Escher with auxiliary lines demonstrating the strong perspective.

artist M.C. Escher in which these construction principles are carried out in a very strict sense are reproduced in Figure 2.2<sup>1</sup>.

A first systematic treatment of the mathematical laws of perspective drawings was undertaken by the French architect and engineer Girard Desargues (1591–1661) [32] and later by his student Blaise Pascal (1623–1662). They laid foundations of the discipline that we today call *projective geometry*. Unfortunately, many of their geometric investigations had not been anticipated by the mathematicians of their time, since approximately at the same time René Descartes (1596–1650) published his groundbreaking work *La géométrie*, which for the first time intimately related the concepts of algebra and geometry by introducing a *coordinate system* (this is why we speak of “Cartesian coordinates”). It was almost 150 years later that large parts of projective geometry were rediscovered by the Frenchman Gaspard Monge (1746–1818), who was, among other occupations, draftsman, lecturer, minister and a strong supporter of Napoleon Bonaparte and his revolution. His mathematical investigations had very practical backgrounds, since they were at least partially directly related to mechanics, architecture, and military applications. In 1799 Monge wrote a book [89] on what we today would call *constructive* or *descriptive geometry*. This discipline deals with the problem of making exact two-dimensional construction sketches of three-dimensional objects. Monge introduced a method (which in essence is still used today by architects and mechanical engineers) of providing different interrelated

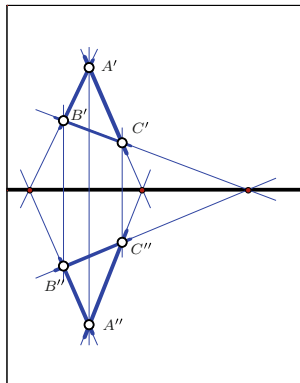
<sup>1</sup> M.C. Escher’s “Delft: Town Hall” ©2010 The M.C. Escher Company-Holland. All rights reserved.—M.C. Escher’s “Tower of Babel” ©2010 The M.C. Escher Company-Holland. All rights reserved. [www.mcescher.com](http://www.mcescher.com).



**Fig. 2.3** Monge view of a square in space.

perspective drawings of a three-dimensional object in a well-defined way, such that the three-dimensional object is essentially determined by the sketches. Monge's method usually projects an object by parallel rays orthogonally to two or three distinct canvases that are orthogonal to each other. Thus the planar sketch contains, for instance a front view, a side view, and a top view of the same object. The line in which the two canvases intersect is identified and commonly used in both perspective drawings. For an example of this method consider Figure 2.3.

Monge made the exciting observation that relations between geometric objects in space and their perspective drawings may lead to genuinely planar theorems. These planar theorems can be entirely interpreted in the plane and need no further reference to the original spatial object. For instance, consider the triangle in space (see Figure 2.4). Assume that a triangle  $A, B, C$  is projected to two different mutually perpendicular projection planes. The vertices of the triangle are mapped to points  $A', B', C'$  and  $A'', B'', C''$  in the projection planes. Furthermore, assume that the plane that supports the triangle contains the line  $\ell$  in which the two projection planes meet. Under this condition the images  $ab'$  and  $ab''$  of the line supporting the edge  $AB$  will also intersect in the line  $\ell$ . The same holds for the images  $ac'$  and  $ac''$  and for  $bc'$  and  $bc''$ . Now let us assume that we are trying to construct such a descriptive geometric drawing without reference to the spatial triangle. The fact that  $ab'$  and  $ab''$  meet in  $\ell$  can be interpreted as the fact that the spatial line  $AB$  meets  $\ell$ . Similarly, the fact that  $ac'$  and  $ac''$  meet in  $\ell$  corresponds to the fact that the spatial line  $AC$  meets  $\ell$ . However, this already implies that the plane that supports the triangle contains  $\ell$ . Hence, line  $BC$  has to meet  $\ell$  as well and therefore  $bc'$  and  $bc''$  also will meet in  $\ell$ . Thus the last coincidence in the theorem will occur automatically. In other words, in the drawing the last coincidence of lines occurs automatically. In fact, this special situation



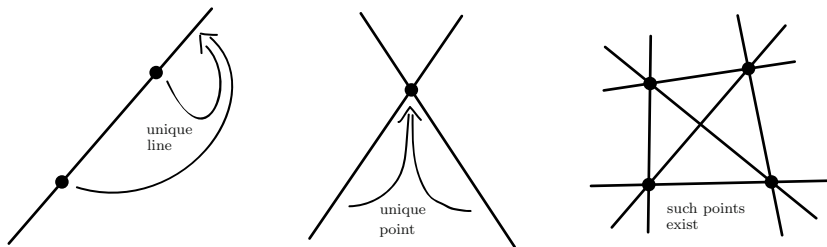
**Fig. 2.4** Monge view of a triangle in space.

is nothing other than Desargues's theorem, which was discovered almost 200 years earlier.

Our starting point, and the last person of our little historical review, was Monge's student Jean-Victor Poncelet (1788–1867). He took up Monge's ideas and elaborated on them on a more abstract level. In 1822 he finished his *Traité des propriétés projectives des figures* [103]. In this monumental work (about 1200 big folio pages) he investigated those properties that remain invariant under projection. This two-volume work contains fundamental ideas of projective geometry, such as the cross-ratio, perspective, involution, and the circular points at infinity, that we will meet in many situations throughout the rest of this book. Poncelet was consequently the first to make use of *elements at infinity*, which form the basis of all the elegant treatments that we will encounter later on.

## 2.2 The Axioms

What happens if we try to untangle planar Euclidean geometry by eliminating special cases arising from parallelism? In planar Euclidean geometry two distinct lines intersect unless they are parallel. Now in the setup of projective geometry one enlarges the geometric setup by claiming that two distinct lines will always intersect. Even if they are parallel, they have an intersection—we just do not see it. In the axiomatic approach a *projective plane* is defined in the following way.



**Fig. 2.5** The axioms of projective geometry.

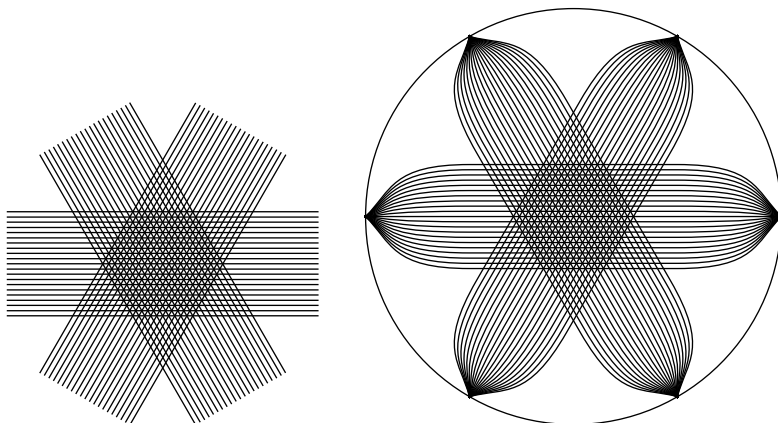
**Definition 2.1.** A *projective plane* is a triple  $(\mathcal{P}, \mathcal{L}, \mathbf{I})$ . The set  $\mathcal{P}$  consists of the *points*, and the set  $\mathcal{L}$  consists of the *lines* of the geometry. The inclusion  $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$  is an incidence relation satisfying the following three axioms:

- (i) For any two distinct points, there is exactly one line incident with both of them.
- (ii) For any two distinct lines, there is exactly one point incident with both of them.
- (iii) There are four distinct points such that no line is incident with more than two of them.

Observe that the first two axioms describe a completely symmetric relation of points and lines. The second axiom simply states that (without any exception) two distinct lines will always intersect in a unique point. The first axiom states that (without any exception) two distinct points will always have a line joining them. The third axiom merely ensures that the structure is not a degenerate trivial case in which most of the points are collinear.

It is the aim of this and the following section to give various models for this axiom system. Let us first see how the usual Euclidean plane can be extended to a projective plane in a natural way by including elements at infinity. Let  $\mathbb{E} = (\mathcal{P}_{\mathbb{E}}, \mathcal{L}_{\mathbb{E}}, \mathbf{I}_{\mathbb{E}})$  be the usual Euclidean plane with points  $\mathcal{P}_{\mathbb{E}}$ , lines  $\mathcal{L}_{\mathbb{E}}$ , and the usual incidence relation  $\mathbf{I}_{\mathbb{E}}$  of the Euclidean plane. We can easily identify  $\mathcal{P}_{\mathbb{E}}$  with  $\mathbb{R}^2$ . Now let us introduce the elements at infinity. For a line  $l$  consider the equivalence class  $[l]$  of all lines that are parallel to  $l$ . For each such equivalence class we define a new point  $p_{[l]}$ . This point will play the role of the point at infinity in which all the parallels contained in the equivalence class  $[l]$  shall meet. This point is supposed to be incident with all lines of  $[l]$ . Furthermore, we define one *line at infinity*  $l_{\infty}$ . All points  $p_{[l]}$  are supposed to be incident with this line. More formally, we set

- $\mathcal{P} = \mathcal{P}_{\mathbb{E}} \cup \{p_{[l]} \mid l \in \mathcal{L}_{\mathbb{E}}\},$
- $\mathcal{L} = \mathcal{L}_{\mathbb{E}} \cup \{l_{\infty}\},$
- $\mathbf{I} = \mathbf{I}_{\mathbb{E}} \cup \{(p_{[l]}, l) \mid l \in \mathcal{L}_{\mathbb{E}}\} \cup \{(p_{[l]}, l_{\infty}) \mid l \in \mathcal{L}_{\mathbb{E}}\}.$



**Fig. 2.6** Sketch of some lines in the projective extension of Euclidean geometry.

It is easy to verify that this system  $(\mathcal{P}, \mathcal{L}, \mathbf{I})$  satisfies the axioms of a projective plane. Let us start with axiom (ii). Two distinct lines  $l_1$  and  $l_2$  have a point in common: If  $l_1$  and  $l_2$  are nonparallel Euclidean lines, then this intersection is simply their usual Euclidean intersection. If they are parallel, it is the corresponding unique point  $p_{[l_1]}$  (which is identical to  $p_{[l_2]}$ ). The intersection of  $l_\infty$  with a Euclidean line  $l$  is the point at infinity  $p_{[l]}$  “on” that line. The first axiom is also easy to check: the unique line incident to two Euclidean points  $p_1$  and  $p_2$  is simply the Euclidean line between them. The line that joins a Euclidean point  $p$  and an infinite point  $p_\infty$  is the unique line  $l$  through  $p$  with the property that  $p_\infty = p_{[l]}$ . Last but not least, the line incident to two distinct infinite points is the line at infinity  $l_\infty$  itself. This completes the considerations for axiom (i) and axiom (ii). Axiom (iii) is evidently satisfied. For this one has simply to pick four points of an arbitrary proper rectangle.

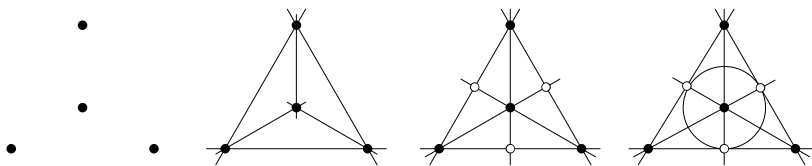
Figure 2.6 (left) symbolizes three bundles of parallels in the Euclidean plane. Figure 2.6 (right) indicates how these lines projectively meet in a point and how all these points lie together on the line at infinity (drawn as a large circle for which antipodal points are assumed to be identified). Looking at the process of extending the Euclidean plane to a projective plane, it may seem that the points at infinity and the line at infinity play a special role. We will see later on that this is by far not the case. In a certain sense the projective extension of a Euclidean plane is even more symmetric than the usual Euclidean plane itself, since it allows for even more automorphisms.



## 2.3 The Smallest Projective Plane

The concept of projective planes as set up by our three axioms is a very general one. The projective extension of the real Euclidean plane is by far not the only model of the axiom system. In fact, even today there is no final classification or enumeration of all possible projective planes. Projective planes do not even have to be infinite objects. There are interesting systems of finitely many points and lines that fully satisfy the axioms of a projective plane. To get a feeling for these structures we will briefly construct and encounter a few small examples.

What is the smallest projective plane? Axiom (iii) tells us that it must contain at least four points, no three of which are collinear. So let us start with four points and search for the smallest system of points and lines that contains these points and at the same time satisfies axioms (i) and (ii). Let the four points be  $A$ ,  $B$ ,  $C$ , and  $D$ . By axiom (ii) any pair of these points has to be connected by a line. This generates exactly  $\binom{4}{2} = 6$  lines. Axiom (i) requires that any pair of such lines intersect. There are exactly three missing intersections, namely those of the pairs of lines  $(\overline{AB}, \overline{CD})$ ,  $(\overline{AC}, \overline{BD})$ , and  $(\overline{AD}, \overline{BC})$ . This gives an additional three points that must necessarily exist. Now again axiom (i) requires that any pair of points be joined by a line. The only pairs of points that are not joined so far are those formed by the most recently added three points. We can satisfy the axioms by simply adding one line that contains exactly these three points.



**Fig. 2.7** Construction of the smallest projective plane.

The final construction contains seven points and seven lines and is called the *Fano plane*. There are a few interesting observations that can be made in this example:

- There are exactly as many lines as there are points in the drawing.
- On each line there is exactly the same number of points (here 3).
- Through each point passes exactly the same number of lines.

Each of these statements generalizes to general finite projective planes, as the following propositions show. We first fix some notation. Let  $(\mathcal{P}, \mathcal{L}, \mathcal{I})$  be

a projective plane. For a line  $l \in \mathcal{L}$  let  $p(l) = \{p \in \mathcal{P} \mid pIl\}$  be the points on  $l$ , and for a point  $p \in \mathcal{P}$  let  $l(p) = \{l \in \mathcal{L} \mid pIl\}$  be the lines through  $p$ . Furthermore, we agree on a few linguistic conventions. Since in a projective plane the line  $l$  that is at the same time incident to two points  $p$  and  $q$  is by axiom (i) uniquely determined, we will use a language that is more functional than set-theoretic and simply speak of the *join* of the two points. We will express this join operation by  $p \vee q$  or by **join**( $p, q$ ). Similarly, we will call the unique point incident with two lines  $l$  and  $m$  the *meet* or *intersection* of these lines and denote the corresponding operation by  $l \wedge m$  or by **meet**( $l, m$ ). We also say that a line  $l$  *contains* a point  $p$  if it is incident with it.

**Lemma 2.1.** *If for  $p, q \in \mathcal{P}$  and  $l, m \in \mathcal{L}$  we have  $pIl$ ,  $qIl$ ,  $pIm$ , and  $qIm$ , then either  $p = q$  or  $l = m$ .*

*Proof.* Assume that  $pIl$ ,  $qIl$ ,  $pIm$ , and  $qIm$ . If  $p \neq q$ , axiom (i) implies that  $l = m$ . Conversely, if  $l \neq m$ , axiom (ii) implies that  $p = q$ .  $\square$

**Lemma 2.2.** *Every line of a projective plane is incident with at least three points.*

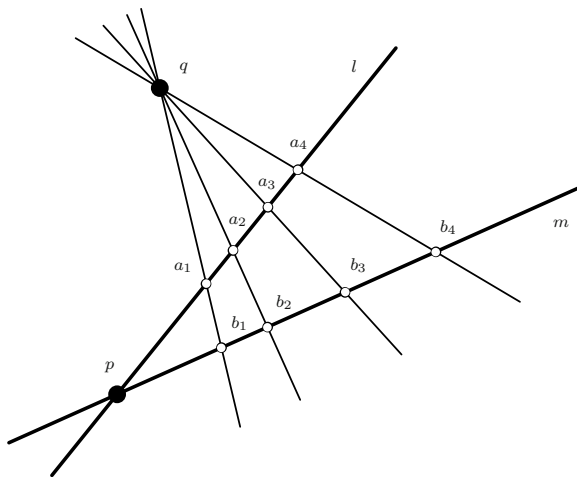
*Proof.* Let  $l \in \mathcal{L}$  be any line of the projective plane and assume to the contrary that  $l$  does contain fewer than three points. Let  $a, b, c$ , and  $d$  be the points of axiom (iii). Assume without loss of generality that  $a$  and  $b$  are not on  $l$ . Consider the lines  $a \vee b$ ,  $a \vee c$ ,  $a \vee d$ . Since these all pass through  $a$ , they must be distinct by axiom (iii) and must by Lemma 2.1 have three distinct intersections with  $l$ .  $\square$

**Lemma 2.3.** *For every point  $p$  there is at least one line not incident with  $p$ .*

*Proof.* Let  $p$  be any point. Let  $l$  and  $m$  be arbitrary lines. Either one of them does not contain  $p$  (then we are done), or we have  $p = l \wedge m$ . By the last lemma there is a point  $p_l$  on  $l$  distinct from  $p$ , and a point  $p_m$  on  $m$  distinct from  $p$ . The join of these two points cannot contain  $p$ , since this would violate axiom (i).  $\square$

**Theorem 2.1.** *Let  $(\mathcal{P}, \mathcal{L}, I)$  be a projective plane with finite sets  $\mathcal{P}$  and  $\mathcal{L}$ . Then there exists a number  $n \in \mathbb{N}$  such that  $|p(l)| = n + 1$  for any  $l \in \mathcal{L}$  and  $|l(p)| = n + 1$  for any  $p \in \mathcal{P}$ .*

*Proof.* Let  $l$  and  $m$  be two distinct lines. Assume that  $l$  contains  $k$  points. We will prove that both lines contain the same number of points. Let  $p = l \wedge m$  be their intersection and let  $\ell$  be a line through  $p$  distinct from  $l$  and  $m$ . Now consider a point  $q$  on  $\ell$  distinct from  $p$ , which exists by Lemma 2.2. Let  $\{a_1, a_2, \dots, a_n\} = p(l) - \{p\}$  be the points on  $l$  distinct from  $p$  and consider



**Fig. 2.8** The proof that all lines have the same number of points.

the  $n - 1$  lines  $l_i = p_i \vee q$ ;  $i = 1, \dots, n$ . Each of these lines intersects the line  $m$  in a point  $b_i = l_i \wedge m$ . All these points have to be distinct, since otherwise there would be lines  $l_i, l_j$  that intersect twice, in contradiction to Lemma 2.1. Thus the number of points on  $m$  is at least as big as the number of points on  $l$ . Similarly, we can argue that the number of points on  $l$  is at least as big as the number of points on  $m$ . Hence both numbers have to be equal. Thus the number of points on a line is the same for any line (see Figure 2.8).

Now let  $p$  be any point and let  $l$  be a line that does not contain  $p$ . Let  $\{p_1, p_2, \dots, p_{n+1}\}$  be the  $n + 1$  points on  $l$ . Joining these points with  $p$  generates  $n + 1$  lines through  $p$ . In fact, these lines must be all lines through  $p$  since any line through  $p$ , must have an intersection with  $l$  by axiom (ii). Hence the number of lines that pass through our (arbitrarily chosen) point  $p$  must also be equal to  $n + 1$ .  $\square$

The number  $n$  of the last proposition (which was the number of points on a line minus one) is usually called the *order* of the projective plane. The following proposition relates the order and the overall number of points and lines in a finite projective plane.

**Theorem 2.2.** *Let  $(\mathcal{P}, \mathcal{L}, \mathcal{I})$  be a projective plane with finite sets  $\mathcal{P}$  and  $\mathcal{L}$  of order  $n$ . Then we have  $|\mathcal{P}| = |\mathcal{L}| = n^2 + n + 1$ .*

*Proof.* The last proposition proved that the number of points on each line is  $n + 1$  and the number of lines through each point is also  $n + 1$ . Let  $p$  be any point of the projective plane. Each of the  $n + 1$  lines through  $p$  contains  $n$  additional points. They must all be distinct, since otherwise two of these lines

intersect twice. We have altogether  $(n + 1) \cdot n + 1 = n^2 + n + 1$  points. A similar count proves that the number of lines is the same.  $\square$

So far we have met two examples of a projective plane. One is the finite Fano plane of order 2; the other (infinite example) was the projective extension of the real Euclidean plane. Our next chapter will show that both can be considered as special examples of a construction that generates a projective plane for every number field.

# Homogeneous Coordinates

*Ich habe bei den folgenden Entwicklungen nur die Absicht gehabt [...] zu zeigen, dass die neue Methode [...] zum Beweise einzelner Sätze und zur Darstellung allgemeiner Theorien sich sehr geschmeidig zeigt.*

*Julius Plücker, Ueber ein neues Coordinatensystem, 1829*

## 3.1 A Spatial Point of View

Let  $\mathbb{K}$  be any field.<sup>1</sup> And let  $\mathbb{K}^3$  be the vector space of dimension three over this field. We will prove that if we consider the one-dimensional subspaces of  $\mathbb{K}^3$  as *points* and the two-dimensional subspaces as *lines*, then we obtain a projective plane by defining *incidence* as subspace containment.

We will prove this fact by creating a more concrete coordinate representation of the one- and two-dimensional subspaces of  $\mathbb{K}^3$ . This will allow us to calculate with these objects easily. For this we first form equivalence classes of vectors by identifying all vectors  $v \in \mathbb{K}^3$  that differ by a nonzero multiple:

$$[v] := \{v' \in \mathbb{K}^3 \mid v' = \lambda \cdot v \text{ for } \lambda \in \mathbb{K} \setminus \{0\}\}.$$

The set of all such equivalence classes could be denoted by  $\frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$ : all nonzero vectors modulo scalar nonzero multiples. Replacing a vector by its

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<sup>1</sup> This is almost the only place in this book where we will refer to an arbitrary field  $\mathbb{K}$ . All other considerations will be much more “down to earth” and refer to specific fields—mostly the real numbers  $\mathbb{R}$  or the complex numbers  $\mathbb{C}$ .

equivalence class preserves many interesting structural properties. In particular, two vectors  $v_1, v_2$  are orthogonal if their scalar product vanishes:  $\langle v_1, v_2 \rangle = 0$ . This relation remains stable if we replace the two vectors by any vectors taken from the corresponding equivalence classes. We define orthogonality of equivalence classes  $[p]$  and  $[l]$  in a canonic way by

$$[p] \perp [l] \iff \langle p, l \rangle = 0.$$

Now we set  $\mathcal{P}_{\mathbb{K}} = \frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$  and let  $\mathcal{L}_{\mathbb{K}} = \frac{\mathbb{K}^3 \setminus \{(0,0,0)\}}{\mathbb{K} \setminus \{0\}}$  as well (we consider  $\mathcal{P}_{\mathbb{K}}$  and  $\mathcal{L}_{\mathbb{K}}$  as disjoint copies of the same kind of space). Furthermore, we define the incidence relation  $\mathbf{I}_{\mathbb{K}} \subseteq \mathcal{P}_{\mathbb{K}} \times \mathcal{L}_{\mathbb{K}}$  for  $[p] \in \mathcal{P}$  and  $[l] \in \mathcal{L}$  by

$$[p] \mathbf{I}_{\mathbb{K}} [l] \iff [p] \perp [l].$$

Before we prove that the triple  $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathbf{I}_{\mathbb{K}})$  is indeed a projective plane, we clarify what this has to do with one- and two-dimensional subspaces. There is a bijection of the set of one-dimensional subspaces of  $\mathbb{K}^3$  and  $\mathcal{P}_{\mathbb{K}}$ . Each subspace can be represented by a single nonzero vector  $p$  in it. In fact, exactly all vectors in the equivalence class  $[p]$  represent the same one-dimensional subspace. We observe, that  $[p]$  itself is this subspace with the zero vector taken out. A two-dimensional vector space  $\{(x, y, z) \mid ax + by + cz = 0\}$  in  $\mathbb{K}^3$  is in our setup represented by its normal vector  $(a, b, c)$ . Since normal vectors that differ only by a scalar multiple describe the same two-dimensional subspace, the set  $\mathcal{L}_{\mathbb{K}}$  is appropriate for representing them. Finally, a one-dimensional subspace represented by  $[p]$  is contained in a two-dimensional subspace represented by  $[l]$  if and only if  $[p] \perp [l]$ . This is consistent with our incidence operator  $\mathbf{I}_{\mathbb{K}}$ .

**Theorem 3.1.** *With the above definitions and notation for any field  $\mathbb{K}$ , the triple  $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathbf{I}_{\mathbb{K}})$  is a projective plane.*

*Proof.* We simply have to verify the three axioms. Let  $[p]$  and  $[q]$  be two distinct elements in  $\mathcal{P}_{\mathbb{K}}$ . In order to verify axiom (i) we must prove that there is a vector  $l$  that is simultaneously orthogonal to  $p$  and  $q$ . Furthermore, we must show that all nonzero vectors with this property must be scalar multiples of  $l$ . Since  $[p]$  and  $[q]$  are distinct, the vectors  $(p_1, p_2, p_3)$  and  $(q_1, q_2, q_3)$  do not differ just by a nonzero scalar multiple. In other words, the matrix

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

has rank 2. Thus the solution space of

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is one-dimensional. This is exactly the desired claim. For any nonzero solution  $(l_1, l_2, l_3)$  of this system the equivalence class  $[(l_1, l_2, l_3)]$  is the desired join of the points.

In a completely similar way, we can verify axiom (ii), which states that for any pair of distinct lines there is exactly one point incident to both.

For axiom (iii) observe that any field  $\mathbb{K}$  must contain a zero element and a one element. It is easy to check that the equivalence classes of the four vectors  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(1, 0, 1)$ , and  $(1, 1, 1)$  satisfy the requirements of non-collinearity of axiom (iii).  $\square$

Although the message of the last theorem is simple, it is perhaps the central point of this entire book. It is the link of geometry and algebra. Its power stems from the fact that we can recover our construction of projectively extending the Euclidean plane directly in the representation of points by three-dimensional vectors. This will be shown in the next section. This representation of points as well as lines of a projective plane by three-dimensional vectors is called homogeneous coordinates. We will later on see that the adjective *homogeneous* is very appropriate, since these coordinates at the same time unify the role of usual lines and the line at infinity and give three coordinates of  $\mathbb{K}^3$  a completely symmetric interpretation. We will see that by introducing this coordinate system we can easily deal with the Euclidean plane and its projective extension (the points and the line at infinity) in a completely algebraic manner.

The use of homogeneous coordinates can be considered an extension of *barycentric coordinates*, which were introduced by August Ferdinand Möbius (1790–1868). Homogeneous coordinates were first introduced by Julius Plücker in his article “Ueber ein neues Coordinatensystem” in 1830 [99]. There he writes

*Ich habe bei den folgenden Entwicklungen nur die Absicht gehabt [...] zu zeigen, dass die neue Methode [...] zum Beweise einzelner Sätze und zur Darstellung allgemeiner Theorien sich sehr geschmeidig zeigt.*<sup>2</sup>

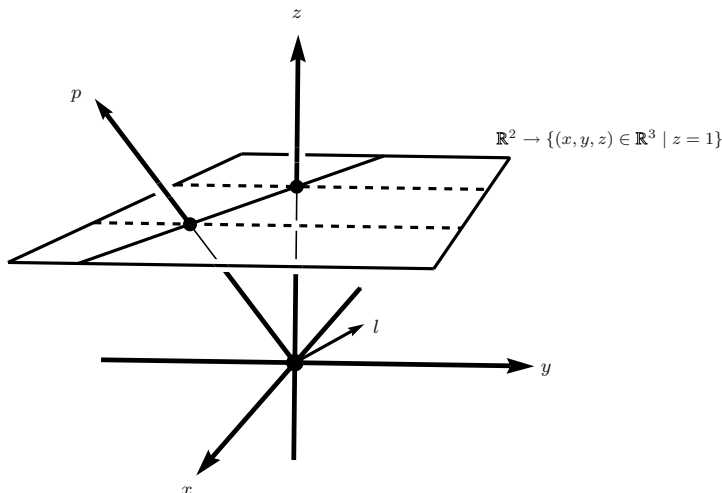
In fact, it is this elegance that we will encounter repeatedly throughout this book. I hope that after finishing this book the reader will finally agree on Plücker’s point of view.

## 3.2 The Real Projective Plane with Homogeneous Coordinates

Let us now analyze how the projective extension of the Euclidean plane fits into the picture of homogeneous coordinates. For this we start with a

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<sup>2</sup> My intention for making the following developments was to demonstrate that this new method turns out to be very pliable for proving specific theorems or for representing general theories.



**Fig. 3.1** Embedding the Euclidean plane in  $\mathbb{R}^3$ .

coordinate representation of the Euclidean plane  $\mathbb{E}$ . As usual, we identify the Euclidean plane with  $\mathbb{R}^2$ . Each point in the Euclidean plane can be represented by a two-dimensional vector of the form  $(x, y) \in \mathbb{R}^2$ . A line can be considered as the set of all points  $(x, y)$  satisfying the equation  $a \cdot x + b \cdot y + c = 0$ . However, since we will treat lines as individual objects rather than sets of points, we will consider the parameters  $(a, b, c)$  themselves as a representation of the line. Observe that for nonzero  $\lambda$  the vector  $(\lambda \cdot a, \lambda \cdot b, \lambda \cdot c)$  represents the same line as  $(a, b, c)$ . Furthermore, the vector  $(0, 0, 1)$  does not represent a real line at all, since then the above equation would read  $1 = 0$ .

Now we make the step to homogeneous coordinates. For this we consider our Euclidean plane embedded affinely in the three-dimensional space  $\mathbb{R}^3$ . It is convenient to consider the plane to be the  $z = 1$  plane. Each point  $(x, y)$  of the Euclidean plane will now be represented by the point  $(x, y, 1)$ . How should we interpret all other points in  $\mathbb{R}^3$ ? In fact, for any point that does not have a zero  $z$ -component we can easily assign a corresponding Euclidean point. For  $(x, y, z) \in \mathbb{R}^3$  we consider the one-dimensional subspace spanned by this point. If  $z \neq 0$  this subspace intersects the embedded Euclidean plane at a unique single point. We can calculate this point simply by dividing by the  $z$ -coordinate. Thus for  $z \neq 0$  the vector  $p = (x, y, z)$  represents the Euclidean point  $(x/z, y/z, 1)$ . Note that all vectors in the equivalence class  $[p]$  represent the same Euclidean point. So if we are interested only in Euclidean points, we do not have to care about nonzero scalar factors.

How about the remaining points of  $\mathbb{R}^3$ , those with  $z$ -coordinate equal to 0? These points will correspond to the *points at infinity* of the projective completion of the Euclidean plane. To see this, we consider a limit process that



dynamically moves a point to infinity and observe what will happen with the Euclidean coordinates. We start in the Euclidean picture. Assume we have a point  $p = (p_1, p_2)$  in the usual Euclidean plane. Furthermore, we have a direction  $r = (r_1, r_2)$ . If we consider  $q_\alpha := p + \alpha \cdot r$  and start to increase  $\alpha$  from 0 to a larger and larger value, the point  $q_\alpha$  will move away in direction  $r$ . What does this situation look like in homogeneous coordinates? Point  $q_\alpha$  is represented by the homogeneous coordinates  $(p_1 + \alpha \cdot r_1, p_2 + \alpha \cdot r_2, 1)$ . Since in homogeneous coordinates we do not care about nonzero multiples, we can (for  $\alpha \neq 0$ ) equivalently represent the point  $q_\alpha$  by  $(p_1/\alpha + r_1, p_2/\alpha + r_2, 1/\alpha)$ . What happens in the limit case  $\alpha \rightarrow \infty$ ? In this case our vector representing  $q_\alpha$  degenerates to the vector  $(r_1, r_2, 0)$ . Let us reinterpret this process geometrically. “No matter with which point we start, if we move it in direction  $r$  further and further out, then in the limit case, we will end up at a point with homogeneous coordinates  $(r_1, r_2, 0)$ .” In other words, we can consider the vector  $(r_1, r_2, 0)$  as a representation of the point at infinity in direction  $r$ . (Perhaps it is a good exercise for the reader to convince himself/herself that we arrive at exactly the same point if we decrease  $\alpha$  starting at  $\alpha = 0$  and ending at  $\alpha = -\infty$ .) Also for infinite points it is possible to neglect scalar multiples and take any point of the corresponding equivalence class  $[(r_1, r_2, 0)]$  to represent the same point at infinity.

The only vector that does not fit into our considerations so far is the zero vector  $(0, 0, 0)$ . This is, however, no problem at all, since the space  $\frac{\mathbb{R}^3 \setminus \{(0,0,0)\}}{\mathbb{R} \setminus \{0\}}$  excludes this vector explicitly. We will see later on that whenever the zero-vector pops up in a calculation we will have encountered a degenerate situation, for instance intersecting two identical lines.

How about the lines? We already saw that a Euclidean line is nicely represented by the parameters  $(a, b, c)$  of the linear equation  $a \cdot x + b \cdot y + c = 0$ . We also observed that multiplying  $(a, b, c)$  by a nonzero scalar does not change the line represented. If we view the line equation in homogeneous coordinates, it becomes

$$a \cdot x + b \cdot y + c \cdot z = 0.$$

If we consider a point on this line with homogeneous coordinates  $(x, y, 1)$ , this form degenerates to the Euclidean version. However, whenever we have a point  $(x, y, z)$  that satisfies the equation, it will still satisfy the equation if we replace it by  $(\lambda x, \lambda y, \lambda z)$ . Thus, this form is stable under our representation of points and lines by equivalence classes. If we interpret this equation in three dimensions, we see that the vector  $(a, b, c)$  is the normal vector of the plane that contains all vectors  $(x, y, z) \in \mathbb{R}^3$  that satisfy the equation. If we intersect this plane with our embedded Euclidean plane, we obtain a line in the Euclidean plane that corresponds to the Euclidean counterpart of our line under consideration (compare Figure 3.1).

There is only one type of vector that does not correspond to a Euclidean line. If we consider the vector  $(0, 0, c)$  with  $c \neq 0$ , the orthogonal vector space is the  $xy$ -plane through the origin. This plane does not intersect the

embedded Euclidean plane. However all points at infinity (remember, they have the form  $(x, y, 0)$ ) are orthogonal to this vector, since  $0 \cdot x + 0 \cdot y + c \cdot 0 = 0$ . We call this line the *line at infinity*. It is incident to all points at infinity.

Let us summarize what we have achieved so far. In Section 2.2 we discussed how we can extend the Euclidean plane by introducing elements at infinity: one point at infinity for each direction and one global line at infinity that contains all these points. Now we have a concrete coordinate representation of these objects. The Euclidean points correspond to points of the form  $(x, y, 1)$ ; the infinite points correspond to points of the form  $(x, y, 0)$ . The Euclidean lines have the form  $(a, b, c)$  with  $a \neq 0$  or  $b \neq 0$  (or both). The line at infinity has the form  $(0, 0, 1)$ . All the vectors are considered modulo nonzero scalar multiples. We will refer to this setup of the real projective plane later on as  $\mathbb{RP}^2$ . This notion stands for Real Projective 2-dimensional space. Later on we will also deal with spaces such as  $\mathbb{RP}^1$ ,  $\mathbb{RP}^d$ ,  $\mathbb{CP}^1$ ,  $\mathbb{CP}^2$ .

From the three-dimensional viewpoint the distinction of infinite and finite elements is completely unnatural: all elements are simply represented by vectors. This resembles the situation in the axiom system for projective planes. There we also do not distinguish between finite and infinite elements. This distinction is only a kind of artifact that arises when we interpret the Euclidean plane in a projective setup. In a sense, if we consider the projective plane as an extension of the Euclidean plane, we break the nice symmetry of projective planes by (artificially) singling out one line to play the role of the line at infinity. Nevertheless, it is a very fruitful exercise to interpret Euclidean theorems in a projective framework or to interpret projective theorems in a Euclidean framework. Usually, a whole group of theorems in Euclidean geometry corresponds to just one theorem in projective geometry and turns out to be just different specializations for different lines at infinity. We will make these kinds of investigations very often in the following chapters, and we will see how nicely projective geometry generalizes different Euclidean concepts.

### 3.3 Joins and Meets

This section is dedicated to a way of easily carrying out elementary operations in geometry by algebraic calculation. In Chapter 2 we saw that the axiom system for projective planes immediately motivates two operations, the *join of two points* and the *meet of two lines*. We will now get to know the algebraic counterparts of these operations. From now on we will (by slight abuse of notation) no longer explicitly refer to the equivalence classes of points that arise from multiplication by nonzero scalars. Rather than that we will do the calculations with explicit representatives of these classes. Essentially all operations that will be described can be simply carried out on this level of representatives. So, from now on the reader should always have in mind that the vectors  $(x, y, z)$  and  $(\lambda x, \lambda y, \lambda z)$  represent the same geometric point.

The crucial point for representing the join and meet operations algebraically is that if (in homogeneous coordinates) the point  $(x, y, z)$  is contained in the line  $(a, b, c)$ , the equation

$$a \cdot x + b \cdot y + c \cdot z = 0$$

holds. If the equation holds, then these two vectors are orthogonal. Now, if two points  $p = (p_1, p_2, p_3)$  and  $q = (q_1, q_2, q_3)$  are given, then the coordinates  $l = (l_1, l_2, l_3)$  of a line incident to both points must be orthogonal to both vectors  $p$  and  $q$ . In Section 3.1 we argued that there is a solution to this problem by explicitly writing down a system of two linear equations. However, there is also a way to obtain a specific solution explicitly. For this consider the vector-product operator “ $\times$ ” from linear algebra. This operator is defined as follows:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} +p_2q_3 - p_3q_2 \\ -p_1q_3 + p_3q_1 \\ +p_1q_2 - p_2q_1 \end{pmatrix}.$$

An easy calculation shows that this operator generates a vector that is simultaneously orthogonal to  $p$  and  $q$ . For instance, for  $p$  we get, after term expansion,

$$p_1 \cdot (p_2q_3 - p_3q_2) + p_2 \cdot (-p_1q_3 + p_3q_1) + p_3 \cdot (p_1q_2 - p_2q_1) = 0.$$

(We will soon see a more structural approach to the vector product that explains this relation.) Thus we can express the join operation of two points simply by the cross product:

$$\mathbf{meet}(p, q) := p \times q.$$

We can deal in a completely similar fashion with the problem of intersecting two lines  $l = (l_1, l_2, l_3)$  and  $m = (m_1, m_2, m_3)$ . A point that is simultaneously incident with both lines must be represented by a vector that is orthogonal to both  $l$  and  $m$ . We can generate such a vector simply by forming the vector product. Thus we get

$$\mathbf{join}(l, m) := l \times m.$$

It is instructive to see these operators in work at a Euclidean example. Let  $A, B, C, D$  (see Figure 3.2, left) be four points in the Euclidean plane given by the following (Euclidean) coordinates:

$$\begin{aligned} A &= (1, 1), \\ B &= (3, 2), \\ C &= (3, 0), \\ D &= (4, 1). \end{aligned}$$

What are the coordinates of the intersection of the lines  $AB$  and  $CD$ ? The homogeneous coordinates of the points are  $A = (1, 1, 1), B = (3, 2, 1)$ ,

$C = (3, 0, 1)$ ,  $D = (4, 1, 1)$ . We can calculate the homogeneous coordinates of the two lines simply by taking the vector products:

$$\begin{aligned} l_{AB} &= (1, 1, 1) \times (3, 2, 1) \\ &= (1 \cdot 1 - 1 \cdot 2, -1 \cdot 1 + 1 \cdot 3, 1 \cdot 2 - 1 \cdot 3) \\ &= (-1, 2, -1), \end{aligned}$$

$$\begin{aligned} l_{CD} &= (3, 0, 1) \times (4, 1, 1) \\ &= (0 \cdot 1 - 1 \cdot 1, -3 \cdot 1 + 1 \cdot 4, 3 \cdot 1 - 0 \cdot 4) \\ &= (-1, 1, 3). \end{aligned}$$

The meet  $E$  of these lines is again calculated by the vector product:

$$\begin{aligned} E &= (-1, 2, -1) \times (-1, 1, 3) \\ &= (2 \cdot 3 - (-1) \cdot 1, -(-1) \cdot 3 + (-1) \cdot (-1), (-1) \cdot 1 - 2 \cdot (-1)) \\ &= (7, 4, 1). \end{aligned}$$

These are the homogeneous coordinates of the Euclidean point  $(7, 4)$ . (The fact that the  $z$ -coordinate turned out to be 1 was, in fact, only a lucky coincidence. In general we would have to divide by this coordinate to get the Euclidean values.) It is somehow amazing that with a projective point of view we get an explicit and straightforward way to calculate with joins and intersections. The calculations even automatically take care of the coordinates if elements at infinity are involved. We consider the same example but now with point  $D$  located at  $(5, 1)$  (see Figure 3.2, right). The calculation above becomes

$$\begin{aligned} l_{AB} &= (1, 1, 1) \times (3, 2, 1) \\ &= (1 \cdot 1 - 1 \cdot 2, -1 \cdot 1 + 1 \cdot 3, 1 \cdot 2 - 1 \cdot 3) \\ &= (-1, 2, -1), \end{aligned}$$

$$\begin{aligned} l_{CD} &= (3, 0, 1) \times (5, 1, 1) \\ &= (0 \cdot 1 - 1 \cdot 1, -3 \cdot 1 + 1 \cdot 5, 3 \cdot 1 - 0 \cdot 5) \\ &= (-1, 2, 3). \end{aligned}$$

$$\begin{aligned} E &= (-1, 2, -1) \times (-1, 2, 3) \\ &= (2 \cdot 3 - (-1) \cdot 2, -(-1) \cdot 3 + (-1) \cdot (-1), (-1) \cdot 2 - 2 \cdot (-1)) \\ &= (8, 4, 0). \end{aligned}$$

Point  $E$  is now an infinite point, since its  $z$ -coordinate is zero. In particular, it is the infinite point in direction  $(8, 4)$  (or equivalently in direction  $(2, 1)$ ). This is the point in which the two parallel lines meet.

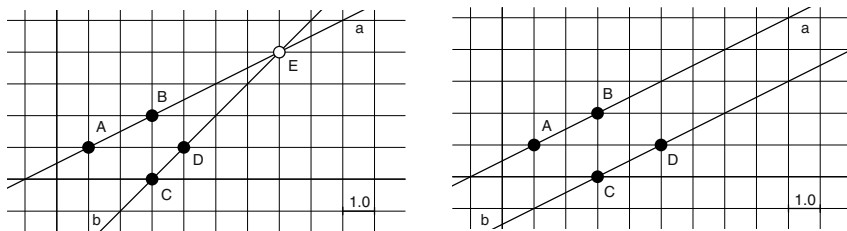


Fig. 3.2 Working with meet and join.

### 3.4 Parallelism

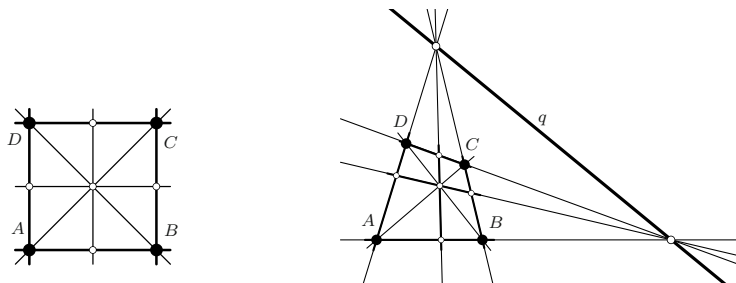
The only operations and relations we have modeled so far are incidence, join, and meet. We will see that many other geometric operations (such as measuring distances, calculating angles, creating perpendiculars) will require special treatment if we want to model them in a projective setup. Nevertheless, there is at least one operation of Euclidean geometry that can be easily modeled in a projective framework: *drawing a parallel to a line through a point*. For this, start with the real projective plane with our usual setup in homogeneous coordinates. We then have to single out a line at infinity. Usually we use the standard line at infinity with homogeneous coordinates  $(0, 0, 1)$ , but we are not forced to do so.

Let  $l_\infty \in \mathcal{L}_\mathbb{R}$  be the line at infinity. With respect to this line we can define an operator  $\text{parallel}(p, l) : \mathcal{P}_\mathbb{R} \times \mathcal{L}_\mathbb{R} \rightarrow \mathcal{L}_\mathbb{R}$  that takes as input a line  $l$  and a point  $p$  and calculates a line parallel to  $l$  and through  $p$ . We define this operator by

$$\text{parallel}(p, l) := \text{join}(p, \text{meet}(l, l_\infty)) = p \times (l \times l_\infty).$$

How does this operator work? First it calculates the intersection of  $l$  with the line at infinity. This is the point at infinity that is contained in  $l$  and on any parallel to  $l$ . So, if we want to obtain a parallel to  $l$  through  $p$ , we have simply to join this point with  $p$ . This is how the operator works.

It is interesting to see what happens if we select a finite Euclidean line as the line at infinity. As an example consider the situation of a square and the task of constructing its two diagonals, its' center, and two lines through this center that are parallel to the quadrangles sides (Figure 3.3). If we had chosen four arbitrary (nonsquare) points  $A, B, C, D$  as corners, the construction could still be performed. For this let  $A, B, C, D$  be the corners of the quadrangle in cyclic order. The joins  $d_1 = \text{join}(A, C)$  and  $d_2 = \text{join}(B, D)$  are the diagonals of the “quadrangle.” Their meet  $m = \text{meet}(d_1, d_2)$  is the center. To get the two parallels we first have to know where the line at infinity is. If we consider (by definition) the four points as corners of a square, we know that



**Fig. 3.3** Parallelism with a finite line as line at infinity.

opposite sides must be parallel. Hence the intersection of the lines supporting opposite sides gives us two ways of constructing a point at infinity, namely  $p_1 = \text{meet}(\text{join}(A, B), \text{join}(C, D))$  and  $p_2 = \text{meet}(\text{join}(B, C), \text{join}(D, A))$ . Joining these two points gives us the position of the line at infinity. We finally want to construct the two lines through the center, parallel to the sides. These are simply the joins  $\text{join}(m, p_1)$  and  $\text{join}(m, p_2)$ . What we finally obtain is a perspectively correct drawing of the quadrangle together with the required points and lines.

### 3.5 Duality

We will here briefly touch upon a topic that we will encounter later in greater depth and detail. You may have observed that if we are in a projective setup, points and lines play a completely symmetric role. We want to point out a few points where this becomes transparent.

- In the axiom system for projective planes axiom (i) transfers to axiom (ii) if one interchanges the words line and point.
- At first sight, axiom (iii) seems to break symmetry. However, one can prove a similar (and equivalent) statement with the role of points and lines interchanged as a consequence of the three axioms.
- In the homogeneous coordinate setup the spaces  $\mathcal{P}_{\mathbb{K}}$  and  $\mathcal{L}_{\mathbb{K}}$  are algebraically identical.
- In the incidence relation  $ax + by + cz = 0$  the vectors  $(a, b, c)$  and  $(x, y, z)$  play a completely symmetric role.
- Joins and meets can both be calculated by the vector product.

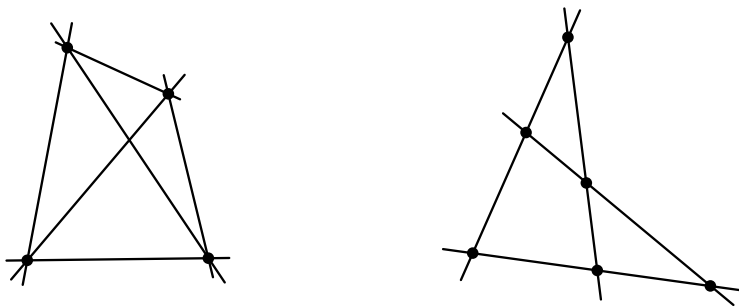
So, every true statement in projective geometry that involves only the vocabulary we have developed so far is again transferred to a true statement if we exchange the terms:

$$\begin{aligned}
 \text{point} &\leftrightarrow \text{line} \\
 \text{join} &\leftrightarrow \text{meet} \\
 \mathcal{P} &\leftrightarrow \mathcal{L}
 \end{aligned}$$

We call this effect *duality*. So we can say that the very basis of projective geometry is dual. This implies that for every concept we will develop further on there will be a corresponding dual counterpart. For every theorem in projective geometry there will be a corresponding dual theorem. For every definition in projective geometry there will be a corresponding dual definition, and so forth. The reader is invited to dualize the rest of this book (i.e., it is useful to question for every concept/theorem/definition/drawing introduced in the book what the corresponding dual would be).

We will exemplify duality with a small construction of projective geometry (compare Figure 3.4). We first describe the primal construction. We start with four points of which no three are collinear in  $\mathbb{RP}^2$ . There are altogether six lines that can be drawn between these four points. Dually this reads thus: Start with four lines. These lines will have altogether six points of intersection. The primal and dual situations are drawn in Figure 3.4.

One has to be aware that the analogy of primal and dual situations goes far beyond the combinatorial level. We can literally take the homogeneous coordinates of a point and interpret them as homogeneous coordinates of a line, and vice versa. Incidences are preserved under this exchange. Figure 3.5 represents an example of three collinear points in the standard embedding of the Euclidean plane on the  $z = 1$  plane. Coordinates of the points and of the line are given. The second picture shows the corresponding dual situation in which the coordinates are interpreted as line coordinates, three lines that meet in a point. The line equations are given, and it is easy to check that the homogeneous coordinates of the points in one picture are exactly the homogeneous coordinates of the lines in the other picture.



**Fig. 3.4** A pair of primal and dual configurations.

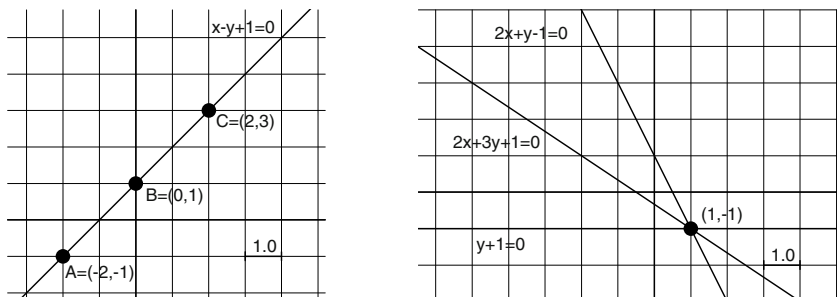


Fig. 3.5 A pair of primal and dual configurations with coordinates.

### 3.6 Projective Transformations

Transformations are a fundamental concept all over geometry. There are different aspects under which one can consider transformations. On the one hand, they are a change of the frame of reference. After a transformation, the same objects are represented within a new coordinate system. Hence a transformation is a (bijective) map of the ambient space onto itself. The other way one can look at transformations is that they take the objects and move (or even deform) them to end up in another position. No matter which picture one prefer describing a transformation, the crucial point is that they leave certain properties of the objects unchanged.

We will first introduce transformations in an abstract setup and become more and more specific further on. In general, one can equip reasonable collections of transformations with a group structure. For this let us consider an object space  $\mathcal{O}$ . This object space will later on be, for instance, the set of points  $\mathcal{P}_{\mathbb{R}}$  of the real projective plane. In general, a transformation is a bijective map  $T: \mathcal{O} \rightarrow \mathcal{O}$ . We obtain the group structure by requiring that collections of transformations be closed under reasonable operations. If one applies two transformations  $T_1$  and  $T_2$  one after another, one can consider the result as a single transformation  $(T_2 \circ T_1): \mathcal{O} \rightarrow \mathcal{O}$ . For this book we make the convention that  $T_2 \circ T_1$  is interpreted as first applying  $T_1$  and then  $T_2$ . Thus if we have a specific object  $o \in \mathcal{O}$ , we have  $(T_2 \circ T_1)(o) = T_2(T_1(o))$ . The identity  $\text{Id}: \mathcal{O} \rightarrow \mathcal{O}$  that maps every element of the object space to itself is a transformation. Since transformations are assumed to be bijective maps in the object space, we can for any transformation  $T$  consider its inverse operation  $T^{-1}$  as a transformation as well. We have  $T \circ T^{-1} = \text{Id}$ . It is also not difficult to check that transformations are in general associative. For this we have to show that if we have three transformations  $T_1, T_2, T_3$ , the relation  $(T_3 \circ T_2) \circ T_1 = T_3 \circ (T_2 \circ T_1)$  holds. In order to see this, consider a concrete object  $o$ . We have



$$\begin{aligned}
((T_3 \circ T_2) \circ T_1)(o) &= (T_3 \circ T_2)(T_1(o)) \\
&= T_3(T_2(T_1(o))) \\
&= T_3((T_2 \circ T_1)(o)) \\
&= (T_3 \circ (T_2 \circ T_1))(o).
\end{aligned}$$

Taking all this together, one obtains the properties that ensure that we have a *group structure*.

Let us be a little more concrete and consider the usual transformations of Euclidean geometry (we will now recall a few facts from linear algebra). For this let  $\mathbb{R}^2$  again represent the coordinates of the Euclidean plane. The points of the Euclidean plane will be our objects; thus  $\mathbb{R}^2$  plays the role of the object space. The usual transformations in Euclidean geometry are *translations*, *rotations*, *reflections*, and *glide reflections*. These transformations can easily be expressed by algebraic operations. A translation by a vector  $(t_x, t_y)$  can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}.$$

A rotation about the origin by an angle  $\alpha$  can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

A rotation about an arbitrary point  $(r_x, r_y)$  can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x - r_x \\ y - r_y \end{pmatrix} + \begin{pmatrix} r_x \\ r_y \end{pmatrix}.$$

Reflections and glide reflections have a similar representation. Any of the above Euclidean transformations can be written in the form

$$p \mapsto M(p - v) + w$$

for suitable choices of a  $2 \times 2$  matrix  $M$  and vectors  $v$  and  $w$ . For rotations the matrix  $M$  has to be a rotation matrix. This means it has the form  $\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$ . For reflections or glide reflections the matrix must be a re-

flection matrix of the form  $\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$ . The group of Euclidean transformations leaves fundamental properties and relations within the object space invariant. For instance, if  $p$  and  $q$  are Euclidean points, then the distance between them is the same before or after a Euclidean transformation. Also the absolute values of angles are not altered by Euclidean transformations. In general, the *shape* and *size* of an object is not altered by a Euclidean transformation. If one maps a circle (or line, or quadrangle) pointwise by a Euclidean transformation, one ends up again with a circle (or line, or quadrangle) of the same size. It may just have moved to another location.

In the above form  $M(p - v) + w$ , one may allow for more general transformations (where  $M$  is *any* invertible  $2 \times 2$  matrix). By this one can also describe scalings, similarities, and affine transformations. In this case the group of transformations becomes larger and the set of properties that is not altered by these transformations becomes smaller. For instance, similarities will still preserve the absolute value of angles but no longer distances. An affine transformation will not even preserve angles. However, an affine transformation still maps a pair of parallel lines to another pair of parallel lines.

From the point of view of computer implementations it is inherently difficult and error-prone to calculate with the above representation of Euclidean transformations. The fact that the rotational or reflectional part is expressed by a matrix multiplication while the translational part is expressed by a vector addition makes it cumbersome to calculate the inverses or the composition of two transformations. Again we get a structurally much clearer approach if we focus on a projective setup and an approach via homogeneous coordinates.

If we represent a Euclidean point  $(x, y)$  by homogeneous coordinates  $(x, y, 1)$ , we can express rotations as well as translations by a multiplication by a  $3 \times 3$  matrix. Translations take the following form (assuming for a moment that the  $z$ -coordinate is chosen to be 1):

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}.$$

Rotations about the origin can be expressed thus:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Applying two transformations in succession is now nothing but multiplication of the corresponding matrices. Inverting a transformation corresponds to matrix inversion. One should notice that the above matrices were chosen in a way that a vector with  $z$ -coordinate equal to one is again mapped to a vector with  $z$ -coordinate equal to one. Hence, the first two entries of the homogeneous coordinate vector directly show the Euclidean position of the mapped point (in our standard embedding). From a conceptual point of view, it is, even if one deals only with Euclidean transformations, often much more useful to work in this more general representation, since here translations, rotations, and reflections arise in a unified way. Moreover, we will gain even more advantage from this representation, since it is the key to an even wider class of transformations: the projective transformations. First, if we consider matrices with nonzero determinant of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix},$$

then we get all affine planar transformations. Still we have not used the whole freedom of an invertible  $3 \times 3$  matrix. A general *projective transformation* is a multiplication by an invertible  $3 \times 3$  matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We now want to investigate the properties of such a general type of projective transformation. We first make a notational convention. Since for any  $p \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$  the product of a  $3 \times 3$  matrix  $M$  and any member of the equivalence class  $[p]$  ends up within the same equivalence class  $[M \cdot p]$ , the action of  $M$  on these equivalence classes is well defined. Thus we can simply interpret  $M$  as acting on our object space (of equivalence classes)  $\mathcal{P}_{\mathbb{R}}$ . Thus we can interpret the multiplication by  $M$  on the level of representatives taken from  $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$  or on the level of equivalence classes  $\frac{\mathbb{R}^3 \setminus \{(0, 0, 0)\}}{\mathbb{R} \setminus \{0\}}$ . Thus for a projective point in  $[p] \in \mathcal{P}_{\mathbb{R}}$  we will write  $M \cdot [p]$  and mean by this the projective point  $[M \cdot p]$ .

Since in the context of projective geometry the input vector and the output vector of our matrix multiplication are determined only up to multiplication by a nonzero scalar, the matrices  $M$  and  $\lambda M$  represent the same projective transformation (for nonzero  $\lambda$ ). Thus we have overall only eight degrees of freedom that determine such a transformation.

One fundamental property of projective transformations is given by the following statement.

**Theorem 3.2.** *A projective transformation maps collinear points to collinear points.*

*Proof.* It suffices to prove the theorem for a generic triple of points. Let  $[a], [b], [c] \in \mathcal{P}_{\mathbb{R}}$  be three collinear points represented by homogeneous coordinates  $a, b, c$ . In this case there exists a line  $[l] \in \mathcal{L}_{\mathbb{R}}$  with  $\langle l, a \rangle = \langle l, b \rangle = \langle l, c \rangle = 0$ . We assume that all homogeneous coordinates are represented by column vectors. We have to show that under these conditions the points represented by  $a' = M \cdot a$ ,  $b' = M \cdot b$ ,  $c' = M \cdot c$ , are also collinear. For this simply consider the line  $[l']$  represented by  $l' := (M^{-1})^T l$ . We have

$$\langle l', a' \rangle = (l')^T a' = ((M^{-1})^T l)^T M a = l^T ((M^{-1})^T)^T M a = l^T a = \langle l, a \rangle = 0.$$

A similar calculation applies also to the other two points. Thus the line represented by  $l'$  is simultaneously incident to all three points represented by  $a', b'$  and,  $c'$ . Hence these points are collinear.  $\square$

Implicitly, the calculations in the last proof describe how a projective transformation  $M: \mathcal{P}_{\mathbb{R}} \rightarrow \mathcal{P}_{\mathbb{R}}$  represented by a  $3 \times 3$  matrix  $M$  acts on the space of lines  $\mathcal{L}_{\mathbb{R}}$ . The homogeneous coordinates of a line must be mapped in such a way that incidences of points and lines are preserved under the mapping. This implies that a line has to be mapped according to  $l \mapsto (M^{-1})^T l$ . If  $p$  and  $l$  are incident before a transformation, they will be incident after the transformation as well.

In fact, the property of Theorem 3.2 is characteristic of projective transformations over the field of real numbers. One can prove the following:

**Theorem 3.3.** *If  $\Phi: \mathcal{P}_{\mathbb{R}} \rightarrow \mathcal{P}_{\mathbb{R}}$  is any bijective map that preserves the collinearity of points, then  $\Phi$  can be expressed as multiplication by a  $3 \times 3$  matrix.*

In fact, this theorem is so crucial that it is sometimes called the *fundamental theorem of projective geometry*. Its proof is a bit subtle, and requires some elementary results from field theory. The proof makes use of the fact that the real numbers do not have any field automorphisms except the identity. The generalization of the above theorem to arbitrary fields involves a proper discussion of field automorphisms. A proof will be postponed to Section 5, where we will discuss the relations of projective geometry and elementary arithmetic operations. For now, we will collect more properties of projective transformations that can be expressed as multiplication by a  $3 \times 3$  matrix.

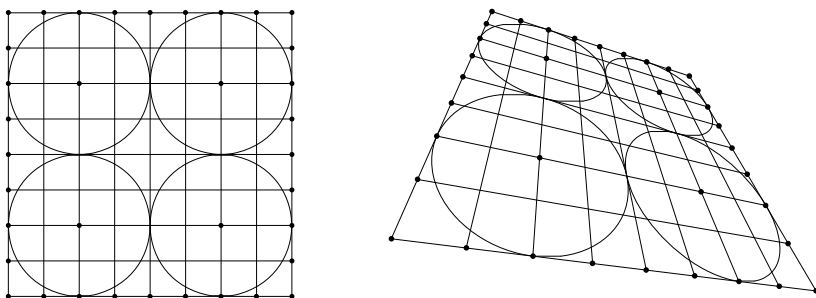
The most fundamental property of projective transformations that we will need (which is also of invaluable practical importance) is the following fact.

**Theorem 3.4.** *Let  $[a], [b], [c], [d] \in \mathcal{P}_{\mathbb{R}}$  be four points of which no three are collinear and let  $[a'], [b'], [c'], [d'] \in \mathcal{P}_{\mathbb{R}}$  be another four points of which no three are collinear then there exists a  $3 \times 3$  matrix  $M$  such that  $[M \cdot a] = [a']$ ,  $[M \cdot b] = [b']$ ,  $[M \cdot c] = [c']$ , and  $[M \cdot d] = [d']$ .*

*Proof.* We assume that  $a, b, c, d, a', b', c', d' \in \mathbb{R}^3$  are representatives of the corresponding equivalence classes. We first prove the theorem for the special case that  $a = (1, 0, 0)$ ,  $b = (0, 1, 0)$ ,  $c = (0, 0, 1)$ , and  $d = (1, 1, 1)$ . Since the columns of a matrix are the images of the unit vectors, the matrix must have the form  $(\lambda \cdot a', \mu \cdot b', \tau \cdot c')$ . (In other words, the image of  $a$  must be a multiple of vector  $a'$  and so forth.) Hence the image of  $d$  is  $\lambda \cdot a' + \mu \cdot b' + \tau \cdot c'$ . This must be a multiple of  $d'$ . We have only to adjust the parameters  $\lambda, \mu, \tau$  accordingly. For this we have to solve the system of linear equations

$$\begin{pmatrix} | & | & | \\ a' & b' & c' \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} | \\ d' \\ | \end{pmatrix}.$$

This system is solvable, by our nondegeneracy assumptions ( $a', b', c'$  are not collinear). Furthermore, none of the parameters is zero (as a consequence of



**Fig. 3.6** The image of a grid under a projective transformation.

the remaining nondegeneracy assumptions). This proves the theorem for the special case.

In order to prove the general case of the theorem one uses the above fact to find a transformation  $T_1$  that maps  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ,  $(1, 1, 1)$  to  $a, b, c, d$ , and to find a transformation  $T_2$  that maps  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ,  $(1, 1, 1)$  to  $a', b', c', d'$ . The desired transformation is then  $T_2 \cdot T_1^{-1}$ .  $\square$

*Remark 3.1. (A note on implementations):* The last theorem is not only of theoretical interest. The proof gives also a practical recipe for calculating a projective transformation that maps  $a, b, c, d$  to  $a', b', c', d'$  (as usual up to scalar multiple). The basic operations that are required for this are matrix multiplication and matrix inversion. One has simply to follow the different calculation steps in the above proof.

The fact that projective transformations preserve collinearities and incidences of points and lines relates them intimately to the topic of perspectively correct drawings. Figure 3.6 shows a drawing of a checkerboard-like grid and four circles and its image under a projective transformation. The projectively transformed picture is completely determined by the image of four corner points. Observe that, for instance, the grid points along the diagonals are again collinear in the transformed image. One can also see that angles and distances are not preserved under a projective transformation. Not even ratios of distances are preserved: an equidistant chain of points in the original picture will in general no longer be equidistant after the projective transformation (later on, we will see that *cross-ratios* are preserved under projective transformations). We also see that circles are not necessarily mapped to circles again. The picture also indicates that tangency relations of curves are preserved under projective transformations.

Throughout the entire book we will very often return to the topic of projective transformations under various aspects.

### 3.7 Finite Projective Planes

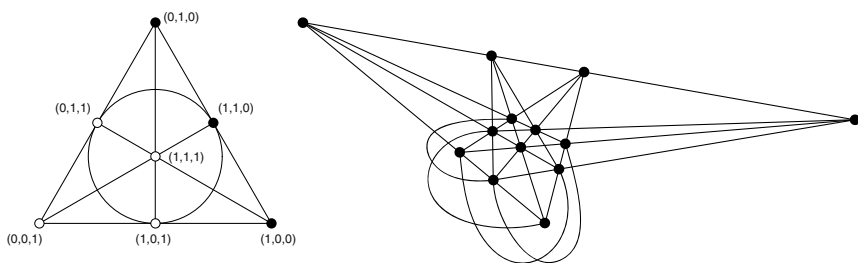
Before we will continue our study of geometric situations over the real (and over the complex) numbers we will have a very brief look at projective spaces over finite fields. Without providing proofs we will report on a few basic facts. The construction of Section 3.1 was a general method of constructing a projective plane starting from a field  $\mathbb{K}$ . Points correspond to one-dimensional subspaces; lines correspond to two-dimensional subspaces. If  $\mathbb{K}$  is a finite field we end up with a projective plane consisting of only finitely many points and lines. Let us consider the smallest cases explicitly. First we study the case  $\mathbb{K} = \text{GF}_2$ , the field of characteristic 2 that consists of a 0 and a 1 only. All nonzero vectors of  $\mathbb{K}^3$  are listed below:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Over this field there are no nontrivial scalar multiples of these vectors (the only nonzero scalar is  $\lambda = 1$ ). Hence each of these vectors corresponds to one point of the corresponding projective plane. These seven points are nothing but the seven points of the Fano plane that we encountered in Section 2.3.

An assignment of coordinates to the points is given in Figure 3.7. Three points are collinear in this plane if and only if there is a line vector  $(a, b, c)$  that is simultaneously orthogonal to all three points. For instance, the circle in the center corresponds to the line  $(1, 1, 1)$ .

Alternatively, one can view the Fano plane in the following way: The  $\text{GF}_2$  analogue of the Euclidean plane  $\mathbb{R}^2$  is the space  $(\text{GF}_2)^2$  that has exactly four elements. We can homogenize them by embedding them in the  $z = 1$  plane of  $(\text{GF}_2)^3$  (white points in the picture). In addition, we have to consider all *points at infinity* with a  $z$ -coordinate 0 (the black points). They lie on a common line—the line at infinity. Observe that each projective line contains



**Fig. 3.7** The Fano plane with coordinates over  $\text{GF}_2$ . And the projective plane over  $\text{GF}_3$

exactly  $n+1$  elements. We can calculate the number of points in two different ways. If  $n$  is the number of elements of the field then we have  $n^2$  finite points and  $n+1$  infinite points. This makes  $n^2 + n + 1$  points altogether. We obtain the same number if we consider the  $(n^3 - 1)$  nonzero vectors in  $\mathbb{K}^3$ . Each equivalence class consists of  $n - 1$  vectors. And we have  $(n^3 - 1)/(n - 1) = n^2 + n + 1$  points.

The next, more complicated, example is a projective plane over the three-element field  $\text{GF}_3$ . Here we have 4 points on each line, and the overall number of points is  $3^2 + 3 + 1 = 13$ . The corresponding incidence structure is shown in Figure 3.7 (right): also here we could divide the points into a finite and an infinite part and single out a line at infinity. However, one should be aware that by construction there are no a priori distinguished lines: As in the case of the Euclidean plane, *any* line can play the role of the line at infinity.

Since there is a finite field for every prime power  $p$ , our general construction immediately yields the following result:

**Theorem 3.5.** *For any prime power  $n$  there is a projective plane that consists of  $n^2 + n + 1$  points and  $n^2 + n + 1$  lines. Each line contains exactly  $n + 1$  points and each point lies on exactly  $n + 1$  lines.*

The parameter  $n$  is called the *order* of the finite projective plane. There is a famous conjecture that the order of a projective plane is always a prime power. However, experts in the field have tried to prove this conjecture now for several decades, and the status of the conjecture remains open. We briefly want to review the state of this conjecture. A priori there is no reason why for  $n > 1$  there should not be a projective plane of order  $n$ . The sharpest result that rules out several cases is the theorem of Bruck and Ryser, which was first proven in 1949 [20] (which we quote without proof here).

**Theorem 3.6.** *If a projective plane of order  $n$  exists, and  $n \equiv 1$  or  $2 \pmod{4}$ , then  $n$  is the sum of two squares.*

Let us see what the situation looks like for orders up to 14:

$2 = 2^1 = 1 + 1$	Fano plane;
$3 = 3^1$	Plane over $\text{GF}_3$ ;
$4 = 2^2$	Plane over $(\text{GF}_2)^2$ ;
$5 = 5^1 = 4 + 1$	Plane over $\text{GF}_5$ ;
6	Not sum of two squares; no projective plane of this order;
$7 = 7^1$	Plane over $\text{GF}_7$ ;
$8 = 2^3$	Plane over $(\text{GF}_2)^3$ ;
$9 = 3^2 = 9 + 0$	Plane over $(\text{GF}_3)^2$ ;
$10 = 9 + 1$	No prime power, but Bruck-Ryser does also not apply;
$11 = 11^1$	Plane over $\text{GF}_{11}$ ;
12	No prime power, but Bruck-Ryser does also not apply;
$13 = 13^1$	Plane over $\text{GF}_{13}$ ;
14	Not sum of two squares; no projective plane of this order;

The table reveals two interesting values of the order for which the Theorem of Bruck and Ryser does not rule out the existence of a projective plane, nor does our field construction apply: the orders 10 and 12.

The case of order 10 was settled in 1989 by H.W.C. Lam, L. Thiel, and S. Swiercz [78, 79]. They proved the nonexistence of a projective plane of order 10 by a clever but in essence still brute-force computer proof. The exhaustive computer proof took the equivalent of 2000 hours on a Cray 1 supercomputer. (In order to get an impression of the problem state it in the following way: Mark the places in a  $111 \times 111$  array such that the following conditions are satisfied: In each row and each column there are exactly 11 marks. Furthermore, each pair of rows must have exactly one mark in the same column.)

The case of order 12 is still wide open. No method seems to be known to break down the difficulty of enumerating all possible cases to a reasonable size that would fit on contemporary computing devices.

One might wonder whether the only way to obtain a finite projective plane is via our field construction. This is not the case. The first case for which such nonstandard planes occur is that of order nine. There are 4 nonisomorphic projective planes of this order. There are even 193 (known) finite projective planes of order 25. A general method of classification seems to be far beyond reach.



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## Lines and Cross-Ratios

*Upside down  
boy you turn me  
inside out  
and round and round*  
*Song text, Diana Ross*

At this stage of this monograph we enter a significant didactic problem. There are three concepts that are intimately related and that unfold their full power only if they play together. These concepts are *performing calculations with geometric objects*, *determinants and determinant algebra*, and *geometric incidence theorems*. The reader should understand that in a beginner's text that makes few assumptions about prior knowledge, these concepts must be introduced sequentially. Therefore we will sacrifice some mathematical beauty for clarity of exposition. Still, we highly recommend that the following chapters be read (at least) twice, so that the reader may obtain an impression of the interplay of the different concepts.

This and the next section are dedicated to the relationship between  $\mathbb{RP}^2$  and calculations in the underlying field  $\mathbb{R}$ . For this we will first find methods to relate points in a projective plane to the coordinates over  $\mathbb{R}$ . Then we will show that elementary operations like addition and multiplication can be mimicked in a purely geometric fashion. Finally, we will use these facts to derive interesting statements about the structure of projective planes and projective transformations.

## 4.1 Coordinates on a Line

Assume that two distinct points  $[p]$  and  $[q]$  in  $\mathcal{P}_{\mathbb{R}}$  are given. How can we describe the set of all points on the line through these two points? It is clear that we can implicitly describe them by first calculating the homogeneous coordinates of the line through  $p$  and  $q$  and then selecting all points that are incident to this line. However, there is also a very direct and explicit way of describing these points, as the following lemma shows:

**Lemma 4.1.** *Let  $[p]$  and  $[q]$  be two distinct points in  $\mathcal{P}_{\mathbb{R}}$ . The set of all points on the line through these points is given by*

$$\{[\lambda \cdot p + \mu \cdot q] \mid \lambda, \mu \in \mathbb{R} \text{ with } \lambda \text{ or } \mu \text{ nonzero}\}.$$

*Proof.* The proof is an exercise in elementary linear algebra. For  $\lambda, \mu \in \mathbb{R}$  (with  $\lambda$  or  $\mu$  nonzero) let  $r = \lambda \cdot p + \mu \cdot q$  be a representative of a point. We have to show that this point is on the line through  $[p]$  and  $[q]$ . In other words, we must prove that  $\langle \lambda \cdot p + \mu \cdot q, p \times q \rangle = 0$ . This is an immediate consequence of the arithmetic rules for the scalar and vector products. We have

$$\begin{aligned} \langle \lambda \cdot p + \mu \cdot q, p \times q \rangle &= \langle \lambda \cdot p, p \times q \rangle + \langle \mu \cdot q, p \times q \rangle \\ &= \lambda \langle p, p \times q \rangle + \mu \langle q, p \times q \rangle \\ &= \lambda \cdot 0 + \mu \cdot 0 \\ &= 0. \end{aligned}$$

The first two equations hold by multilinearity of the scalar product. The third equation comes from the fact that  $\langle p, p \times q \rangle$  and  $\langle q, p \times q \rangle$  are always zero.

Conversely, assume that  $[r]$  is a point on the line spanned by  $[p]$  and  $[q]$ . This means that there is a vector  $l \in \mathbb{R}^3$  with

$$\langle l, p \rangle = \langle l, q \rangle = \langle l, r \rangle = 0.$$

The points  $[p]$  and  $[q]$  are distinct, and thus  $p$  and  $q$  are linearly independent. Consider the matrix  $M$  with row vectors  $p, q, r$ . This matrix cannot have full rank, since the product  $M \cdot l$  is the zero vector. Thus  $r$  must lie in the span of  $p$  and  $q$ . Since  $r$  itself is not the zero vector, we have a representation of the form  $r = \lambda \cdot p + \mu \cdot q$  with  $\lambda$  or  $\mu$  nonzero.  $\square$

The last proof is simply an algebraic version of the geometric fact that we consider a line as the linear span of two distinct points on it. In the form  $r = \lambda \cdot p + \mu \cdot q$  we can simultaneously multiply both parameters  $\lambda$  and  $\mu$  by the same factor  $\alpha$  and still obtain the same point  $[r]$ . If one of the two parameters is nonzero we can normalize this parameter to 1. Using this fact we can express almost all points on the line through  $[p]$  and  $[q]$  by the expression  $\lambda \cdot p + q$ ;  $\lambda \in \mathbb{R}$ . The only point we miss is  $[p]$  itself. Similarly,

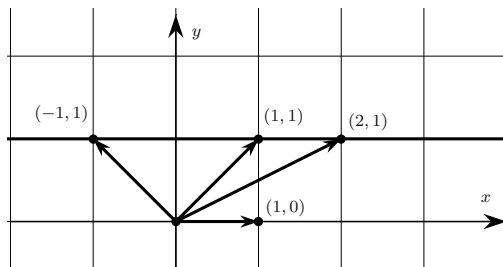
we obtain all points except  $[q]$  by the expression  $p + \mu \cdot q$ . Let us interpret these relations within the framework of concrete coordinates of points in the standard embedding of the Euclidean plane. For this we set  $o = (0, 0, 1)$  (the corresponding point  $[o]$  represents the origin of the coordinate system of  $\mathbb{R}^2$  embedded in the  $z = 1$  plane) and  $x_\infty = (1, 0, 0)$  (the corresponding point  $[x_\infty]$  represents the infinite point in the direction of the  $x$ -axis). The points represented by vectors  $\lambda \cdot x_\infty + o = (\lambda, 0, 1)$  are the finite points on the line joining  $[o]$  and  $[x_\infty]$  (this is the embedded  $x$ -axis). Each such point  $(\lambda, 0, 1)$  is bijectively associated to a real parameter  $\lambda \in \mathbb{R}$ .

It is important to notice that this way of assigning real numbers to points in the projective plane is heavily dependent on the choice of the reference points. It will be our next aim to reconstruct this relation of real parameters in a purely projective setup.

## 4.2 The Real Projective Line

The last section focused on viewing a single line in  $\mathcal{P}_{\mathbb{R}}$  from the projective viewpoint. In the expression  $\lambda \cdot p + \mu \cdot q$  the parameters  $(\lambda, \mu)$  themselves can be considered homogeneous coordinates on the one-dimensional projective line spanned by  $p$  and  $q$ . In this section we want to step back from our considerations of the projective plane and study the situation of a (self-contained) projective line. We will do this in analogy to the homogeneous setup for the projective plane. For the moment, we again restrict ourselves to the real case.

A real (Euclidean) line is a one-dimensional object that can be isomorphically associated to the real numbers  $\mathbb{R}$ . Each point on the line uniquely corresponds to exactly one real number. Increasing this real number more and more, we will move the corresponding point farther and farther out. Decreasing the parameter will move the point farther and farther out in the opposite direction. In a projective setup we will compactify this situation by adding *one* point at infinity on this line. If we increase or decrease the real



**Fig. 4.1** Homogeneous coordinates on the real projective line.

parameter, we will in the limit reach this unique infinite point. Algebraically we can model this process again by introducing homogeneous coordinates. A finite point with parameter  $\lambda$  on the line will be represented by a two-dimensional vector  $(\lambda, 1)$  (or any nonzero multiple of this vector). The unique infinite point corresponds to the vector  $(1, 0)$  (or any nonzero multiple of this vector). Formally we can describe this space as  $\frac{\mathbb{R}^2 - \{(0,0)\}}{\mathbb{R} - \{0\}}$ . Figure 4.1 above gives an impression of the situation. The original line is now embedded in the line  $y = 1$ . Each two-dimensional vector represents a one-dimensional subspace of  $\mathbb{R}^2$ . For finite points the intersection of this subspace with the line gives the corresponding point on the line. The infinite point is represented by any vector on the  $x$ -axis. (The reader should notice that this setup is completely analogous to the setup for the real projective plane that we described in Section 3.1.) Topologically a projective line has the shape of a circle—a one-dimensional road on which we return to the starting-point if we travel long enough in one direction. We call this space  $\mathbb{RP}^1$ . In the projective plane  $\mathbb{RP}^2$  we can consider any line as an isomorphic copy of  $\mathbb{RP}^1$ .

In analogy to the real projective plane we define a projective transformation by the multiplication of the homogeneous coordinate vector by a matrix. This time it must be a  $2 \times 2$  matrix:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we consider our points  $(\lambda, 1)$  represented by a real parameter  $\lambda$ , then matrix multiplication induces the following action on the parameter  $\lambda$ :

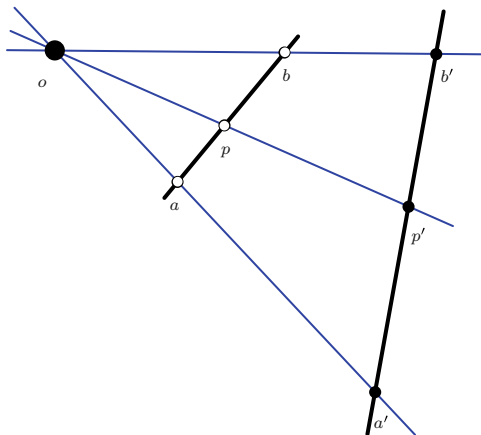
$$\lambda \mapsto \frac{a \cdot \lambda + b}{c \cdot \lambda + d}.$$

A point gets mapped to infinity if the denominator of the above ratio vanishes. An argument completely analogous to the proof of Theorem 3.4 proves the following result:

**Theorem 4.1.** *Let  $[a], [b], [c] \in \mathbb{RP}^1$  be three points no two of which are coincident and let  $[a'], [b'], [c'] \in \mathbb{RP}^1$  be another three points no two of which are coincident, then there exists a  $2 \times 2$  matrix  $M$  such that  $[M \cdot a] = [a']$ ,  $[M \cdot b] = [b']$ , and  $[M \cdot c] = [c']$ .*

In other words, three points and their images uniquely determine a projective transformation. The projective transformations arise in a natural way if we represent points on the line with respect to two different sets of reference vectors, as the following lemma shows.

**Lemma 4.2.** *Let  $\ell$  be the line spanned by two points  $[p]$  and  $[q]$  in  $\mathbb{RP}^2$ . Let  $[a]$  and  $[b]$  be two other distinct points on  $\ell$ . Consider the vector  $\lambda p + \mu q$  (which represents a point on  $\ell$ ). This vector can also be written as  $\alpha a + \beta b$  for certain  $\alpha, \beta$ . The parameters  $(\alpha, \beta)$  can be expressed in terms of  $(\lambda, \mu)$  by a linear transformation that depends only on  $a, b, p$ , and  $q$ .*



**Fig. 4.2** Projective scales under projections.

*Proof.* Theorem 4.1 ensures that the point represented by  $\lambda p + \mu q$  can also be expressed in the form  $\alpha a + \beta b$ . Since  $p$  is in the span of  $a$  and  $b$ , it can be written as  $p = \alpha_p a + \beta_p b$ . Similarly,  $q$  can be written as  $q = \alpha_q a + \beta_q b$ . Thus the expression  $\lambda p + \mu q$  can be written as  $\lambda(\alpha_p a + \beta_p b) + \mu(\alpha_q a + \beta_q b)$ . Thus we have

$$\alpha a + \beta b = (\alpha_p \lambda + \alpha_q \mu) a + (\beta_p \lambda + \beta_q \mu) b.$$

Since  $a$  and  $b$  are linearly independent, we have

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_p & \alpha_q \\ \beta_p & \beta_q \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.$$

□

Finally, we want to describe how perspectivities from one line to another induce projective maps on the coordinates of the lines. For this let  $\ell$  and  $\ell'$  be two lines and let  $o$  be a projection point not incident to either of them. Furthermore, assume that  $[a]$  and  $[b]$  are points on  $\ell$  and that  $[a']$  and  $[b']$  are the corresponding projected images in a projection through  $o$  from  $\ell$  to  $\ell'$ . The situation is illustrated in Figure 4.2. With these settings we obtain the following:

**Lemma 4.3.** *There exists a number  $\tau \in \mathbb{R}$  such that the image of a point  $\alpha a + \beta b$  under the projection is  $\alpha' a' + \beta' b'$ , with  $(\alpha', \beta') = (\alpha \tau, \beta)$ .*

*Proof.* One way to geometrically express the desired result is to say that the line  $(\alpha a + \beta b) \times (\alpha' a' + \beta' b')$  is incident to  $o$ . This happens if and only if the scalar product of  $o$  and this line is zero. In this case we have

$$\begin{aligned}
0 &= \langle (\alpha a + \beta b) \times (\alpha' a' + \beta' b'), o \rangle \\
&= \langle (\alpha a \times \alpha' a') + (\beta b \times \beta' b') + (\alpha a \times \beta' b') + (\beta b \times \alpha' a'), o \rangle \\
&= \alpha \alpha' \langle (a \times a'), o \rangle + \beta \beta' \langle (b \times b'), o \rangle + \alpha \beta' \langle (a \times b'), o \rangle + \beta \alpha' \langle (b \times a'), o \rangle \\
&= \alpha \beta' \langle (a \times b'), o \rangle + \beta \alpha' \langle (b \times a'), o \rangle.
\end{aligned}$$

The first and second equalities just expand the cross product by distributivity. The third equality holds since  $o$  is on  $a \times a'$  and  $o$  is on  $b \times b'$ . The last line being equal to zero can also be written as

$$\frac{\alpha}{\beta} \cdot \left( -\frac{\langle (a \times b'), o \rangle}{\langle (b \times a'), o \rangle} \right) = \frac{\alpha'}{\beta'}.$$

Setting  $\tau = -\frac{\langle (a \times b'), o \rangle}{\langle (b \times a'), o \rangle}$  gives the desired claim.  $\square$

### 4.3 Cross-Ratios (a First Encounter)

In the previous sections we have seen that many geometric magnitudes (among them seemingly natural magnitudes such as distances and ratios of distances) do not remain invariant under projective transformations.

*Cross-ratios* are the simplest magnitudes that are invariant under projective transformations. Cross-ratios will play an important role throughout each of the following chapters.

Before we introduce cross-ratios, we will set up a little notation that will help us to abbreviate many of the formulas we will have to consider from now on. If  $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  and  $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$  are two-dimensional vectors, we will use

$$[a, b] := \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

as an abbreviation for the determinant of the  $2 \times 2$  matrix formed by these two vectors. We will also use

$$[a, b, c] := \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

as an abbreviation of a  $3 \times 3$  determinant if  $a, b, c$  are three-dimensional vectors. The reader should be careful not to confuse these brackets with the notation we use for equivalence classes.

A cross-ratio is assigned to an ordered quadruple of points on a line. We first restrict ourselves to the case of calculating the cross-ratio for four points in  $\mathbb{RP}^1$ . Later on we will define the cross-ratio for four arbitrary points on a line in the projective plane  $\mathbb{RP}^2$ .

We first define the cross-ratio on the level of homogeneous coordinates and then prove that the cross-ratio actually depends only on the projective points represented by these coordinates.

**Definition 4.1.** Let  $a, b, c, d$  be four nonzero vectors in  $\mathbb{R}^2$ . The *cross-ratio*  $(a, b; c, d)$  is the following magnitude:

$$(a, b; c, d) := \frac{[a, c][b, d]}{[a, d][b, c]}.$$

We will now show that the value of the cross-ratio does not change under various transformations.

**Lemma 4.4.** For any real nonzero parameters  $\lambda_a, \lambda_b, \lambda_c, \lambda_d \in \mathbb{R}$  we have

$$(a, b; c, d) = (\lambda_a a, \lambda_b b; \lambda_c c, \lambda_d d).$$

*Proof.* Since  $[p, q]$  represents a determinant with columns  $p$  and  $q$ , we have  $[\lambda_p p, \lambda_q q] = \lambda_p \lambda_q [p, q]$ . Applying this to the definition of cross-ratios, we get

$$\frac{[\lambda_a a, \lambda_c c][\lambda_b b, \lambda_d d]}{[\lambda_a a, \lambda_d d][\lambda_b b, \lambda_c c]} = \frac{\lambda_a \lambda_b \lambda_c \lambda_d [a, c][b, d]}{\lambda_a \lambda_b \lambda_c \lambda_d [a, d][b, c]} = \frac{[a, c][b, d]}{[a, d][b, c]}.$$

Canceling all the  $\lambda$ 's is feasible, since they were assumed to be nonzero. The equality of the leftmost and the rightmost terms is exactly the claim.  $\square$

This lemma proves that it makes sense to speak of *the* cross-ratio

$$([a], [b]; [c], [d])$$

of four points in  $\mathbb{RP}^1$ , since the concrete choices of the representatives are irrelevant for the value of the cross-ratio. Cross-ratios are also invariant under projective transformations, as the following lemma shows:

**Lemma 4.5.** Let  $M$  be a  $2 \times 2$  matrix with nonvanishing determinant and let  $a, b, c, d$  be four vectors in  $\mathbb{R}^2$ . Then we have

$$(a, b; c, d) = (M \cdot a, M \cdot b; M \cdot c, M \cdot d).$$

*Proof.* We have  $[M \cdot p, M \cdot q] = \det(M) \cdot [p, q]$ . This gives:

$$\frac{[M \cdot a, M \cdot c][M \cdot b, M \cdot d]}{[M \cdot a, M \cdot d][M \cdot b, M \cdot c]} = \frac{\det(M)^2 [a, c][b, d]}{\det(M)^2 [a, d][b, c]} = \frac{[a, c][b, d]}{[a, d][b, c]}.$$

Canceling all determinants is feasible, since  $M$  was assumed to be invertible. The equality of the leftmost and the rightmost terms is exactly the claim.  $\square$

Taking the last two lemmas together highlights a remarkable robustness of the cross-ratio. Not only is it independent of the vectors representing the points. It is invariant even under projective transformations. This in turn has the consequence that if we have a line in  $\mathbb{RP}^2$  with two points  $p$  and  $q$  such that the points on the line are represented by  $\lambda p + \mu q$ , the cross-ratio of four points on this line can be calculated using the parameters  $(\lambda, \mu)$  as one-dimensional homogeneous coordinates. The value of this cross-ratio is well defined according to Lemma 4.2, Lemma 4.4, and Lemma 4.5.

Thus we have encountered our first genuine projective measure: the cross-ratio. From now on we will only very rarely have to distinguish between a point  $[p]$  in projective space and its representation in homogeneous coordinates. The reason is that whenever we want to link projective entities to measures we can do this via cross-ratios. If no confusion can arise we will from now on identify a point  $[p]$  with the homogeneous coordinate vector  $p$  representing it. Whenever we speak of the *point*  $p$  we mean the equivalence class  $[p]$ , and if we speak of the *vector*  $p$  we mean the element of  $\mathbb{R}^d$  representing it.

## 4.4 Elementary Properties of the Cross-Ratio

In this section we will collect a few elementary facts that are useful whenever one calculates with cross-ratios.

**Cross-ratios and the real numbers line:** Readers already familiar with cross-ratios may have noticed that our approach to cross-ratios is not the one taken most often by textbooks. Usually cross-ratios are introduced by expressions concerning the oriented distances of points on a line. For reference we will briefly also present this approach.

For this let  $\ell$  be any line and let  $a, b, c, d$  be four points on this line. We assume that  $\ell$  is equipped with an orientation (a preferred direction) and we denote by  $|a, b|$  the directed (Euclidean) distance from  $a$  to  $b$  (this means that  $|a, b| = -|b, a|$ ). If  $\ell$  represents the real number line, each point  $a$  corresponds to a number  $x_a \in \mathbb{R}$  and we can simply set  $|a, b| = x_b - x_a$ . Now the cross-ratio is usually defined as

$$(a, b; c, d) = \frac{|a, c|}{|a, d|} \bigg/ \frac{|b, c|}{|b, d|}.$$

(In German literature the cross-ratio is called *Doppelverhältnis*: a “ratio of ratios.”) It is easy to see that this definition agrees with our setup for all finite points  $a, b, c, d$ . We can introduce homogeneous coordinates  $\begin{pmatrix} a \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} b \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} c \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} d \\ 1 \end{pmatrix}$  for the points. The determinant then becomes

$$\det \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix} = a - b = -|a, b|.$$



An easy calculation shows the identity of both setups. Compared to this approach via oriented lengths the approach taken in the last section has the advantage that it also treats infinite points correctly. Sometimes the form above provides a nice shortcut for calculating the cross-ratio for finite points.

For some positions of the input values the cross-ratio becomes infinite. This happens whenever either  $a$  and  $d$  coincide or  $b$  and  $c$  coincide. It will later on turn out to be useful not to consider this as an unpleasant special case. We simply can consider the results of the cross-ratio themselves as points on a projectively closed line. The infinite value is then nothing but a representation of the infinite point. If we assume in this interpretation that three of the entries (say  $a$ ,  $b$ , and  $c$ ) are distinct and fixed, then the map

$$d \mapsto (a, b; c, d)$$

itself becomes a projective transformation. If one wants to calculate with infinite numbers, the following rules will be consistent with all operations throughout this book:

$$1/\infty = 0; \quad 1/0 = \infty; \quad 1 + \infty = \infty.$$

**Permutations of cross-ratios:** The cross-ratio is not independent of the order of the entries. However, if we know the cross-ratio  $(a, b; c, d) = \lambda$ , we can reconstruct the cross-ratio for any permutation of  $a, b, c$ , and  $d$ . We obtain the following theorem:

**Theorem 4.2.** *Let  $a, b, c, d$  be four points on a projective line with cross-ratio  $(a, b; c, d) = \lambda$ . Then we have*

- (i)  $(a, b; c, d) = (b, a; d, c) = (c, d; a, b) = (d, c; b, a),$
- (ii)  $(a, b; d, c) = 1/\lambda,$
- (iii)  $(a, c; b, d) = 1 - \lambda,$
- (iv) *the values for the remaining permutations are a consequence of these three rules.*

*Proof.* Statement (i) is clear from Definition 4.1 and the anticommutativity of the determinant. Statement (ii) is obvious from the definition, since it just exchanges numerator and denominator.

Statement (iii) requires a little elementary calculation. The expression we want to prove is

$$(a, c; b, d) = 1 - (a, b; c, d).$$

On the determinant level this reads

$$\frac{[a, b][c, d]}{[a, d][c, b]} = 1 - \frac{[a, c][b, d]}{[a, d][b, c]}.$$

Multiplying by  $[a, d][b, c]$ , this translates to

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0.$$

Since the cross-ratio is invariant under projective transformations, we may assume that all points are finite and we can represent them by numbers  $\lambda_a$ ,  $\lambda_b$ ,  $\lambda_c$ , and  $\lambda_d$ . The determinants then become differences, and our expression reads

$$(\lambda_a - \lambda_b)(\lambda_c - \lambda_d) - (\lambda_a - \lambda_c)(\lambda_b - \lambda_d) + (\lambda_a - \lambda_d)(\lambda_b - \lambda_c) = 0.$$

Expanding all terms, we get

$$\begin{aligned} & (\lambda_a \lambda_c + \lambda_b \lambda_d - \lambda_a \lambda_d - \lambda_b \lambda_c) \\ & - (\lambda_a \lambda_b + \lambda_c \lambda_d - \lambda_a \lambda_d - \lambda_c \lambda_b) \\ & + (\lambda_a \lambda_b + \lambda_d \lambda_c - \lambda_a \lambda_c - \lambda_d \lambda_b) = 0. \end{aligned}$$

This is obviously true, since all summands cancel.

Finally, it is obvious that we can generate all possible permutations of points by application of the three rules. The first rule allows us to bring any letter to the front position. The second and third equations describe two specific transpositions from which all permutations of the last three letters can be generated.  $\square$

*Remark 4.1.* If  $(a, b; c, d) = \lambda$ , the six values of the cross-ratio for permutations of these points are

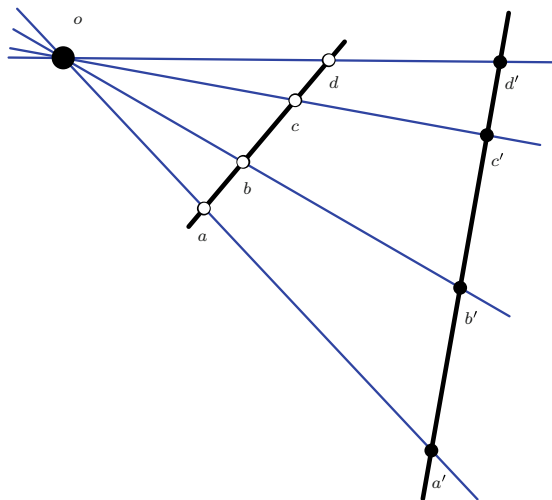
$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{1 - \lambda}, \quad \frac{1 - \lambda}{\lambda}.$$

In particular, these six functions form a group isomorphic to  $S_3$ .

**Cross-ratios and perspectivities:** We now want to demonstrate how cross-ratios are invariant under geometric projections. This is an immediate corollary of the fact that projections induce a projective transformation (Lemma 4.3) and the invariance of the cross-ratio under projective transformations:

**Corollary 4.1.** *Let  $o$  be a point and let  $\ell$  and  $\ell'$  be two lines not passing through  $o$ . If four points  $a, b, c, d$  on  $\ell$  are projected by the viewpoint  $o$  to four points  $a', b', c', d'$  on  $\ell'$ , then the cross-ratios satisfy  $(a, b; c, d) = (a', b'; c', d')$ .*

This corollary justifies another concept. We can assign to any quadruple of lines that pass through one point  $o$  a cross-ratio. We can assign this cross-ratio in the following way. We cut the four lines by an arbitrary line  $\ell$  (not through  $o$ ). The four points of intersection define a cross-ratio. The



**Fig. 4.3** Cross-ratios under projections. We have  $(a, b; c, d) = (a', b'; c', d')$ .

last corollary shows that the value of this cross-ratio is independent of the specific choice of  $\ell$ . Thus we can call it the *cross-ratio of the lines*. This fact is nothing but a consequence of the fact that in projective geometry every concept must have a reasonable dual. So, if one can assign a cross-ratio to four points on a line, one must also be able to assign a cross-ratio to four lines through a point.

**Cross-ratios in  $\mathbb{RP}^2$ :** Sometimes it is very inconvenient to calculate cross-ratios of four points on a line in the real projective plane  $\mathbb{RP}^2$  by first introducing a projective scale on the line. However, there is a possibility to calculate the cross-ratio much more directly using quotients of  $3 \times 3$  determinants.

**Lemma 4.6.** *Let  $a, b, c, d$  be four collinear points in the projective plane  $\mathbb{RP}^2$  and let  $o$  be a point not on this line. Then one can calculate the cross-ratio via*

$$(a, b; c, d) = \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]}.$$

*Proof.* Similar to the proof of Lemmas 4.4 and 4.5, it is easy to see that the value of this expression does not depend on the specific choice of the representing vectors, and that it is invariant under projective transformations. Hence we may assume without loss of generality that we have

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad o = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Under these assumptions the points  $c$  and  $d$  have coordinates

$$c = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ 0 \end{pmatrix}.$$

All determinants reduce to  $2 \times 2$  determinants, and the theorem follows immediately.  $\square$

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## Calculating with Points on Lines

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Newsgroups: sci.math
Subject: Re: I'm looking for axioms and proof in math texts
Date: 6 Aug 2003 02:53:12 -0700
From: euclid@softcom.net (prometheus666)
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```
[...]
and being passing familiar with the number line -- I'm sure if I don't
say passing familiar somebody here will say, "You have to know vector
tensor shmeliaculus in 15 triad synergies to really understand the
number line. It's not even called that, it's called the real
torticular space." or something to that effect --
[...]
```

*Found on the WWW*

In the previous section we have seen that cross-ratios form a universal link from projective geometry to the underlying coordinate field. In this chapter we want to elaborate more on this topic. We will see that one can recover the structure of the real numbers from the purely geometric setup of the real projective plane.

A line has (seemingly) far less geometric structure than the plane. In the plane there can be collinear and non collinear points, and we have seen that collinearity is a crucial property that is invariant under projective transformations. On a line all triples of points are automatically collinear, so we cannot hope for this to be a crucial characterizing property of projective transformations. The simplest nontrivial property that is invariant under a projective transformation on a line is the specific value of the cross-ratio of a quadruple of points. It will be the ultimate goal of this chapter to show that a map that fixes a certain value for the cross-ratio of four points automatically fixes all cross-ratios and will automatically be a projective transformation. From there we will prove Theorem 3.3, which claims that any bijective transformation in  $\mathbb{RP}^2$  that preserves collinearity is automatically a projective transformation.

## 5.1 Harmonic Points

We will now focus on the geometric properties of quadruples of points on a line that have a cross-ratio of  $(a, b; c, d) = -1$ . It is clear that if we fix points  $a, b, c$  on a line then this condition uniquely determines the position of the last point  $d$ . So, from a projective viewpoint there is essentially only one point set (up to projective transformations) that satisfies this condition. However, under the aspect of coordinates there are many interesting cases that can be considered.

**Definition 5.1.** Two pairs of points  $(a, b)$  and  $(c, d)$  are in *harmonic position* with respect to each other if the cross-ratio  $(a, b; c, d)$  is  $-1$ .

First we observe that the property of being in harmonic position is well defined. This means that it really is invariant under transposition of the points in the pairs.

**Lemma 5.1.** *If  $(a, b; c, d) = -1$  we also have  $(b, a; c, d) = (a, b; d, c) = (c, d; a, b) = -1$ .*

*Proof.* This lemma is an immediate consequence of Theorem 4.2 and the fact that  $1/(-1) = -1$ .  $\square$

In fact, if  $a \neq b$  and  $c \neq d$  and the cross-ratio  $(a, b; c, d)$  is invariant under interchange of the last two letters, the points must be in harmonic position. This is the case, since we then have

$$(a, b; c, d) = \frac{1}{(a, b; d, c)} = \frac{1}{(a, b; c, d)}.$$

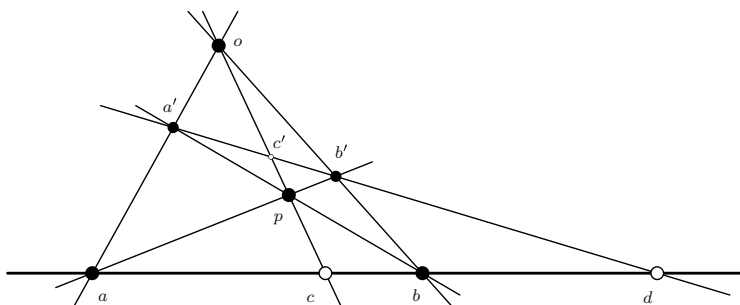
The only values for the cross-ratio that satisfy this equality are 1 and  $-1$ . Since  $a \neq b$  and  $c \neq d$ , this excludes the first case.

If  $a, b, c$  on a line  $\ell$  are given, the fourth harmonic point can be constructed in the following way:

- Start with an auxiliary point  $o$ ,
- connect this point to  $a, b$ , and  $c$ ,
- on  $\mathbf{join}(o, c)$  choose another auxiliary point  $p$ ,
- construct  $d$  by

$$d = \mathbf{meet}(\ell, \mathbf{join}(\mathbf{meet}(\mathbf{join}(o, a), \mathbf{join}(p, b)), \mathbf{meet}(\mathbf{join}(o, b), \mathbf{join}(p, a)))).$$

Figure 5.1 shows the construction. The construction is not feasible when  $a$  and  $b$  coincide.



**Fig. 5.1** Construction of harmonic points.

**Lemma 5.2.** *Independently of the choice of the auxiliary points  $o$  and  $p$ , the construction presented above will end up with the same point  $d$ . This point satisfies  $(a, b; c, d) = -1$ .*

*Proof.* For the labeling we refer to Figure 5.1. The point  $o$  can be considered the center of projection from  $\ell$  to the line spanned by  $b'$  and  $a'$ . Thus we get  $(a, b; c, d) = (a', b'; c', d)$ . Similarly,  $p$  can be considered a center of projection, which implies  $(a', b'; c', d) = (b, a; c, d)$ . Taking these two relations together, we obtain  $(a, b; c, d) = (b, a; c, d)$ , which implies  $(a, b; c, d) = -1$ . This also implies the independence of  $d$  of the specific choice of  $o$  and  $p$ .  $\square$

There are a few remarkable collections of relative positions of points that generate a harmonic position. The following collection of equations collects some of them. For the sake of simplicity we identify points on a line with the corresponding real numbers on  $\mathbb{R}$  (including the point  $\infty$  for the point at infinity).

**Lemma 5.3.** *Let  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  be arbitrary numbers. We have*

- (i)  $(-x, x; 0, \infty) = -1$ ,
- (ii)  $(0, 2x; x, \infty) = -1$ ,
- (iii)  $(x, y; \frac{x+y}{2}, \infty) = -1$ ,
- (iv)  $(-1, 1; x, 1/x) = -1$ ,
- (v)  $(-x, x; 1, x^2) = -1$ .

*Proof.* We start with a proof of property (v). We can simply check it via a calculation representing determinants as differences of numbers (as described in Section 4.4.1). We can calculate

$$\frac{(-x-1)(x-x^2)}{(-x-x^2)(x-1)} = \frac{x(-x-1)(1-x)}{x(-1-x)(x-1)} = -1.$$

Statement (iv) is a consequence of statement (v); it arises by dividing all entries by  $x$  (this is a projective transformation and does not change the cross-ratio). Statement (i) can be considered a limit case of (iv) if  $x \rightarrow 0$ . Statement (ii) arises from (i) by adding  $x$  to all entries (this is again a projective transformation). Statement (iii) arises from (i) also by scaling and shifting.  $\square$

## 5.2 Projective Scales

We now fix three distinct points on a line, and call them  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$ . Every point  $\mathbf{x}$  on the line can be associated with a unique cross-ratio  $(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})$ . This construction allows us to equip the line with a scale that behaves “as if” point  $\mathbf{0}$  were the origin,  $\mathbf{1}$  were a unit point, and point  $\infty$  were infinitely far away. In particular we have the following:

**Lemma 5.4.** *The following equations hold:*

- (i)  $(\mathbf{0}, \infty; \mathbf{0}, \mathbf{1}) = 0$ ,
- (ii)  $(\mathbf{0}, \infty; \mathbf{1}, \mathbf{1}) = 1$ ,
- (iii)  $(\mathbf{0}, \infty; \infty, \mathbf{1}) = \infty$ .

*Proof.* The lemma is a direct application of the definition of the cross-ratio.  $\square$

The next lemma shows that the cross-ratios with respect to these three points can be used to reconstruct the coordinates of a point from its geometric location. For this we set  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  to the positions 0, 1, and  $\infty$  on a real number line.

**Lemma 5.5.** *Assume that we have the specific homogeneous coordinates  $\mathbf{0} = (0, 1)$ ,  $\mathbf{1} = (1, 1)$ ,  $\infty = (1, 0)$ ,  $\mathbf{x} = (x, 1)$  for  $x \in \mathbb{R}$ . Then we have  $(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1}) = x$ .*

*Proof.* We can proof this fact by direct calculation:

$$(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1}) = \frac{\det \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}} = x.$$

$\square$



If we single out three distinct points on a line, we can always consider them a projective basis and refer all measurements to these three points. By this we can assign to every point on the line a real number. The resulting number is then invariant under projective transformations. In a way this is like reconstructing the original scenery from a perspectively distorted photograph if the original positions of three points are given.

## 5.3 From Geometry to Real Numbers

From 1850 to 1856 the mathematician Karl Georg Christian von Staudt (1798–1867) published a series of books under the title *Beiträge zur Geometrie der Lage*. In these books von Staudt develops a completely synthetic (this means there are no calculations) setup for projective geometry. In his setup for projective geometry he works on a similar axiomatic level as Euclid did for Euclidean geometry. One of the major achievements of von Staudt's work was to provide a method that starts with a purely projective setup and reconstructs an underlying algebraic structure. In particular, he was able to reconstruct the field structure of the underlying coordinate field from properties of geometric constructions.

We will not follow exactly these lines of thought here.<sup>1</sup> However, we will demonstrate how intimately the concepts of real numbers, cross-ratios, and projective transformations are interwoven. In particular, we will provide the promised proof of Theorem 3.3, which claims that whenever we have a bijective map in  $\mathbb{RP}^2$  that maps collinear points to collinear points, then it must be a projective transformation. We will even work a little harder and provide the proof for a similar fact on the projective line from which Theorem 3.3 will easily follow.

**Theorem 5.1.** *Let  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  be a bijective map with the property that harmonic quadruples of points are mapped to harmonic quadruples of points. Then  $\tau$  is a projective transformation.*

We subdivide the proof of this rather strong theorem into several smaller parts. A bijective map with the property that harmonic quadruples of points are mapped to harmonic quadruples of points will be called a *harmonic map*. Our strategy will be first to show that harmonic maps are very rigid objects. We will see that harmonic maps that preserve the base points of a projective scale will induce a field automorphism. For this we will first fix a projective scale  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  on  $\mathbb{RP}^1$ . We can identify each point  $\mathbf{x}$  of  $\mathbb{RP}^1 \setminus \{\infty\}$  by its real coordinate  $(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1}) \in \mathbb{R}$ . If  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  is any bijective map with  $\tau(\infty) = \infty$ , this induces a bijection  $f_\tau: \mathbb{R} \rightarrow \mathbb{R}$  according to

$$\tau(p) = q \iff f_\tau((\mathbf{0}, \infty; p, \mathbf{1})) = (\mathbf{0}, \infty; q, \mathbf{1}).$$

---

<sup>1</sup> In fact, our approach closely follows Blaschke's presentation in [6].

The behavior of  $f$  is mainly determined by the fact that  $\tau$  is harmonic:

**Lemma 5.6.** *Let  $\tau$  be a harmonic map that in addition satisfies  $\tau(\mathbf{0}) = \mathbf{0}$ ,  $\tau(\mathbf{1}) = \mathbf{1}$ ,  $\tau(\infty) = \infty$ . Then the function  $f_\tau: \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following relations:*

- (i)  $f_\tau(0) = 0$ ,
- (ii)  $f_\tau(1) = 1$ ,
- (iii)  $f_\tau\left(\frac{x+y}{2}\right) = \frac{f_\tau(x)+f_\tau(y)}{2}$ ,
- (iv)  $f_\tau(2x) = 2f_\tau(x)$ ,
- (v)  $f_\tau(x+y) = f_\tau(x) + f_\tau(y)$ ,
- (vi)  $f_\tau(-x) = -f_\tau(x)$ ,
- (vii)  $f_\tau(x^2) = f_\tau(x)^2$ ,
- (viii)  $f_\tau(x \cdot y) = f_\tau(x) \cdot f_\tau(y)$ .

*Proof.* We can prove the statements in order. In doing so, we make heavy use of the fact that if the positions of points  $a, b, c$  on a line are fixed, then the fourth harmonic point  $d$  is determined uniquely. Within this proof we will freely identify points on  $\mathbb{RP}^1$  and the corresponding real numbers with respect to the basis  $\mathbf{0}, \mathbf{1}, \infty$ .

Statements (i) and (ii) are direct reformulations of the facts that  $\tau(\mathbf{0}) = \mathbf{0}$  and  $\tau(\mathbf{1}) = \mathbf{1}$ . Statement (iii) holds for the following reason. For any  $x, y$  the pairs of points  $(x, y)$  and  $((x+y)/2, \infty)$  are in harmonic position (compare Lemma 5.3 (iii)). Thus if we apply a harmonic map the image pairs  $(f(x), f(y))$  and  $(f((x+y)/2), f(\infty))$  are harmonic again. Since we have  $f(\infty) = \infty$ , the point  $f((x+y)/2)$  must be  $\frac{f_\tau(x)+f_\tau(y)}{2}$ . Several of the other statements follow by similar reasoning. In these cases we refer only to the corresponding harmonic sets. Statement (iv) holds, since the pair  $(0, 2x)$  is harmonic with the pair  $(x, \infty)$ . Statement (v) is now a consequence of (iii) and (iv). Statement (vi) is a consequence of (v) and (i). Statement (vii) holds, since  $(-x, x; 1, x^2)$  is harmonic together with (ii) and (vi). Statement (viii) requires a little calculation. By (vii), (v), and (iv) we can conclude that

$$\begin{aligned}
 f_\tau(x)^2 + 2f_\tau(x \cdot y) + f_\tau(y)^2 &= f_\tau(x^2) + 2f_\tau(x \cdot y) + f_\tau(y^2) \\
 &= f_\tau(x^2 + 2xy + y^2) \\
 &= f_\tau((x+y)^2) \\
 &= (f_\tau(x+y))^2 \\
 &= f_\tau(x)^2 + 2f_\tau(x) \cdot f_\tau(y) + f_\tau(y)^2.
 \end{aligned}$$

Comparing the first and the last expressions in this chain of equalities, we obtain  $f_\tau(x \cdot y) = f_\tau(x) \cdot f_\tau(y)$ .  $\square$

The most important statements in the last theorem were the facts that we have  $f_\tau(x + y) = f_\tau(x) + f_\tau(y)$  and  $f_\tau(x \cdot y) = f_\tau(x) \cdot f_\tau(y)$ . In other words, the function  $f_\tau$  is a field automorphism of  $\mathbb{R}$ . However, the only field automorphism of  $\mathbb{R}$  is the identity, as the next lemma shows:

**Lemma 5.7.** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function that satisfies  $f(x+y) = f(x) + f(y)$  and  $f(x \cdot y) = f(x) \cdot f(y)$ . Then  $f$  is the identity.*

*Proof.* Since 0 is the only element in  $\mathbb{R}$  that satisfies  $x + x = x$  and we have  $f(0) + f(0) = f(0 + 0) = f(0)$ , we must have  $f(0) = 0$ . Similarly,  $f(1) \cdot f(1) = f(1 \cdot 1) = f(1)$  implies  $f(1) = 1$ . Now consider an integer  $n \in \mathbb{N}$ . We can write  $n = \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}$  and obtain

$$f(n) = f(\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}}) = \underbrace{f(1) + f(1) + \cdots + f(1)}_{n\text{-times}} = \underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} = n.$$

The fact that  $f(x \cdot y) = f(x) \cdot f(y)$  implies that for any rational number  $q = n/m$ ,  $n, m \in \mathbb{N}$ , we have  $f(q) = q$ , since  $q$  is the only number that satisfies the equation  $f(m) \cdot q = f(n)$  and we have  $f(n) = n$  and  $f(m) = m$ . For similar reasons we have  $f(-x) = -f(x)$  for any  $x \in \mathbb{R}$ .

Now the key observation is that one can characterize positivity in  $\mathbb{R}$  by multiplication. A number is positive if it can be written as  $x^2$  for  $x \neq 0$ . Thus from  $f(x^2) = f(x)^2$  we obtain that  $f$  maps positive numbers to positive numbers. This implies that  $x < y$  implies  $f(x) < f(y)$  (to see this, observe that  $x < y$  is equivalent to  $y - x > 0$ ).

If  $f$  were not the identity, then we could find a real number  $a$  such that  $f(a) \neq a$ . Then we would have either  $a < f(a)$  or  $a > f(a)$ . Consider the first case. Since the set of rational numbers is dense in  $\mathbb{R}$ , the open interval  $(a, f(a))$  contains a rational number  $q$ . This leads to a contradiction to the order-preservation property of  $f$ , since we then have  $a < q$ , but  $f(a) > q = f(q)$ . The case  $a > f(a)$  is analogous.  $\square$

Taking together Lemma 5.6 and Lemma 5.7, we can finally prove Theorem 5.1, the fact that harmonic maps can be expressed by matrix multiplication.

*Proof of Theorem 5.1.* Assume that  $\psi: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  is a harmonic map. The special points  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  are mapped by  $\psi$  to some points  $\mathbf{0}'$ ,  $\mathbf{1}'$ , and  $\infty'$ . Theorem 4.1 implies that there exists a projective transformation  $\phi: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  with  $\phi(\mathbf{0}') = \mathbf{0}$ ,  $\phi(\mathbf{1}') = \mathbf{1}$ ,  $\phi(\infty') = \infty$ . The composition  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  defined by  $\tau(x) = \phi(\psi(x))$  is again a harmonic map, which in addition leaves  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  invariant. By Lemma 5.6 it induces a field automorphism on  $\mathbb{R}$ , which (by Lemma 5.7) can only be the identity. Thus  $\tau$  itself must have been the identity map. This in turn implies that the original harmonic map  $\psi$  must be the inverse of the projective transformation  $\phi$ . Hence it is itself a projective transformation.  $\square$

## 5.4 The Fundamental Theorem

Theorem 5.1 is of fundamental importance. It links a structural geometric concept (harmonic point quadruples) to the algebraic structure of the underlying field: *On the level of coordinates every harmonic map can be expressed as a matrix multiplication.* Von Staudt originally went even one step further: He started with an incidence system that satisfies the axioms of a projective plane and required the additional presence of another incidence property (namely that Pappos's theorem holds within the incidence structure) and reconstructed an underlying field  $\mathbb{K}$  from this incidence structure. One can show that the original projective plane is isomorphic to  $\frac{\mathbb{K}^3 - \{(0,0,0)\}}{\mathbb{K} - \{0\}}$ . We will not prove this here. For a proof of this fact see for instance [44, 58, 3]. However, we will explain later the relevance of Pappos's theorem in this context: The presence of Pappos's theorem results in the *commutativity* of the underlying field.

Theorem 5.1 is a one-dimensional counterpart of Theorem 3.3, for which we still have not presented a proof. This theorem states that in  $\mathbb{RP}^2$  any bijective map that maps collinear points to collinear points can be expressed by a projective transformation. We will obtain this in a similar way as Theorem 5.1. However, we will have to find a way to encode two-dimensional positions in terms of cross-ratios. A *harmonic map* in  $\mathbb{RP}^2$  is any bijection that maps harmonic points on a line to harmonic points (perhaps on another line). A *collineation* in  $\mathbb{RP}^2$  will be any bijection that maps collinear points to collinear points. It is obvious that a harmonic map is a collineation (collinear points are mapped to collinear points). However, the opposite is also true:

**Lemma 5.8.** *Any collineation in  $\mathbb{RP}^2$  is a harmonic map.*

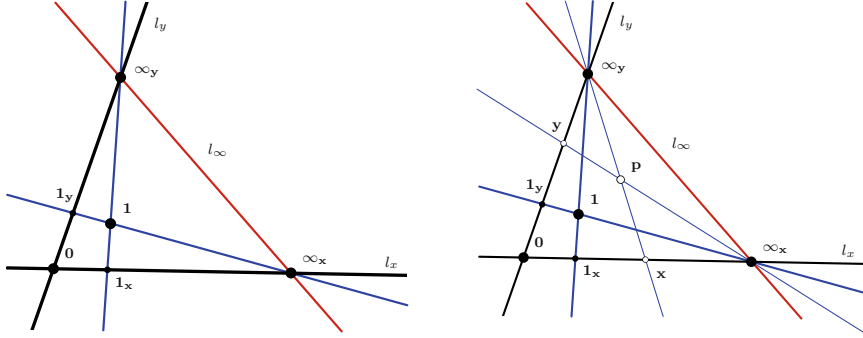
*Proof.* Consider four harmonic points  $a, b, c, d$  on some line  $l$ . By Lemma 5.2 there exist four auxiliary points not on  $l$  that form (together with  $a, b, c, d$ ) the incidence pattern of Figure 5.1. Under a collineation this incidence pattern is mapped to an incidence pattern with the same combinatorial structure. This implies that (again by Lemma 5.2) the image points are harmonic as well.  $\square$

We now use two values of cross-ratios in  $\mathbb{RP}^2$  as “coordinates” for two-dimensional points. For this we single out a projective basis in  $\mathbb{RP}^2$  consisting of four distinct points  $\mathbf{0}$ ,  $\infty_x$ ,  $\infty_y$ , and  $\mathbf{1}$ . These four points will play the role of the origin, an infinite point on the  $x$ -axis, an infinite point on the  $y$ -axis, and a point with coordinates  $(1, 1)$  respectively. Furthermore, we define

$$\mathbf{1}_x = \text{meet}(\text{join}(\mathbf{0}, \infty_x), \text{join}(\mathbf{1}, \infty_y))$$

and

$$\mathbf{1}_y = \text{meet}(\text{join}(\mathbf{0}, \infty_y), \text{join}(\mathbf{1}, \infty_x)).$$



**Fig. 5.2** Fixing a projective framework in  $\mathbb{RP}^2$ .

The triple  $(\mathbf{0}, \mathbf{1}_x, \infty_x)$  forms a projective basis on the line  $l_x$  spanned by  $\mathbf{0}$  and  $\infty_x$ , and a point  $\mathbf{x}$  on  $l_x$  is uniquely determined by the cross-ratio

$$x := (\mathbf{0}, \infty_x; \mathbf{x}, \mathbf{1}_x).$$

Similarly, the triple  $(\mathbf{0}, \mathbf{1}_y, \infty_y)$  forms a projective basis on the line  $l_y$  spanned by  $\mathbf{0}$  and  $\infty_y$ , and a point  $\mathbf{y}$  on  $l_y$  is uniquely determined by the cross-ratio

$$y := (\mathbf{0}, \infty_y; \mathbf{y}, \mathbf{1}_y).$$

Any point  $\mathbf{p}$  of  $\mathbb{RP}^2$  that does not lie on the line  $l_\infty := \mathbf{join}(\infty_x, \infty_y)$  defines uniquely two points

$$\mathbf{x} = \mathbf{meet}(l_x, \mathbf{join}(\mathbf{p}, \infty_y)) \quad \text{and} \quad \mathbf{y} = \mathbf{meet}(l_y, \mathbf{join}(\mathbf{p}, \infty_x)),$$

from which it can be reconstructed by means of

$$\mathbf{p} = \mathbf{meet}(\mathbf{join}(\mathbf{x}, \infty_y), \mathbf{join}(\mathbf{y}, \infty_x)).$$

In other words, the pair of numbers  $(x, y)$  with  $x := (\mathbf{0}, \infty_x; \mathbf{x}, \mathbf{1}_x)$  and  $y := (\mathbf{0}, \infty_y; \mathbf{y}, \mathbf{1}_y)$  uniquely determines the position of  $\mathbf{p}$ .

Now we are ready for the proof of Theorem 3.3: Every collineation is a projective transformation.

*Proof of Theorem 3.3* Assume that  $\psi: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  is a collineation. The special points  $\mathbf{0}, \mathbf{1}, \infty_x$ , and  $\infty_y$  are mapped by  $\psi$  to some points  $\mathbf{0}', \mathbf{1}', \infty_x'$ , and  $\infty_y'$ . Theorem 3.4 implies that there exists a projective transformation  $\phi: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  with  $\phi(\mathbf{0}') = \mathbf{0}$ ,  $\phi(\mathbf{1}') = \mathbf{1}$ ,  $\phi(\infty_x') = \infty_x$ ,  $\phi(\infty_y') = \infty_y$ . The composition  $\tau(x) = \phi(\psi(x))$  is again a collineation, which in addition leaves  $\mathbf{0}, \mathbf{1}, \infty_x$ , and  $\infty_y$  invariant.

By Lemma 5.6 it induces a field automorphism on  $\mathbb{R}$ , which (by Lemma 5.7) can only be the identity. Thus  $\tau$  itself must have been the identity map on the lines  $l_x$  and  $l_y$ . Our above considerations imply that all points of  $\mathbb{RP}^2$  that are not on  $l_\infty$  must be invariant. As a consequence, all points on  $l_\infty$  must be invariant as well (any such point can be encoded as the intersection of  $l_\infty$  and a line through two points not on  $l_\infty$ ). Thus  $\tau$  is the identity. This in turn implies that the original harmonic map  $\psi$  must be the inverse of the projective transformation  $\phi$ . Hence it is itself a projective transformation.  $\square$

## 5.5 A Note on Other Fields

The last three sections established a close connection of the field that underlies a projective geometry and projective transformations. In our approach we entirely focused on the *real* projective plane. Almost all our constructions (such as projective transformations, cross-ratios, harmonic points, collineations) could as well be carried out over arbitrary fields. However, with respect to the fundamental theorem a little care is necessary.

One crucial ingredient for Theorem 5.1 and Theorem 3.3 was to derive a field automorphism from a harmonic map. The fact that  $\mathbb{R}$  has only have the trivial field automorphism translates into the statement that *every* collineation can be expressed as a matrix multiplication. Over other fields (such as  $\mathbb{C}$ ) we have nontrivial field automorphisms. These field automorphisms induce collineations that cannot be expressed as a projective transformation. Let us focus for a moment on the *complex projective plane*  $\mathbb{CP}^2$ . In complete analogy to  $\mathbb{RP}^2$  we can define this space by  $\mathcal{P}_{\mathbb{C}} = \frac{\mathbb{C}^2 - \{(0,0,0)\}}{\mathbb{C} - \{0\}}$  and  $\mathcal{L}_{\mathbb{C}} = \frac{\mathbb{C}^2 - \{(0,0,0)\}}{\mathbb{C} - \{0\}}$ . A point  $(x, y, z)$  is on a line  $(a, b, c)$  if  $a \cdot x + b \cdot y + c \cdot z = 0$ . Matrix multiplications are again the projective transformations and induce collineations. However, over  $\mathbb{C}$  we have additional collineations. We obtain the simplest such collineation by taking the map that assigns to each coordinate entry its complex conjugate:

$$(x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z}) \quad \text{and} \quad (a, b, c) \mapsto (\bar{a}, \bar{b}, \bar{c}).$$

We have  $a \cdot x + b \cdot y + c \cdot z = 0 \iff \bar{a} \cdot \bar{x} + \bar{b} \cdot \bar{y} + \bar{c} \cdot \bar{z} = 0$ .

The generalization of the fundamental theorem that applies to all fields can be stated as follows.

**Theorem 5.2.** *Let  $\tau: \mathbb{KP}^1 \rightarrow \mathbb{KP}^1$  be a collineation. Then  $\tau$  can be factored as  $\phi \circ \psi$ , where  $\phi$  is a projective transformation and  $\psi$  is induced by a field automorphism.*

## 5.6 Von Staudt's Original Constructions

In Section 5.3 we introduced a way to mimic multiplication and addition by geometric constructions. Since we wanted to stay within  $\mathbb{RP}^1$ , our basic primitive was harmonic point quadruples rather than collinearity. The constructions performed in Lemma 5.6 can also be turned into geometric constructions for addition and multiplication using just join and meet operations. For this every harmonic quadruple of points is forced to be harmonic using the construction of Figure 5.1. From an “economic” point of view this way of encoding geometric addition and multiplication is far too complicated. Each harmonic set used in the construction requires four additional points. There is a much more direct way to encode geometric multiplication and geometric addition by a few construction steps only. These were the constructions originally given by von Staudt. They require only four additional points for addition or multiplication. Since these constructions are nice little gadgets that are useful for many geometric constructions, we will present them here.

As a “warmup” we start with a construction that from  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  (which are assumed to be given on a line  $\ell$ ), constructs the points corresponding to the integers.

We start with one more arbitrary point  $p$  not on  $\ell$  and a point  $q$  on the line joining  $\mathbf{0}$  and  $p$ . We construct a point  $r$  by

$$r = \text{meet}(\text{join}(p, \infty), \text{join}(q, \mathbf{1})).$$

Then we construct a point  $q_1$  according to

$$q_1 = \text{meet}(\text{join}(q, \infty), \text{join}(p, \mathbf{1}))$$

and a point  $\mathbf{2}$  according to

$$\mathbf{2} = \text{meet}(\ell, \text{join}(r, q_1)).$$

For the special choice of coordinates  $\mathbf{0} = (0, 0, 1)$ ,  $\mathbf{1} = (1, 0, 1)$ , and  $\infty = (1, 0, 0)$ , Figure 5.3 shows the corresponding situation (in the usual

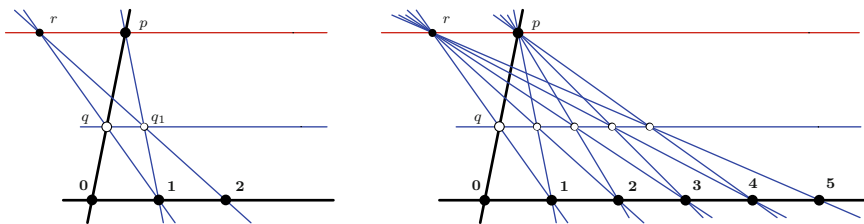
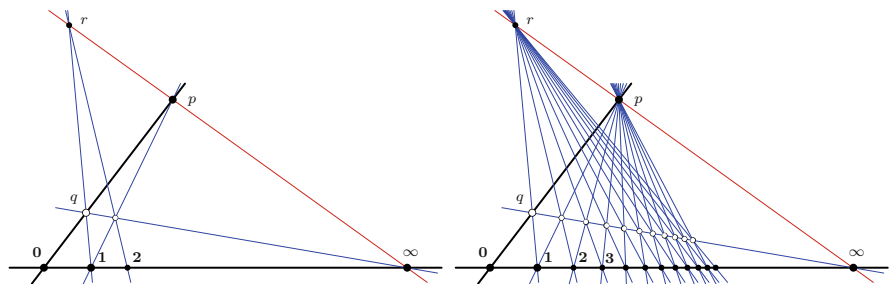


Fig. 5.3 Constructing the integers.



**Fig. 5.4** Constructing the integers projectively.

view of the Euclidean plane). Since  $\infty$  is a point at infinity, the three lines  $\ell$ ,  $\text{join}(q, \infty)$ , and  $\text{join}(p, \infty)$  are parallel in the picture. Thus the (oriented) segment lengths  $|0, 1|$ ,  $|1, 2|$ , and  $|q, q_1|$  are related by

$$\frac{|0, 1|}{|q, q_1|} = \frac{|1, 2|}{|q, q_1|}.$$

Multiplying both sides by  $|q, q_1|$  we obtain

$$|0, 1| = |1, 2|.$$

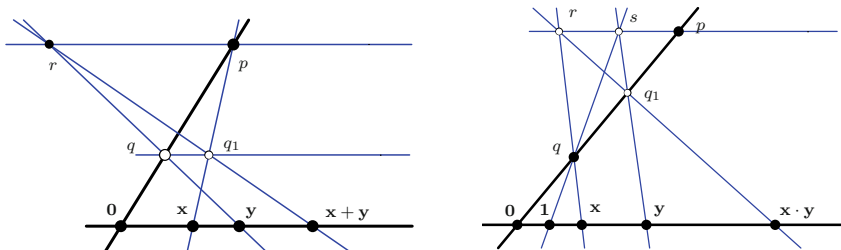
Hence, the location of **2** is independent of the choice of  $p$  and  $q$ . It is as far from **1** as **1** is from **0**. Hence it must have homogeneous coordinates  $(2, 0, 1)$ . Iterative application of this construction can be used to construct a sequence of points with homogeneous coordinates  $(n, 0, 1)$ ,  $n \in \mathbb{N}$ . Figure 5.4 shows the result of the construction if  $\infty$  is chosen to be a finite point. Still, our considerations imply that the positions of **0**, **1**, and  $\infty$  determine the positions of **2**, **3**, **4**,  $\dots$  uniquely, independently of the choice of  $p$  and  $q$ .

As a next aim we will model the operations of *addition* and *multiplication* by geometric constructions. For this we again fix points **0**, **1**, and  $\infty$  and choose points  $p$  and  $q$  as before. The constructions for addition and multiplication are given in Figure 5.5.

We assume that on the line  $\ell$  two additional points  $\mathbf{x}$  and  $\mathbf{y}$  are given with homogeneous coordinates  $(x, 0, 1)$  and  $(y, 0, 1)$ , respectively. We demonstrate how to construct the points  $\mathbf{x} + \mathbf{y}$  and  $\mathbf{x} \cdot \mathbf{y}$  with homogeneous coordinates  $(x + y, 0, 1)$  and  $(x \cdot y, 0, 1)$ , respectively. The constructions can be read off in a straightforward way from the pictures. Using the parallel lines in each picture and considering length ratios we obtain for the first picture the relation

$$\frac{|0, \mathbf{x}|}{|q, q_1|} = \frac{|\mathbf{y}, \mathbf{x} + \mathbf{y}|}{|q, q_1|}.$$





**Fig. 5.5** Geometric addition and multiplication.

Thus the distance from  $0$  to  $x$  is the same as the distance from  $y$  to  $x + y$ . This implies the desired relation for addition. For the picture on the right we obtain the relations

$$\frac{|0, 1|}{|0, x|} = \frac{|p, s|}{|r, p|} = \frac{|0, y|}{|0, x \cdot y|}.$$

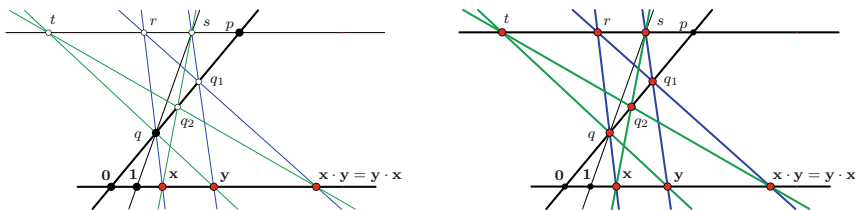
The first relation comes from the relations induced by the lines passing through  $q$ , the second comes from the lines passing through  $q_1$ ; Comparing the first and the third terms gives the desired relation.

If we alter the role of the points, the same configurations can be used to perform geometric subtraction or geometric division. By combining several von Staudt constructions, the evaluation of arbitrary quotients of polynomials can be modeled geometrically.

## 5.7 Pappos's Theorem

As a final topic of this chapter, we want to demonstrate the important role that is played by Pappos's theorem in relation to underlying fields of a projective plane. We will see that Pappos's theorem is equivalent to the *commutativity* of the underlying field. To see this, we will have a closer look at the von Staudt constructions for addition and multiplication.

If we interchange the roles of  $x$  and  $y$  in the von Staudt addition (see Figure 5.5 left) we end up with a drawing that has exactly the same combinatorics. This implies the commutativity of *addition*. This is not the case for von Staudt *multiplication*. Figure 5.6 overlays different constructions for the point  $x \cdot y$ . The blue lines are exactly the same as presented as “von Staudt multiplication” in the last section. The green construction corresponds to the “von Staudt multiplication” for  $y \cdot x$  (the roles of  $x$  and  $y$  are interchanged). The black lines belong to both constructions. Since over real numbers the multiplication is commutative and we have  $x \cdot y = y \cdot x$ , both constructions



**Fig. 5.6** Commutativity of multiplication and Pappos's theorem.

must result in the same point. This fact can be interpreted as a purely geometric incidence theorem (without ever referring to algebra).

In fact, the decisive incidence theorem that hides behind commutativity of multiplication is nothing but Pappos's theorem (recall the first chapter of this book). In Figure 5.6 the nine lines that correspond to Pappos's theorem are drawn in bold, and the nine points are emphasized in red. If we look back at the collection of proofs for Pappos's theorem that we presented in the first chapter of this book, then we will recognize that each of these proofs at one point made use of commutativity of the underlying coordinate field. The reason for this is that the presence of Pappos's theorem in a projective plane implies that the plane can be considered as  $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathbf{I}_{\mathbb{K}})$  for some (commutative) field  $\mathbb{K}$ . Projective planes in which Pappos's theorem holds are called Pappian planes. With this terminology we can reformulate the above fact in the following manner:

**Theorem 5.3.** *A projective plane is Pappian if and only if it is of the form  $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathbf{I}_{\mathbb{K}})$  for some field  $\mathbb{K}$ .*

We do not prove this theorem here (for proofs see [44, 58, 3]), since the main focus of this book is not the distinction of projective planes that come from fields from those that do not come from fields. All projective planes used hereinafter will come from a field, and hence we can (and will) freely make use of Pappos's theorem.

# Determinants

---

*One person's constant is another person's variable.*

*Susan Gerhart*

While the previous chapters had their focus on the exploration of the logical and structural properties of projective planes, this chapter will focus on the following question: *What is the easiest way to calculate with geometry?* Many textbooks on geometry introduce coordinates (or homogeneous coordinates) and base all analytic computations on calculations on this level. An analytic proof of a geometric theorem is carried out in the parameter space. For a different parameterization of the theorem the proof may look entirely different.

In this chapter we will see that this way of thinking is very often not the most economical one. The reason for this is that the coordinates of a geometric object are in a way a basis-dependent artifact and carry not only information on the geometric object but also on the relation of this object to the basis of the coordinate system. For instance, if a point is represented by its homogeneous coordinates  $(x, y, z)$ , we have encoded its relative position to a frame of reference. From the perspective of projective geometry the perhaps most important fact that one can say about the point is simply that it *is a point*. All other properties are not projectively invariant. Similarly, if we consider three points  $p_1 = (x_1, y_1, z_1)$ ,  $p_2 = (x_2, y_2, z_2)$ ,  $p_3 = (x_3, y_3, z_3)$ , the statement that these points are collinear reads

$$x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1 = 0,$$

a  $3 \times 3$  determinant. Again from a structural point of view this expression is far too complicated. It would be much better to encode the collinearity directly into a short algebraic expression and deal with this. The simplest way to

do this is to change the role of primary and derived algebraic objects. If we consider the *determinants* themselves as “first-class citizens,” the statement of collinearity simply reads  $\det(p_1, p_2, p_3) = 0$ , where the determinant is considered an unbreakable unit rather than just a shorthand for the above expanded formula. In this chapter we will explore the roles determinants play within projective geometry. For further reading on this fascinating topic we recommend [16, 33, 126, 132].

## 6.1 A “Determinantal” Point of View

Before we start with the treatment of determinants on a more general level we will review and emphasize the role of determinants in topics we have treated so far.

One of our first encounters of determinants occurred when we expressed the collinearity of points in homogeneous coordinates. Three points  $p_1, p_2, p_3$  are collinear in  $\mathbb{RP}^2$  if and only if  $\det(p_1, p_2, p_3) = 0$ . One can interpret this fact either geometrically (if  $p_1, p_2, p_3$  are collinear, then the corresponding vectors of homogeneous coordinates are coplanar) or algebraically (if  $p_1, p_2, p_3$  are collinear, then the system of linear equations  $\langle p_1, l \rangle = \langle p_2, l \rangle = \langle p_3, l \rangle = 0$  has a nontrivial solution  $l \neq (0, 0, 0)$ ). Dually, we can say that the determinant of three lines  $l_1, l_2, l_3$  vanishes if and only if these lines have a point in common (this point may be at infinity).

A second instance in which determinants played a less obvious role occurred when we calculated the join of two points  $p$  and  $q$  by the cross product  $p \times q$ . We will give an algebraic interpretation of this. If  $(x, y, z)$  is *any* point on the line  $p \vee q$ , then it satisfies

$$\det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ x & y & z \end{pmatrix} = 0.$$

If we develop the determinant by the last row, we can rewrite this as

$$\det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix} \cdot x - \det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix} \cdot y + \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \cdot z = 0.$$

Or expressed as a scalar product,

$$\left\langle \left( \det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix}, -\det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix}, \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right), (x, y, z) \right\rangle = 0.$$

We can geometrically reinterpret this equation by saying that

$$\left( \det \begin{pmatrix} p_2 & p_3 \\ q_2 & q_3 \end{pmatrix}, -\det \begin{pmatrix} p_1 & p_3 \\ q_1 & q_3 \end{pmatrix}, \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right)$$

must be the homogeneous coordinates of the line  $l$  through  $p$  and  $q$ , since every vector  $(x, y, z)$  on this line satisfies  $\langle l, (x, y, z) \rangle = 0$ . This vector is nothing, but the cross product  $p \times q$ .

A third situation in which determinants played a fundamental role was in the definition of cross-ratios. Cross-ratios were defined as the product of two determinants divided by the product of two other determinants.

We will see later on that all three circumstances described here will have nice and interesting generalizations:

- In projective  $d$ -space coplanarity will be expressed as the vanishing of a  $(d+1) \times (d+1)$  determinant.
- In projective  $d$ -space joins and meets will be nicely expressed as vectors of sub-determinants.
- Projective invariants can be expressed as certain rational functions of determinants.

## 6.2 A Few Useful Formulas

We will now see how we can translate geometric constructions into expressions that involve only determinants and base points of the construction. Since from now on we will have to deal with many determinants at the same time, we first introduce a useful abbreviation. For three points  $p, q, r \in \mathbb{RP}^2$  we set

$$[p, q, r] := \det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}.$$

Similarly, we set for two points in  $\mathbb{RP}^1$ ,

$$[p, q] := \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}.$$

We call an expression of the form  $[\dots]$  a *bracket*. Here are a few fundamental and useful properties of  $3 \times 3$  determinants:

Alternating sign changes:

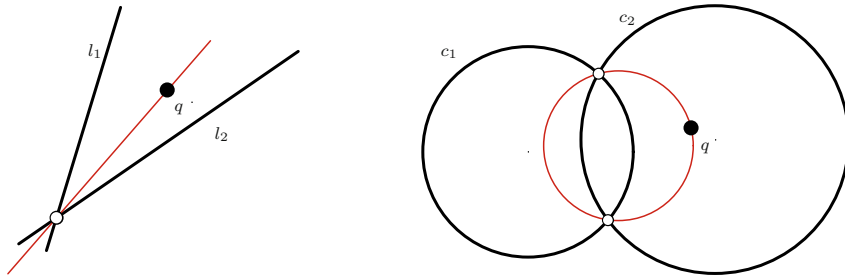
$$[p, q, r] = [q, r, p] = [r, p, q] = -[p, r, q] = -[r, q, p] = -[q, p, r].$$

Linearity (in every row and column):

$$[\lambda \cdot p_1 + \mu \cdot p_2, q, r] = \lambda \cdot [p_1, q, r] + \mu \cdot [p_2, q, r].$$

Plücker's formula:

$$[p, q, r] = \langle p, q \times r \rangle.$$



**Fig. 6.1** Two applications of Plücker's  $\mu$ .

The last formula can be considered a shorthand for our developments on cross products, scalar products, and determinants in the previous section.

### 6.3 Plücker's $\mu$

We now introduce a very useful trick with which one can derive formulas for geometric objects that should simultaneously satisfy several constraints. The trick was frequently used by Plücker and is sometimes called *Plücker's  $\mu$* .

Imagine you have an equation  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  whose zero set describes a geometric object. For instance, think of a line equation  $(x, y, z) \mapsto a \cdot x + b \cdot y + c \cdot z$  or a circle equation in the plane  $(x, y) \mapsto (x - a)^2 + (y - b)^2 - r^2$ . Often one is interested in objects that share intersection points with a given reference object and in addition pass through a third object. If the linear combination  $\lambda \cdot f(p) + \mu \cdot g(p)$  again describes an object of the same type, then one can apply Plücker's  $\mu$ . All objects described by

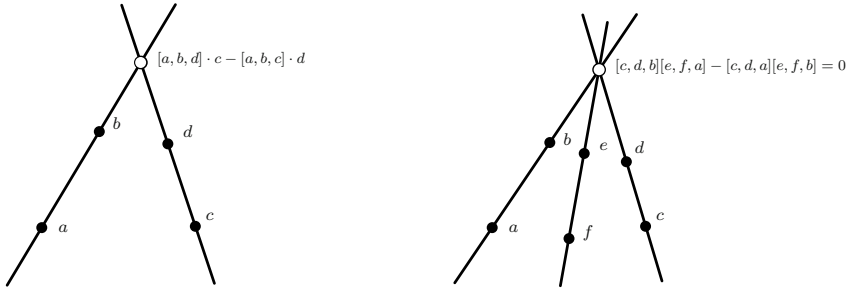
$$p \mapsto \lambda \cdot f(p) + \mu \cdot g(p)$$

will pass through the common zeros of  $f$  and  $g$ . This is obvious, since whenever  $f(p) = 0$  and  $g(p) = 0$ , any linear combination is also 0. If one in addition wants to have the object pass through a specific point  $q$ , then the linear combination

$$p \mapsto g(q) \cdot f(p) - f(q) \cdot g(p)$$

is the desired equation. To see this, simply plug the point  $q$  into the equation. Then one gets  $g(q) \cdot f(q) - f(q) \cdot g(q) = 0$ .

With this trick we can very easily describe the homogeneous coordinates of a line  $\ell$  that passes through the intersection of two other lines  $l_1$  and  $l_2$  and through a third point  $q$  by



**Fig. 6.2** Conditions for lines meeting in a point.

$$\langle l_2, q \rangle l_1 - \langle l_1, q \rangle l_2.$$

Testing whether this line passes through  $q$  yields

$$\langle \langle l_2, q \rangle l_1 - \langle l_1, q \rangle l_2, q \rangle = \langle l_2, q \rangle \langle l_1, q \rangle - \langle l_1, q \rangle \langle l_2, q \rangle = 0,$$

which is obviously true. Later on we will make frequent use of this trick whenever we need a fast and elegant way to calculate a specific geometric object. We will now use Plücker's  $\mu$  to calculate intersections of lines spanned by points.

What is the intersection of the two lines spanned by the point pairs  $(a, b)$  and  $(c, d)$ ? On the one hand, the point has to be on the line  $a \vee b$ ; thus it must be of the form  $\lambda \cdot a + \mu \cdot b$ . It also has to be on  $c \vee d$ ; hence it must be of the form  $\psi \cdot c + \phi \cdot d$ . Identifying these two expressions and solving for  $\lambda, \mu, \psi$ , and  $\phi$  would be one possibility to solve the problem. But we can directly read off the right parameters using (a dual version of) Plücker's  $\mu$ . The property that encodes that a point  $p$  is on the line  $c \vee d$  is simply  $[c, d, p] = 0$ . Thus we immediately obtain that the point

$$[c, d, b] \cdot a - [c, d, a] \cdot b$$

must be the desired intersection. This point is obviously on  $a \vee b$ , and it is on  $c \vee d$ , since we have

$$[c, d, [c, d, b] \cdot a - [c, d, a] \cdot b] = [c, d, b] \cdot [c, d, a] - [c, d, a] \cdot [c, d, b] = 0.$$

We could equivalently have applied the calculation with the roles of  $a \vee b$  and  $c \vee d$  interchanged. Then we can express the same point as

$$[a, b, d] \cdot c - [a, b, c] \cdot d.$$

In fact, it is not a surprise that these two expressions end up at identical points. We will later on, in Section 6.5, see that this is just a reformulation of the well-known Cramer's rule for solving systems of linear equations.

How can we express the condition that three lines  $a \vee b$ ,  $c \vee d$ ,  $e \vee f$  meet in a point? For this we simply have to test whether the intersection  $p$  of  $a \vee b$  and  $c \vee d$ , is on  $e \vee f$ . We can do this by testing whether the determinant of these three points is zero. Plugging in the formula for  $p$ , we get

$$[e, f, [c, d, b] \cdot a - [c, d, a] \cdot b] = 0.$$

After expansion by multilinearity, we obtain

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0.$$

This is the algebraic condition for the three lines meeting in a point. Taking a look at the above formula, we should pause and make a few observations:

- The first and most important observation is that we could write such a projective condition as a polynomial of determinants evaluating to zero.
- In the formula, each term has the same number of determinants.
- Each letter occurs equally often in each term.

All three observations extend very well to much more general cases. In order to see this, we will have first to introduce the notion of *projectively invariant properties*.

Before we do this we want to use this formula to obtain (another) beautiful proof for Pappos's theorem. Consider the drawing of Pappos's theorem in Figure 6.3 (observe the nice 3-fold symmetry). We can state Pappos's theorem in the following way: If for six points  $a, \dots, f$  in the projective plane the lines  $a \vee d$ ,  $c \vee b$ ,  $e \vee f$  meet and the lines  $c \vee f$ ,  $e \vee d$ ,  $a \vee b$  meet, then also  $e \vee b$ ,  $a \vee f$ ,  $c \vee d$  meet. The two hypotheses can be expressed as

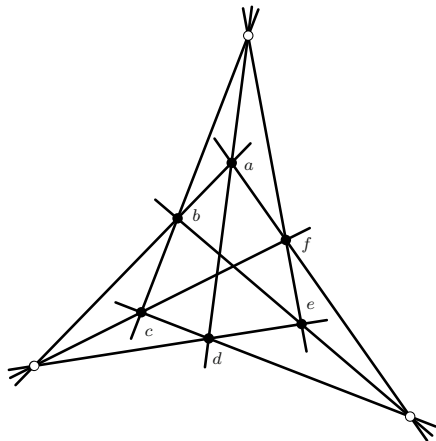
$$\begin{aligned} [b, c, e][a, d, f] &= [b, c, f][a, d, e], \\ [c, f, b][d, e, a] &= [c, f, a][d, e, b]. \end{aligned}$$

Using the fact that a cyclic shift of the points of a  $3 \times 3$  bracket does not change its sign, we observe that the first term of the second equation is identical to the second term of the first equation. So we obtain

$$[f, a, c][e, b, d] = [f, a, d][e, b, c]$$

Which is exactly the desired conclusion of Pappos's theorem.





**Fig. 6.3** Pappos's theorem, once more.

## 6.4 Invariant Properties

How can we algebraically characterize that a certain property of a point configuration is invariant under projective transformations? Properties of such type are for instance *three lines being concurrent* or six points  $a, \dots, f$  such that  $a \vee b$ ,  $c \vee d$ ,  $e \vee f$  meet.

In general, properties of this type can be expressed as functions in the (homogeneous) coordinates of the points that have to be zero when the property holds. Being invariant means that a property holds for a point configuration  $P$  if and only if it also holds for any projective transformation of  $P$ . More formally, let us express the point configuration  $P$  by a matrix whose columns are the homogeneous coordinates of  $n$  points  $p_1, \dots, p_n$ :

$$P = \begin{pmatrix} p_{1x} & p_{2x} & \dots & p_{nx} \\ p_{1y} & p_{2y} & \dots & p_{ny} \\ p_{1z} & p_{2z} & \dots & p_{nz} \end{pmatrix}.$$

A projective transformation is then simply represented by left-multiplication by a  $3 \times 3$  invertible matrix  $T$ . A projectively invariant property should also be invariant when we replace a vector  $p_i$  by a scalar multiple  $\lambda_i \cdot p_i$ . We can express the scaling of the points by right multiplication of  $P$  by an invertible diagonal matrix  $D$ . All matrices obtained from  $P$  via

$$T \cdot P \cdot D$$

represent essentially the same projective configuration. A projectively invariant property is any property of  $P$  that is invariant under such a transformation.

Very often, our invariant properties will be polynomials being zero, but for now we want to keep things more general and consider any map that associates to  $P$  a Boolean value. The matrix  $P$  can be considered an element of  $\mathbb{R}^{3 \cdot n}$ . Thus we make the following definition:

**Definition 6.1.** A *projectively invariant property* of  $n$  points in the real projective plane is a map  $f: \mathbb{R}^{3 \cdot n} \rightarrow \{\mathbf{true}, \mathbf{false}\}$  such that for all invertible real  $3 \times 3$  matrices  $T \in \mathrm{GL}(\mathbb{R}, 3)$  and  $n \times n$  invertible real diagonal matrices  $D \in \mathrm{diag}(\mathbb{R}, n)$ , we have

$$f(P) = f(T \cdot P \cdot D).$$

In a canonical way we can identify each predicate  $P \subseteq X$  on a space  $X$  with its characteristic function  $f: X \rightarrow \{\mathbf{true}, \mathbf{false}\}$ , where  $f(x)$  evaluates to **true** if and only if  $x \in P$ . Thus we can equivalently speak of projectively invariant predicates.

In this sense, for instance,  $[a, b, c] = 0$  defines a projectively invariant property of three points  $a, b, c$  in the real projective plane. Also the property that we encountered in the last section,

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0,$$

which encodes the fact that three lines  $a \vee b$ ,  $c \vee d$ ,  $e \vee f$  meet in a point, is projectively invariant. Before we state a more general theorem we will analyze why this relation is invariant from an algebraic point of view. Transforming the points by a projective transformation  $T$  results in replacing the points  $a, \dots, f$  with  $T \cdot a, \dots, T \cdot f$ . Scaling the homogeneous coordinates results in replacing  $a, \dots, f$  by  $\lambda_a \cdot a, \dots, \lambda_f \cdot f$  with nonzero  $\lambda$ 's. Thus if  $P$  encodes the point configuration, then the overall effect of  $T \cdot P \cdot D$  on the expression  $[c, d, b][e, f, a] - [c, d, a][e, f, b]$  can be written as

$$\begin{aligned} & [\lambda_c \cdot T \cdot c, \lambda_d \cdot T \cdot d, \lambda_b \cdot T \cdot b][\lambda_e \cdot T \cdot e, \lambda_f \cdot T \cdot f, \lambda_a \cdot T \cdot a] \\ & - [\lambda_c \cdot T \cdot c, \lambda_d \cdot T \cdot d, \lambda_a \cdot T \cdot a][\lambda_e \cdot T \cdot e, \lambda_f \cdot T \cdot f, \lambda_b \cdot T \cdot b]. \end{aligned}$$

Since  $[T \cdot p, T \cdot q, T \cdot r] = \det(T) \cdot [p, q, r]$ , the above expression simplifies to

$$(\det(T)^2 \cdot \lambda_a \cdot \lambda_b \cdot \lambda_c \cdot \lambda_d \cdot \lambda_e \cdot \lambda_f) \cdot ([c, d, b][e, f, a] - [c, d, a][e, f, b]).$$

All  $\lambda$ 's were nonzero and  $T$  was assumed to be invertible. Hence the expression  $[c, d, b][e, f, a] - [c, d, a][e, f, b]$  is zero if and only if the above expression is zero. Observe that it was important that each summand of the bracket polynomial had exactly the same number of brackets. This made it possible to factor out a factor  $\det(T)^2$ . Furthermore, in each summand each letter occurred equally often. This made it possible to factor out the  $\lambda$ 's.

This example is a special case of a much more general fact, namely that all *multihomogeneous bracket polynomials* define projectively invariant properties.

**Definition 6.2.** Let  $P = (p_1, p_2, \dots, p_n) \in (\mathbb{R}^3)^n$  represent a point configuration of  $n$  points. A *bracket monomial* on  $P$  is an expression of the form

$$[a_{1,1}, a_{1,2}, a_{1,3}] \cdot [a_{2,1}, a_{2,2}, a_{2,3}] \cdots [a_{k,1}, a_{k,2}, a_{k,3}],$$

where each  $a_{j,k}$  is one of the points  $p_i$ . The *degree*  $\deg(p_i, M)$  of a point  $p_i$  in a monomial is the total number of occurrences of  $p_i$  in  $M$ . A *bracket polynomial* on  $P$  is a sum of bracket monomials on  $P$ . A bracket polynomial  $Q = M_1 + \cdots + M_l$  with monomials  $M_1, \dots, M_l$  is *multihomogeneous* if for each point  $p_i$  we have

$$\deg(p_i, M_1) = \cdots = \deg(p_i, M_l).$$

In other words, a bracket polynomial is multihomogeneous if each point occurs in each summand the same number of times. We can make an analogous definition for points on a projective line. The only difference there is that we have to deal with brackets of length 2 instead of length 3.

As a straightforward generalization of our observations on the multihomogeneous bracket polynomial  $[c, d, b][e, f, a] - [c, d, a][e, f, b]$  we obtain the following theorem:

**Theorem 6.1.** *Let  $Q(P)$  be a multihomogeneous bracket polynomial on  $n$  points  $P = (p_1, p_2, \dots, p_n) \in (\mathbb{R}^3)^n$ . Then  $Q(P) = 0$  defines a projectively invariant property.*

*Proof.* Since  $Q$  is multihomogeneous, each of the summands contains the same number (say  $3k$ ) of points. Therefore each summand is the product of  $k$  brackets. Thus we have for any projective transformation  $T$  the relation

$$Q(T \cdot P) = \det(T)^k \cdot Q(P).$$

Furthermore, the degree of the point  $p_i$  is the same (say  $d_i$ ) in each monomial. Scaling the points by scalars  $\lambda_1 \cdots \lambda_n$  can be expressed as multiplication by the diagonal matrix  $D = \text{diag}(\lambda_1 \cdots \lambda_n)$ . Since each bracket is linear in each point entry, the scaling induces the following action on  $Q$ :

$$Q(P \cdot D) = \lambda_1^{d_1} \cdots \lambda_n^{d_n} \cdot Q(P).$$

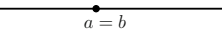
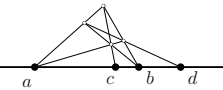
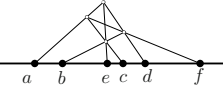
Overall, we obtain

$$Q(T \cdot P \cdot D) = \det(T)^k \cdot \lambda_1^{d_1} \cdots \lambda_n^{d_n} \cdot Q(P).$$

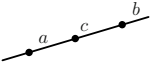
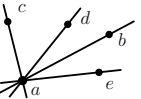
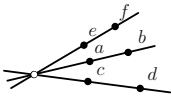
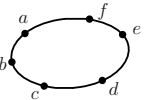
The factors preceding  $Q(P)$  are all nonzero, since  $T$  is invertible and only nonzero  $\lambda_i$  are allowed. Hence  $Q(T \cdot P \cdot D)$  is zero if and only if  $Q(P)$  is zero.  $\square$

Clearly, a similar statement also holds for points on the projective line (and  $2 \times 2$  brackets) and also for projective planes over other fields.

We could now begin a comprehensive study of multihomogeneous bracket polynomials and the projective invariants encoded by them. We will encounter several of them later in the book. Here we just give without further proofs a few examples to exemplify the expressive power of multihomogeneous bracket polynomials. We begin with a few examples on the projective line:

$[ab] = 0$	$a$ coincides with $b$	
$[ac][bd] + [ad][bc] = 0$	$(a, b); (c, d)$ is harmonic	
$[ae][bf][cd] - [af][bd][ce] = 0$	$(a, b); (c, d); (e, f)$ is a quadrilateral set	

Here are some other examples in the projective plane:

$[abc] = 0$	$a, b, c$ are collinear	
$[abd][ace] + [abe][acd] = 0$	The line pairs $(a \vee b, a \vee c); (a \vee d, a \vee e)$ are harmonic	
$[abe][cdf] - [abf][cde] = 0$	$(a \vee b); (c \vee d); (e \vee f)$ meet in a point	
$[abc][aef][dbf][dec] - [def][dbc][aec][abf] = 0$	$a, b, c, d, e, f$ are on a conic	

## 6.5 Grassmann-Plücker relations

When we studied the example of three lines  $a \vee b, c \vee d, e \vee f$  meeting in a point we ended up with the formula

$$[c, d, b][e, f, a] - [c, d, a][e, f, b] = 0.$$

A closer look at this formula shows that the line  $a \vee b$  plays a special role compared to the other two lines. Its points are distributed over the brackets, while the points of the other lines always occur in one bracket. The symmetry of the original property implies that there are two more essentially different ways to encode the property in a bracket polynomial:

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] = 0 \quad \text{and} \quad [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0.$$

The reason for this is that there are multi-homogeneous bracket polynomials that will always evaluate to zero no matter where the points of the configuration are placed. These special polynomials are of fundamental importance whenever one makes calculations in which several determinants are involved. The relations in question are the *Grassmann-Plücker relations*. In principle, such relations exist in any dimension. However, as usual in our exposition we will mainly focus on the case of the projective line and the projective plane, i.e.,  $2 \times 2$  and  $3 \times 3$  brackets. We start with the  $2 \times 2$  case. We state the relations on the level of vectors rather than on the level of projective points.

**Theorem 6.2.** *For any vectors  $a, b, c, d \in \mathbb{R}^2$  the following equation holds:*

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0.$$

*Proof.* If one of the vectors is the zero vector, the equation is trivially true. Thus we may assume that each of the vectors represents a point of the projective line. Since  $[a, b][c, d] - [a, c][b, d] + [a, d][b, c]$  is a multihomogeneous bracket polynomial, we may assume that all vectors are (if necessary after a suitable projective transformation) finite points and normalized to vectors  $\begin{pmatrix} \lambda_a \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \lambda_d \\ 1 \end{pmatrix}$ . The determinants then become simply differences. Rewriting the term gives

$$(\lambda_a - \lambda_b)(\lambda_c - \lambda_d) - (\lambda_a - \lambda_c)(\lambda_b - \lambda_d) + (\lambda_a - \lambda_d)(\lambda_b - \lambda_c) = 0.$$

Expanding all terms, we get equivalently

$$\begin{aligned} & (\lambda_a \lambda_c + \lambda_b \lambda_d - \lambda_a \lambda_d - \lambda_b \lambda_c) \\ & - (\lambda_a \lambda_b + \lambda_c \lambda_d - \lambda_a \lambda_d - \lambda_c \lambda_b) \\ & + (\lambda_a \lambda_b + \lambda_d \lambda_c - \lambda_a \lambda_c - \lambda_d \lambda_b) = 0. \end{aligned}$$

The last equation can be easily checked. □

Grassmann-Plücker relations can be interpreted in many equivalent ways and, thereby this link several branches of geometry and invariant theory. We will here present three more interpretations (or proofs if you want).

**1. Determinant expansion:** The Grassmann-Plücker relation  $[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0$  can be considered a determinant expansion. For

this assume without loss of generality that  $[a, b] \neq 0$ . After a projective transformation we may assume that  $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . The Grassmann-Plücker relation then reads as

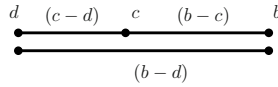
$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} - \begin{vmatrix} 1 & c_1 \\ 0 & c_2 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 \\ 1 & d_2 \end{vmatrix} + \begin{vmatrix} 1 & d_1 \\ 0 & d_2 \end{vmatrix} \cdot \begin{vmatrix} 0 & c_1 \\ 1 & c_2 \end{vmatrix} \\ = 1 \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} - c_2 \cdot (-d_1) + d_2 \cdot (-c_1) = 0$$

The last expression is easily recognized as the expansion formula for the determinant and obviously evaluates to zero.

**2. Area relation:** After a projective transformation and rescaling we can also assume that  $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $b = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}$ ,  $c = \begin{pmatrix} c_1 \\ 1 \end{pmatrix}$  and  $d = \begin{pmatrix} d_1 \\ 1 \end{pmatrix}$ . Then the Grassmann-Plücker relation reads

$$\begin{vmatrix} 1 & b_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & c_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} b_1 & d_1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & d_1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} b_1 & c_1 \\ 1 & 1 \end{vmatrix} \\ = 1 \cdot (c_1 - d_1) - 1 \cdot (b_1 - d_1) + 1 \cdot (b_1 - c_1) = 0.$$

This formula can be affinely (!) interpreted as the relation of three directed length segments of three points  $b, c, d$  on a line:



**3. Cramer's rule:** Let us assume that  $[a, c] \neq 0$ . Cramer's rule gives us an explicit formula to solve the system of equations

$$\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We get

$$\alpha = \frac{[b, c]}{[a, c]} \quad \text{and} \quad \beta = \frac{[a, b]}{[a, c]}.$$

Inserting this into the original equation and multiplying by  $[a, c]$ , we get

$$[b, c] \cdot a + [a, b] \cdot c - [a, c] \cdot b = 0.$$

Here “0” means the zero vector. Thus we can find the following expansion of zero:

$$0 = [[b, c] \cdot a + [a, b] \cdot c - [a, c] \cdot b, d] = [b, c][a, d] + [a, b][c, d] - [a, c][b, d].$$

This is exactly the Grassmann-Plücker relation.

What happens in dimension 3 (i.e., the projective plane)? First of all, we obtain a consequence of the Grassmann-Plücker relation on the line when we add the same point to any bracket:

**Theorem 6.3.** *For any vectors  $a, b, c, d, x \in \mathbb{R}^3$  the following equation holds:*

$$[x, a, b][x, c, d] - [x, a, c][x, b, d] + [x, a, d][x, b, c] = 0.$$

*Proof.* Assuming without loss of generality that  $x = (1, 0, 0)$  reduces all determinants of the expression to  $2 \times 2$  determinants, any of the above proofs translates literally.  $\square$

In the projective plane we get another Grassmann-Plücker relation that involves four instead of three summands.

**Theorem 6.4.** *For any vectors  $a, b, c, d, e, f \in \mathbb{R}^3$  the following equation holds:*

$$[a, b, c][d, e, f] - [a, b, d][c, e, f] + [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0.$$

*Proof.* Applying Cramer's rule to the solution of a  $3 \times 3$  equation

$$\begin{pmatrix} | & | & | \\ c & d & e \\ | & | & | \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} | \\ f \\ | \end{pmatrix},$$

we can prove the identity

$$[f, d, e] \cdot c + [c, f, e] \cdot d + [c, d, f] \cdot e = [c, d, e] \cdot f.$$

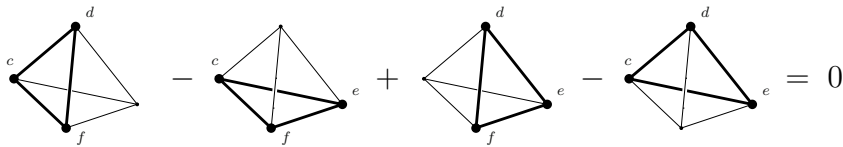
Rearranging the terms yields

$$[d, e, f] \cdot c - [c, e, f] \cdot d + [c, d, f] \cdot e - [c, d, e] \cdot f = 0.$$

Inserting this expansion of the zero vector  $0$  into  $[a, b, 0] = 0$  yields (after expanding the terms by multilinearity) the desired relation.  $\square$

Again, we can also interpret this equation in many different ways. Setting  $(a, b, c)$  to the unit matrix the Grassmann-Plücker relation encodes the development of the  $3 \times 3$  determinant  $(d, e, f)$  by the first column. We get

$$\begin{aligned} & \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \\ 0 & 0 & d_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & e_1 & f_1 \\ 0 & e_2 & f_2 \\ 1 & e_3 & f_3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 0 & 0 & e_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 & f_1 \\ 0 & d_2 & f_2 \\ 1 & d_3 & f_3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & f_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & d_1 & e_1 \\ 0 & d_2 & e_2 \\ 1 & d_3 & e_3 \end{vmatrix} \\ &= 1 \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - d_3 \cdot \begin{vmatrix} e_1 & f_1 \\ e_2 & f_2 \end{vmatrix} + e_3 \cdot \begin{vmatrix} d_1 & f_1 \\ d_2 & f_2 \end{vmatrix} - f_3 \cdot \begin{vmatrix} d_1 & e_1 \\ d_2 & e_2 \end{vmatrix} = 0. \end{aligned}$$



**Fig. 6.4** Grassmann-Plücker relation as area formulas.

Observe that we can express each minor of the determinant  $[d, e, f]$  as a suitable bracket that involves  $a, b, c$ . This point will later be of fundamental importance.

There is also a nice interpretation that generalizes the “area viewpoint.” The determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}$$

calculates twice the oriented area  $\Delta(a, b, c)$  of the affine triangle  $a, b, c$ . After a suitable projective transformation the Grassmann-Plücker relation reads

$$\begin{vmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & d_1 \\ 0 & 1 & d_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & e_1 & f_1 \\ c_2 & e_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & e_1 \\ 0 & 1 & e_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

In terms of triangle areas this equation reads as

$$\Delta(d, e, f) - \Delta(c, e, f) + \Delta(c, d, f) - \Delta(c, d, e) = 0.$$

This formula has again a direct geometric interpretation in terms of affine oriented areas of triangles. Assume that  $c, d, e, f$  are any four points in the affine plane. The convex hull of these four points can be covered in two ways by triangles spanned by three of the points. These two possibilities must both result in the same total (oriented) area. This is the Grassmann-Plücker relation.

Using Grassmann-Plücker relations we can easily explain why the property that  $a \vee b, c \vee d, e \vee f$  meet can be expressed either by

$$[a, b, e][c, d, f] - [a, b, f][c, d, e] = 0$$

or by

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] = 0.$$

Adding the two expressions yields

$$[a, b, c][e, f, d] - [a, b, d][e, f, c] + [a, b, e][c, d, f] - [a, b, f][c, d, e] = 0,$$



which is exactly a Grassmann-Plücker relation. Hence this equation must be zero, which proves the equivalence of the above two expressions.



# More on Bracket Algebra

*Algebra is generous; she often gives more than is asked of her.*

*D'Alembert (1717–1783)*

The last chapter demonstrated that determinants (and in particular multi-homogeneous bracket polynomials) are of fundamental importance in expressing projectively invariant properties. In this chapter we will alter our point of view. What if our “first-class citizens” were not the points of a projective plane but the values of determinants generated by them? We will see that with the use of Grassmann-Plücker relations we will be able to recover a projective configuration from its values of determinants.

We will start our treatment by considering vectors in  $\mathbb{R}^3$  rather than considering homogeneous coordinates of points in  $\mathbb{RP}^2$ . This has the advantage that we can neglect the (only technical) difficulty that the determinant values vary when the coordinates of a point are multiplied by a scalar.

While reading this chapter the reader should constantly bear in mind that all concepts presented in this chapter generalize to arbitrary dimensions. Still we will concentrate on the case of vectors in  $\mathbb{R}^2$  and in  $\mathbb{R}^3$  to keep things conceptually as simple as possible.

## 7.1 From Points to Determinants ...

Assume that we are given a configuration of  $n$  vectors in  $\mathbb{R}^3$  arranged in a matrix:

$$P = \begin{pmatrix} | & | & | & \dots & | \\ p_1 & p_2 & p_3 & \dots & p_n \\ | & | & | & \dots & | \end{pmatrix}.$$

Later on we will consider these vectors as homogeneous coordinates of a point configuration in the projective plane. The matrix  $P$  may be considered an element in  $\mathbb{R}^{3 \cdot n}$ . There is an overall number of  $\binom{n}{3}$  possible  $3 \times 3$  matrix minors that could be formed from this matrix, since there are as many ways to select three points from the configuration. If we know the value of the corresponding determinants we can reconstruct the value of any bracket using permutations of the points and applying the appropriate sign changes, since we have

$$[a, b, c] = [b, c, a] = [c, a, b] = -[b, a, c] = -[a, c, b] = -[c, b, a].$$

We consider the index set  $E$  for the points in  $P$  by

$$E = \{1, 2, \dots, n\}$$

and a corresponding index set for the index triples

$$A(n, 3) := \{(i, j, k) \in \mathbb{E}^3 \mid i < j < k\}.$$

Calculating the determinants for our vector configuration can now be considered as a map

$$\begin{aligned} \Gamma: \mathbb{R}^{3 \cdot n} &\rightarrow \mathbb{R}^{\binom{n}{3}}, \\ P &\mapsto ([p_1, p_2, p_3], [p_1, p_2, p_4], \dots, [p_{n-2}, p_{n-1}, p_n]). \end{aligned}$$

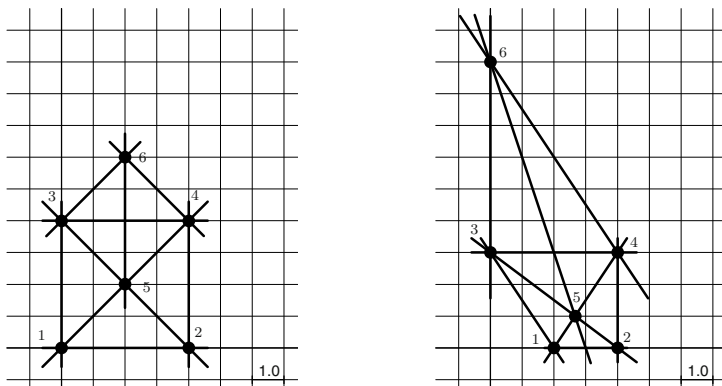
We can consider the vector of determinants  $\Gamma(P)$  itself as a map that assigns to each element  $(i, j, k) \in A(n, 3)$  the value of the corresponding determinant  $[p_i, p_j, p_k]$ . By applying the alternating rule, the values of all brackets can be recovered from the values of  $\Gamma(P)$ . In order to avoid all the information about a bracket being captured by the subscripts, we make the following typographical convention. If  $P$  is a matrix consisting of columns  $p_1, p_2, \dots, p_n$ , then we may also write  $[i, j, k]_P$  instead of  $[p_i, p_j, p_k]$ .

The reader should not be scared of the high dimension of the spaces involved. This high dimensionality comes from the fact that we consider an entire collection of  $n$  vectors now as a *single* object in  $\mathbb{R}^{n \cdot 3}$ . Similarly, an element in the space  $\mathbb{R}^{\binom{n}{3}}$  may be considered a single object that carries the values of all determinants simultaneously. It will be our aim to show that both spaces carry in principle the same information if we are concerned with projective properties.

One of the fundamental relations between  $P$  and  $\Gamma(P)$  is given by the following lemma:

**Lemma 7.1.** *Let  $P \in (\mathbb{R}^3)^n$  be a configuration of  $n$  vectors in  $\mathbb{R}^3$  and let  $T$  be an invertible  $3 \times 3$  matrix. Then  $\Gamma(T \cdot P) = \lambda \cdot \Gamma(P)$  for a suitable  $\lambda \neq 0$ .*

*Proof.* Let  $(i, j, k) \in A(n, 3)$ : Then we have  $\det(T \cdot p_i, T \cdot p_j, T \cdot p_k) = \det(T) \cdot [p_i, p_j, p_k]$ . Thus we have  $\Gamma(T \cdot P) = \det(T) \cdot \Gamma(P)$ .  $\square$



**Fig. 7.1** A configuration and a projective image of it.

The previous lemma states that up to a scalar multiple the vector  $\Gamma(P)$  is invariant under linear transformations. On the level of point configurations in the projective plane this means that if two configurations of points are projectively equivalent, we can assign matrices of homogeneous coordinates  $P$  and  $Q$  to them such that  $\Gamma(P) = \Gamma(Q)$ . A little care has to be taken. Since the homogeneous coordinates of each point are determined only up to a scalar factor, we have to adjust these factors in the right way to get the above relation. We can get rid of the  $\lambda$  in Lemma 7.1 by an overall scaling applied to all homogeneous coordinates.

*Example 7.1.* Consider the two pictures in Figure 7.1. They are related by a projective transformation. We get homogeneous coordinates of the points by simply extending the Euclidean coordinates by a 1. The two coordinate matrices are

$$P = \begin{pmatrix} 0 & 4 & 0 & 4 & 2 & 2 \\ 0 & 0 & 4 & 4 & 2 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 4 & 0 & 4 & \frac{8}{3} & 0 \\ 0 & 0 & 3 & 3 & 1 & 9 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For  $P$  the corresponding determinants indexed by  $\Lambda(6, 3)$  are

$$\begin{array}{llll} [1, 2, 3]_P = 16 & [1, 3, 5]_P = -8 & [2, 3, 4]_P = -16 & [2, 5, 6]_P = -8 \\ [1, 2, 4]_P = 16 & [1, 3, 6]_P = -8 & [2, 3, 5]_P = 0 & [3, 4, 5]_P = -8 \\ [1, 2, 5]_P = 8 & [1, 4, 5]_P = 0 & [2, 3, 6]_P = -16 & [3, 4, 6]_P = 8 \\ [1, 2, 6]_P = 24 & [1, 4, 6]_P = 16 & [2, 4, 5]_P = 8 & [3, 5, 6]_P = 8 \\ [1, 3, 4]_P = -16 & [1, 5, 6]_P = 8 & [2, 4, 6]_P = 8 & [4, 5, 6]_P = -8. \end{array}$$

The corresponding values for  $Q$  are

$$\begin{array}{llll}
[1, 2, 3]_Q = 6 & [1, 3, 5]_Q = -4 & [2, 3, 4]_Q = -16 & [2, 5, 6]_Q = -8 \\
[1, 2, 4]_Q = 6 & [1, 3, 6]_Q = -12 & [2, 3, 5]_Q = 0 & [3, 4, 5]_Q = -8 \\
[1, 2, 5]_Q = 2 & [1, 4, 5]_Q = 0 & [2, 3, 6]_Q = -24 & [3, 4, 6]_Q = 24 \\
[1, 2, 6]_Q = 18 & [1, 4, 6]_Q = 24 & [2, 4, 5]_Q = 4 & [3, 5, 6]_Q = 16 \\
[1, 3, 4]_Q = -12 & [1, 5, 6]_Q = 8 & [2, 4, 6]_Q = 12 & [4, 5, 6]_Q = -16.
\end{array}$$

At first sight these two collections of values seem not to be very related. However, if we simply choose different homogeneous coordinates for the point in the second picture, such as

$$r_1 = 4 \cdot q_1, \quad r_2 = 4 \cdot q_2, \quad r_3 = 2 \cdot q_3, \quad r_4 = 2 \cdot q_4, \quad r_5 = 3 \cdot q_5, \quad r_6 = 1 \cdot q_6,$$

we get

$$\begin{array}{llll}
[1, 2, 3]_R = 192 & [1, 3, 5]_R = -96 & [2, 3, 4]_R = -192 & [2, 5, 6]_R = -96 \\
[1, 2, 4]_R = 192 & [1, 3, 6]_R = -96 & [2, 3, 5]_R = 0 & [3, 4, 5]_R = -96 \\
[1, 2, 5]_R = 96 & [1, 4, 5]_R = 0 & [2, 3, 6]_R = -192 & [3, 4, 6]_R = 96 \\
[1, 2, 6]_R = 288 & [1, 4, 6]_R = 192 & [2, 4, 5]_R = 96 & [3, 5, 6]_R = 96 \\
[1, 3, 4]_R = -192 & [1, 5, 6]_R = 96 & [2, 4, 6]_R = 96 & [4, 5, 6]_R = -96.
\end{array}$$

This is exactly 12 times the values of the determinants obtained from  $P$ . It is also instructive to check the Grassmann-Plücker relations for a few special cases. For instance, we should have

$$[1, 2, 3]_P [4, 5, 6]_P - [1, 2, 4]_P [3, 5, 6]_P + [1, 2, 5]_P [3, 4, 6]_P - [1, 2, 6]_P [3, 4, 5]_P = 0.$$

This can easily be verified:

$$16 \cdot (-8) - 16 \cdot 8 + 8 \cdot 8 - 24 \cdot (-8) = -128 - 128 + 64 + 192 = 0.$$

## 7.2 ... and Back

We will now focus on the other direction. To what extent do the values of  $\Gamma(P)$  already determine the entries of  $P$ ? Let us first start with two observations.

**Observation 1:** *The elements in  $\Gamma(P)$  are not independent of each other.* This comes from the fact that the entries at least have to satisfy the Grassmann-Plücker relations.

**Observation 2:** *The elements in  $\Gamma(P)$  can determine  $P$  only up to a linear transformation.* This is the statement of Lemma 7.1.

These two observations extract the essence of the relation between  $\Gamma(P)$  and  $P$ . We will prove that for every  $\Gamma \in \mathbb{R}^{\binom{n}{3}} - \{\mathbf{0}\}$  that satisfies the Grassmann-Plücker relations there is a  $P \in \mathbb{R}^{3 \cdot n}$  such that  $\Gamma = \Gamma(P)$ . This  $P$  is uniquely determined up to a linear transformation.

We again consider  $\Gamma$  as a map  $\Lambda(n, 3) \rightarrow \mathbb{R}$ . For the rest of this chapter we will denote the value  $\Gamma((i, j, k))$  by  $[i, j, k]$ . Thus the problem of finding a suitable  $P$  can be restated in the following way:

*Find a  $P$  such that  $[i, j, k] = [i, j, k]_P$  for all  $(i, j, k) \in \Lambda(n, 3)$ .*

If  $\Gamma$  is not the zero vector  $\mathbf{0}$ , then we may without loss of generality assume that  $[1, 2, 3] = 1$  (otherwise we simply have to permute the indices in a suitable way, and scale  $\Gamma$  by a suitable factor). If we find any suitable matrix  $P$  with  $\Gamma(P) = \Gamma$ , then the first three vectors form an invertible matrix

$$M = \begin{pmatrix} | & | & | \\ p_1 & p_2 & p_3 \\ | & | & | \end{pmatrix}.$$

If we replace  $P$  by  $P' := \det M \cdot M^{-1} \cdot P$ , we still have  $\Gamma(P') = \Gamma$  but in addition the first three vectors became unit vectors. The matrix  $P'$  has the shape

$$P' = \begin{pmatrix} 1 & 0 & 0 & p_{41} & p_{51} & \dots & p_{n1} \\ 0 & 1 & 0 & p_{42} & p_{52} & \dots & p_{n2} \\ 0 & 0 & 1 & p_{43} & p_{53} & \dots & p_{n3} \end{pmatrix}.$$

The key observation now is that each entry in the matrix corresponds to the value of a suitable bracket. For instance, we have

$$p_{43} = [1, 2, 4]; \quad p_{42} = -[1, 3, 4]; \quad p_{41} = [2, 3, 4].$$

Thus if our matrix  $P'$  exists, we can immediately fill in all the other entries if we know the values of  $\Gamma$ . We get

$$P' = \begin{pmatrix} 1 & 0 & 0 & [2, 3, 4] & [2, 3, 5] & \dots & [2, 3, n] \\ 0 & 1 & 0 & -[1, 3, 4] & -[1, 3, 5] & \dots & -[1, 3, n] \\ 0 & 0 & 1 & [1, 2, 4] & [1, 2, 5] & \dots & [1, 2, n] \end{pmatrix}.$$

So far we have only made sure that brackets of the forms  $[1, 2, 3]$ ,  $[1, 2, i]$ ,  $[1, 3, i]$ , and  $[2, 3, i]$ ,  $i = 4, \dots, n$ , get the right value. How about all the remaining brackets (which are the vast majority)? This is the point where the Grassmann-Plücker relations come into play. If  $\Gamma$  satisfies all identities required by the Grassmann-Plücker relations, then all other bracket values will fit automatically. We can prove this by successively showing that (under the hypothesis that the Grassmann-Plücker relations hold) the values of the brackets are uniquely determined after fixing the brackets of the above form. Let us take the bracket  $[1, 4, 5]$  for instance. By our assumptions on  $\Gamma$  we

know that the relation

$$[1, 2, 3][1, 4, 5] - [1, 2, 4][1, 3, 5] + [1, 2, 5][1, 3, 4] = 0$$

must hold. Except for the value of the bracket  $[1, 4, 5]$ , all other bracket values are already fixed. Since  $[1, 2, 3] \neq 0$ , the value of  $[1, 4, 5]$  is determined uniquely. Similarly, we can show that all values of the brackets of the forms  $[1, i, j]$ ,  $[2, i, j]$ , and  $[3, i, j]$  are uniquely determined. It remains to show that brackets that do not contain any of the indices 1, 2, 3 are also already fixed. As an example we take the bracket  $[4, 5, 6]$ . The following relation must hold:

$$[1, 2, 3][4, 5, 6] - [1, 2, 4][3, 5, 6] + [1, 2, 5][3, 4, 6] - [1, 2, 6][3, 4, 5] = 0.$$

Again all values except  $[4, 5, 6]$  are already determined by our previous considerations. Hence the above expression fixes the value of  $[4, 5, 6]$ . We may argue similarly for any bracket  $[i, j, k]$ . We have finished the essential parts of the proof of the following theorem:

**Theorem 7.1.** *Let  $\Gamma \in \mathbb{R}^{\binom{n}{3}}$ ,  $\Gamma \neq \mathbf{0}$ , be an assignment of bracket values that satisfies all Grassmann-Plücker relations. Then there exists a vector configuration  $P \in \mathbb{R}^{3 \cdot n}$  with  $\Gamma = \Gamma(P)$ .*

*Proof.* For the proof we just summarize what we have done so far. With the above notation we may without loss of generality assume that  $[1, 2, 3] = 1$ . As above, set

$$P = \begin{pmatrix} 1 & 0 & 0 & [2, 3, 4] & [2, 3, 5] & \dots & [2, 3, n] \\ 0 & 1 & 0 & -[1, 3, 4] & -[1, 3, 5] & \dots & -[1, 3, n] \\ 0 & 0 & 1 & [1, 2, 4] & [1, 2, 5] & \dots & [1, 2, n] \end{pmatrix}.$$

In particular, by this choice we have

$$[1, 2, 3] = [1, 2, 3]_P, \quad [1, 2, i] = [1, 2, i]_P, \quad [1, 3, i] = [1, 3, i]_P, \quad [2, 3, i] = [2, 3, i]_P,$$

for any  $i \in \{4, \dots, n\}$ . Since  $P$  is a point configuration, it satisfies all Grassmann-Plücker relations. Since if the Grassmann-Plücker relations are satisfied the values of all brackets are determined uniquely if the above bracket values are fixed, we must have  $\Gamma = \Gamma(P)$ .  $\square$

*Example 7.2.* Assume that we are looking for a vector configuration that gives us the following bracket values (they have been chosen carefully to satisfy all Grassmann-Plücker relations, check it):



$$\begin{array}{llll}
[\mathbf{1}, \mathbf{2}, \mathbf{3}] = 1 & [\mathbf{1}, \mathbf{3}, \mathbf{5}] = -1/2 & [\mathbf{2}, \mathbf{3}, \mathbf{4}] = -1 & [2, 5, 6] = -1/2 \\
[\mathbf{1}, \mathbf{2}, \mathbf{4}] = 1 & [\mathbf{1}, \mathbf{3}, \mathbf{6}] = -1/2 & [\mathbf{2}, \mathbf{3}, \mathbf{5}] = 0 & [3, 4, 5] = -1/2 \\
[\mathbf{1}, \mathbf{2}, \mathbf{5}] = 1/2 & [1, 4, 5] = 0 & [\mathbf{2}, \mathbf{3}, \mathbf{6}] = -1 & [3, 4, 6] = 1/2 \\
[\mathbf{1}, \mathbf{2}, \mathbf{6}] = 3/2 & [1, 4, 6] = 1 & [2, 4, 5] = 1/2 & [3, 5, 6] = 1/2 \\
[\mathbf{1}, \mathbf{3}, \mathbf{4}] = -1 & [1, 5, 6] = 1/2 & [2, 4, 5] = 1/2 & [4, 5, 6] = -1/2.
\end{array}$$

The brackets that are emphasized by bold letters are those we can directly use to obtain the entries in our matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

It is easy to check that the other bracket values are satisfied automatically. The careful reader may have recognized that the brackets were chosen such in a way that they again produce a projectively equivalent copy of the drawing in Figure 7.1.

## 7.3 A Glimpse of Invariant Theory

The essence of the last two chapters can be stated in the following way: *The determinants  $\Gamma(P)$  of a vector configuration  $P$  carry essentially the same information as the configuration itself.* In principle, we can perform all operations of projective geometry entirely on the level of determinants.

There are several ways to exploit this fact, and we want to mention at least a few of them. In this section we want to review a few facts that belong to the context of *classical invariant theory* (see for instance [98, 131]). This theory originated precisely in the efforts of the classical geometers to understand how geometry and algebra are related. Besides a purely theoretical interest there were also very practical motivations for considering this topic: People in the nineteenth century did not have computers. All calculations had to be done by hand. For dealing with geometric constructions of nontrivial size it was absolutely necessary to have advanced algebraic techniques that reduced the amount of hand calculations to a minimum. Plücker, who was (as we already have seen) one of the pioneers in this field, once said (according to Klein) that *one has to read in equations*. This means to draw far-reaching conclusions from simple algebraic facts. Striving for powerful algebraic techniques that are as close to geometry as possible first led to the development of homogeneous coordinates, later to the recognition of the importance of determinants and still later to the development of invariant theory.

One might be tempted to think that nowadays with the help of computers that can do all the “number crunching” it is no longer necessary to care

about sophisticated algebraic techniques, since calculations on the coordinate level can be carried out automatically. In fact, exactly the opposite is the case. The rise of computational power has led to a considerable demand for sophisticated techniques for performing geometric calculations. There are several reasons for this (and we will list only a few of them):

- With computers it is possible to deal with really huge geometric situations that were not previously within reach. So still it is necessary to compute efficiently.
- Practical applications in computer-aided design, robotics, computer vision, and structural mechanics lead to problems that are at the core of projective geometry, and the algebraic techniques that are used there have to be appropriate for the problem. (For instance, consider an autonomous robot that has to use pictures taken by a video camera to obtain an inner model of the environment. This corresponds to the problem of lifting a two-dimensional projective scene to a three-dimensional one.)
- Symbolic calculations on the coordinate level very soon lead to a combinatorial explosion, since usually every single multiplication of two polynomials doubles the number of summands involved.
- The rise of computers opened the new discipline of “automatic deduction in geometry” (see [45, 115, 133, 60, 17]). There one is, for instance, interested in automatically generating proofs for geometric theorems. In order to be able to retranslate the automatically generated proof into geometric statements one has finally to reinterpret algebraic terms geometrically. This is much easier if the algebraic statements are “close” to geometry.

Our investigations of the last two sections showed that brackets form a *functional basis* for projective invariants. To make this notion a bit more precise, we first slightly broaden the scope of Definition 6.1, in which we introduced projectively invariant properties. To avoid technical difficulties that arise from configurations that do not have full rank, we call a configuration of points given by homogeneous coordinates  $P \in \mathbb{R}^{3 \cdot n}$  *proper* if  $P$  has rank 3. If  $P$  is not proper, then all points of  $P$  lie on a line, or even collapse to a single point.

**Definition 7.1.** Let  $M$  be an arbitrary set. A *projective invariant* of  $n$  points in the real projective plane is a map  $f: \mathbb{R}^{3 \cdot n} \rightarrow M$  such that for all invertible real  $3 \times 3$  matrices  $T \in \text{GL}(\mathbb{R}, 3)$  and  $n \times n$  invertible real diagonal matrices  $D \in \text{diag}(\mathbb{R}, n)$  and for any proper configuration  $P$ , we have

$$f(P) = f(T \cdot P \cdot D).$$

In this definition the image range  $M$  is taken to be an arbitrary set. If  $M$  is the set  $\{\mathbf{true}, \mathbf{false}\}$ , then  $f$  is a projectively invariant property as introduced

in Definition 6.1. If  $M = \mathbb{R} \cup \{\infty\}$ , then  $f$  measures some number that is invariant under projective transformations (such as the cross-ratio, for instance). We now want to study under what circumstances one can generate one projective invariant from others. For this we introduce the concept of a functional basis.

**Definition 7.2.** A collection  $f_i: \mathbb{R}^{3 \cdot n} \rightarrow M_i$ ,  $i = 1, \dots, k$ , of functions is a *functional basis* of the set of all projective invariants if for every projective invariant  $f: \mathbb{R}^{3 \cdot n} \rightarrow M$ , there is a function

$$m: M_1 \times \dots \times M_k \rightarrow M$$

such that for all  $X \in \mathbb{R}^{3 \cdot n}$  we have

$$f(X) = m(f_1(X), \dots, f_k(X)).$$

In other words, if  $f_1, \dots, f_k$  is a functional basis then it completely suffices to know the values of these functions to calculate the value of any other invariant. We can also understand the concept of functional basis on an algorithmic level. For every invariant  $f$  there exists an algorithm  $\mathcal{A}$  that takes the values of  $f_1, \dots, f_k$  as input and calculates the value of  $f$ . The last two sections essentially prove the following theorem:

**Theorem 7.2.** *The entries of the determinant vector  $\Gamma(P)$  form a functional basis for all projective invariants.*

*Proof (sketch).* Let  $f$  be any invariant and let  $P$  be an arbitrary proper configuration of points. Since  $P$  is proper, at least one entry of  $\Gamma := \Gamma(P)$  is nonzero (this means that at least one determinant does not vanish). Thus we can use the techniques of Section 7.2 to calculate a configuration  $P' := P'(\Gamma)$  that is projectively equivalent to  $P$ . Thus we must have  $f(P'(\Gamma)) = f(P)$ , which proves the claim.  $\square$

This theorem is a weak version of a much deeper algebraic fact that is known as the *first fundamental theorem* of projective invariant theory (see [131, 126]). Theorem 7.2 states only that we can *compute* any invariant if we know the values of the brackets. It does not state that the algebraic structure of an invariant is preserved by any means. The first fundamental theorem in addition guarantees that the algebraic type of an invariant is essentially preserved.

Unfortunately, at this point we have to pay a price for calculating with homogeneous coordinates. Since we do not distinguish homogeneous coordinates that differ only by a scalar multiple, we have to take care of these equivalence classes in a corresponding algebraic setup. Any polynomial in the entries of the matrix  $P \in \mathbb{R}^{3 \cdot n}$  is thus not invariant under rescaling of the homogeneous coordinates. This implies that the category of *polynomials* is not the

appropriate one for talking about projective invariants. There are essentially two ways out of this dilemma. Both involve quite a few technical difficulties. The first is to introduce the concept of a *relative invariant polynomial*. Such a polynomial  $f(P)$  is not an invariant in the sense of Definition 7.1. Rather than being strictly invariant, one requires that the action of  $T \in \mathrm{GL}(\mathbb{R}, 3)$  and  $D \in \mathrm{diag}(\mathbb{R}, n)$  change  $f(T \cdot P \cdot D)$  in a very predictable way. If  $D$  has diagonal entries  $\lambda_1, \dots, \lambda_n$ , then  $f$  is called a relative invariant if there are exponents  $\tau_1, \dots, \tau_n, \tau$  such that

$$f(T \cdot P \cdot D) = \det(T)^\tau \cdot \lambda_1^{\tau_1} \cdots \lambda_n^{\tau_n} \cdot f(P).$$

This means that rescaling and transforming only results in a controllable factor that essentially depends only on the number of times a point is involved in the function  $f$ .

The other way out is to change the category of functions under consideration and not consider polynomials as the important functions. The simplest category of functions in which projective invariants arise is that of *rational functions* that are quotients of polynomials. We already have encountered such types of invariants. The cross-ratio is the simplest instance of such an invariant. We will follow this path a little later.

For now we want to state at least one version of the first fundamental theorem that avoids all these technical difficulties for the price of not directly making a statement about projective geometry [131]. Our version of the first fundamental theorem is formulated on the level of *vector configurations*. Thus this time we distinguish vectors that differ by scalar multiples. Furthermore, we have to restrict the set of allowed transformations to those that have determinant one (this means we have to study group actions of  $SL(\mathbb{R}, 3)$ ; this still includes all projective transformations).

**Theorem 7.3.** *Let  $f: \mathbb{R}^{3 \cdot n} \rightarrow \mathbb{R}$  be a polynomial with the property that for every  $T \in SL(\mathbb{R}, 3)$  and every  $P \in \mathbb{R}^{3 \cdot n}$  we have*

$$f(P) = f(T \cdot P).$$

*Then  $f$  can be expressed as a polynomial in the  $3 \times 3$  sub-determinants of  $P$ .*

So, the first fundamental theorem does not state that every invariant can be expressed in terms of the determinants. It states that every *polynomial* invariant can be expressed as a *polynomial* in the determinants. To understand the power of this statement we want to emphasize that by this theorem, usually rather long and involved formulas on the level of coordinates (i.e., the entries of  $P$ ) will factor into small and geometrically understandable polynomials on the level of brackets. We will not prove the first fundamental theorem here since the proof requires some nontrivial and extensive technical machinery. (A proof of this classical theorem may be found, for instance, in [131], in [33], or in [126].) However, we at least want to give an example that demonstrates the power of the statement.

*Example 7.3.* We want to analyze the condition that six points 1, 2, 3, 4, 5, 6 lie on a common quadratic curve (a conic). Algebraically this can be stated in the following way. Assume that the points  $1, \dots, 6$  have homogeneous coordinates  $(x_1, y_1, z_1), \dots, (x_6, y_6, z_6)$ . If all points are on a common quadratic curve, there are parameters  $a, b, c, d, e, f$  such that the quadratic equations

$$a \cdot x_i^2 + b \cdot y_i^2 + c \cdot z_i^2 + d \cdot x_i \cdot y_i + e \cdot x_i \cdot z_i + f \cdot y_i \cdot z_i = 0$$

for  $i = 1, \dots, 6$  hold (and at least one of the parameters does not vanish). This defines a system of linear equations

$$\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1 y_1 & x_1 z_1 & y_1 z_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2 y_2 & x_2 z_2 & y_2 z_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3 y_3 & x_3 z_3 & y_3 z_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4 y_4 & x_4 z_4 & y_4 z_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5 y_5 & x_5 z_5 & y_5 z_5 \\ x_6^2 & y_6^2 & z_6^2 & x_6 y_6 & x_6 z_6 & y_6 z_6 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system has a nontrivial solution if and only if the determinant of this matrix does not vanish. Expanding this determinant produces a polynomial  $f(x_1, x_2, \dots, z_6)$  in 18 variables with 720 summands each of degree 12. The polynomial must be a projective invariant, since the condition of six points being on a conic is invariant under projective transformations. The first fundamental theorem tells us that this polynomial must be expressible in terms of determinants. In fact, on the level of determinants the polynomial simplifies to the following form:

$$[1, 2, 3][1, 5, 6][4, 2, 6][4, 5, 3] - [4, 5, 6][4, 2, 3][1, 5, 3][1, 2, 6] = 0.$$

This condition can be easily checked by a computer algebra system. One generates the polynomial  $f$  by expanding the above determinant and then applies to  $f$  an operation like `simplify(f)`. If the computer algebra system is clever enough, it will find a form that resembles the above bracket expression.

If one takes a closer look at the above bracket expression one observes that after expansion back to the coordinate level each of the two summands produces many summands. Each bracket is a determinant with 6 summands, and thus each summand produces  $6^4 = 1296$  summands on the level of coordinates. Altogether we have 2592 summands, and 1872 of these summands cancel pairwise.

Since there are nontrivial dependencies among the determinants (such as Grassmann-Plücker relations), the bracket expression is far from unique. There are  $\binom{6}{3}$  equivalent bracket expressions with 2 summands, and many more with more than two summands. Expanding any of these equivalent expressions to the coordinate level results in the same polynomial  $f$ .

The first fundamental theorem can be proved constructively by providing an explicit algorithm that takes a polynomial

$$f(P) \in \mathbb{R}[x_{1,1}, x_{1,2}, x_{1,3}, \dots, x_{n,1}, x_{n,2}, x_{n,3}] =: \mathbb{R}[\mathbf{X}]$$

on the coordinate level as input and produces a polynomial in the brackets of the points  $1, \dots, n$  whenever  $f$  is an invariant. Here  $\mathbf{X}$  abbreviates the collection of all indeterminates on the coordinate level. One strategy for this is to introduce a generic basis of three independent points  $e_1, e_2, e_3$ . Under the assumption of invariance of  $f$  one may without loss of generality assume that these three vectors are the three unit vectors. Thus each variable on the coordinate level can be expressed as a bracket that involves two unit vectors and one point of the configuration. So we can rewrite  $f$  as a polynomial  $b(\dots)$  in these brackets. Now the difficult part of the proof begins. One can show that using Grassmann-Plücker relations,  $b$  can be rewritten in the form

$$b = [e_1, e_2, e_3]^\tau \cdot b'(\dots).$$

Here  $b'(\dots)$  is a bracket polynomial that involves only points  $1, \dots, n$ . Since  $[e_1, e_2, e_3]^\tau$  is constantly 1, the polynomial  $b'(\dots)$  must be the desired bracket polynomial. The reader interested in a formal proof that follows these lines is referred to [126].

A general problem in the translation of coordinate polynomials to bracket polynomials is that one has almost no control on the length of the resulting bracket polynomial. It is still a difficult open research problem to provide efficient algorithms that generate bracket expressions of provably short length.

## 7.4 Projectively Invariant Functions

In geometry one is often interested in measuring certain values of geometric configurations (lengths, angles, etc.). As we have seen, lengths and angles are not invariant under projective transformations. Thus they are not a reasonable measure within the framework of projective geometry. In contrast, it is appropriate to study functions  $f: \mathbb{RP}^2 \rightarrow \mathbb{R} \cup \{\infty\}$  that are projectively invariant in the sense of Definition 7.1. Here we will focus on rational functions that are projectively invariant.

We have already met one of these functions: the cross-ratio. If we reconsider the proofs of Lemmas 4.4 (invariance under rescaling homogeneous coordinates) and 4.5 (invariance under projective transformations), we immediately see how to obtain other projectively invariant rational functions. The crucial property is that in the numerator and in the denominator we have multihomogeneous functions with the same degree for any letter. Under these conditions, rescaling factors of homogeneous coordinates and determinants of projective transformations will cancel perfectly.

We will see later on (in Part III of this book) how to express angles and distances as invariant functions that relate projective objects with special points or lines (such as the line at infinity).

Under mild nondegeneracy assumptions it can be shown that every rational projectively invariant function can be expressed as a rational function in cross-ratios. The proof of this fact essentially reconstructs the coordinates of the configuration (up to projective equivalence) from the cross-ratios (by techniques similar to those of Section 5.4). Then the invariant functions can simply be expanded in terms of cross-ratios. A formal proof may be found in [18].

## 7.5 The Bracket Algebra

In this section we want to change our point of view and again consider the brackets as “first-class citizens” in preference to coordinates. We have seen in Sections 7.1 and 7.2 that brackets carry all information that is necessary to reconstruct a configuration up to projective equivalence. Furthermore, Section 7.3 showed that as a consequence of this we can compute any projective invariant from the values of the brackets. In particular, Example 7.3 indicated that expressing projective invariants on the level of brackets leads to considerably shorter expressions compared to the coordinate level. Moreover, the bracket expressions are often much easier to interpret geometrically than the coordinate expression.

Taking all this into account, it would be a wise strategy to drop the coordinate level completely and model the entire setup of projective geometry over  $\mathbb{R}$  on the level of brackets.

For this we will in this chapter consider the brackets  $[a, b, c]$  themselves as indeterminates that may take any value in  $\mathbb{R}$ . The set of all brackets on  $n$  points is abbreviated

$$\mathbf{B} := \{[i, j, k] \mid i, j, k \in E\}, \quad E = \{1, \dots, n\}.$$

The polynomial ring  $\mathbb{R}[\mathbf{B}]$  contains all polynomials that we can possibly write with those brackets (remember, the brackets are now variables, not determinants). If the brackets really came from the determinants of a vector configuration they would be far from independent. There would be many relations among them: a bracket that contains a repeated letter would have to be zero, the brackets would satisfy the alternating determinant rules, and, last but not least, the brackets would have to satisfy the Grassmann-Plücker relations.

We can take these dependencies into account by factoring out the corresponding relations from the ring  $\mathbb{R}[\mathbf{B}]$ .

**Definition 7.3.** We define the following three ideals in the ring  $\mathbb{R}[\mathbf{B}]$ :

- $\mathbf{I}_{\text{repeat}} := \left\langle \{[i, j, k] \in \mathbf{B} \mid i = j \text{ or } i = k \text{ or } j = k\} \right\rangle,$
- $\mathbf{I}_{\text{altern}} := \left\langle \{[\lambda_1, \lambda_2, \lambda_3] + \sigma(\pi)[\lambda_{\pi(1)}, \lambda_{\pi(2)}, \lambda_{\pi(3)}] \mid \lambda_1, \lambda_2, \lambda_3 \in E, \sigma \in S_3\} \right\rangle,$
- $\mathbf{I}_{\text{GP}} := \left\langle \{[a, b, c][d, e, f] - [a, b, d][c, e, f] + [a, b, e][c, d, f] - [a, b, f][c, d, e] \mid a, b, c, d, e, f \in E\} \right\rangle.$

In this expression  $\pi$  is a permutation of three elements, and  $\sigma(\pi)$  is its sign. The *bracket ring*  $\mathbf{BR}$  is the ring  $\mathbb{R}[\mathbf{B}]$  factored modulo these three ideals:

$$\mathbf{BR} := \mathbb{R}[\mathbf{B}] / \langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \rangle.$$

Thus the bracket ring is defined in a way that brackets with repeated letters are automatically zero (this is forced by  $\mathbf{I}_{\text{repeat}}$ ). Furthermore, alternating determinant rules are forced by  $\mathbf{I}_{\text{altern}}$ , and finally, the Grassmann-Plücker relations are forced by  $\mathbf{I}_{\text{GP}}$ . See [132] for a more elaborate treatment of the bracket ring. Bracket polynomials that are identical in  $\mathbf{BR}$  turn out to expand to the same expression when we replace the brackets by the corresponding determinants. In order to formulate this precisely, we introduce a ring homomorphism

$$\Phi: \mathbb{R}(\mathbf{B}) \rightarrow \mathbb{R}[\mathbf{X}]$$

that models the expansion of brackets. This homomorphism is uniquely determined by its action on the brackets

$$\Phi([i, j, k]) := \det(x_i, x_j, x_k).$$

We now can prove the following theorem:

**Theorem 7.4.** *Let  $f, g$  be two polynomials in  $\mathbb{R}(\mathbf{B})$  such that  $f \equiv g$  in the bracket ring. Then  $\Phi(f) = \Phi(g)$ .*

*Proof.* If  $f \equiv g$  in the bracket ring, then there is a polynomial  $h \in \langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \rangle$  such that  $f = g + h$  in  $\mathbb{R}[\mathbf{B}]$ . Applying the operator  $\Phi$  to both sides, we obtain

$$\Phi(f) = \Phi(g + h) = \Phi(g) + \Phi(h) = \Phi(g).$$

The last equation holds since every polynomial in the ideals  $\mathbf{I}_{\text{repeat}}, \mathbf{I}_{\text{altern}}, \mathbf{I}_{\text{GP}}$  expands to zero under  $\Phi$  by definition.  $\square$

The converse of the above theorem is true as well:

**Theorem 7.5.** *Let  $f, g$  be two polynomials in  $\mathbb{R}(\mathbf{B})$  with  $\Phi(f) = \Phi(g)$ . Then  $f - g \in \langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \rangle$ .*

Equivalently, we can state this theorem also by saying that every bracket polynomial  $f \in \mathbb{R}(\mathbf{B})$  with  $\Phi(f) = 0$  is automatically in the ideal



$$\langle \mathbf{I}_{\text{repeat}} \cup \mathbf{I}_{\text{altern}} \cup \mathbf{I}_{\text{GP}} \rangle.$$

This theorem is also known as the *second fundamental theorem* of invariant theory: All bracket polynomials that are identical to zero for all point configurations are generated by our three ideals. If we, a little less formally, identify bracket symbols according to the rules

$$[i, j, k] = [j, k, i] = [k, i, j] = -[i, k, j] = -[k, j, i] = -[j, i, k],$$

we can also express this theorem by saying that every bracket expression  $f(\mathbf{B})$  that vanishes on all configurations can be written as

$$f = \sum m_i \cdot \gamma_i,$$

where the  $\gamma_i$  are Grassmann-Plücker relations.



## Part II

# Working and Playing with Geometry



*Need to play is the mother of all invention.*

*Kristina Brenneman, Portland Tribune, 2005*

*You've achieved success in your field when you don't know  
whether what you're doing is work or play.*

*Warren Beatty (b. 1937)*

So far, we have prepared a solid basis for projective geometry with a strong emphasis on relationship to algebra. Now it is time to explore how these concepts interact in several contexts. It is the aim of the following few chapters to demonstrate how a bracket-oriented point of view makes it algebraically easy to describe, prove, and even generate interesting geometric facts. We also want to investigate how homogeneous coordinates can be used to express geometric calculations in an elegant fashion.

The material used in the following sections is intentionally taken from different areas of projective geometry. However, especially the chapters on quadrilateral sets and on conics will play an important role later on. Throughout these chapters we will also try to introduce several standard techniques for proving geometric facts on the level of bracket algebra.



# Quadrilateral Sets and Liftings

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*The mathematics of rhythm are universal.  
They don't belong to any particular culture*

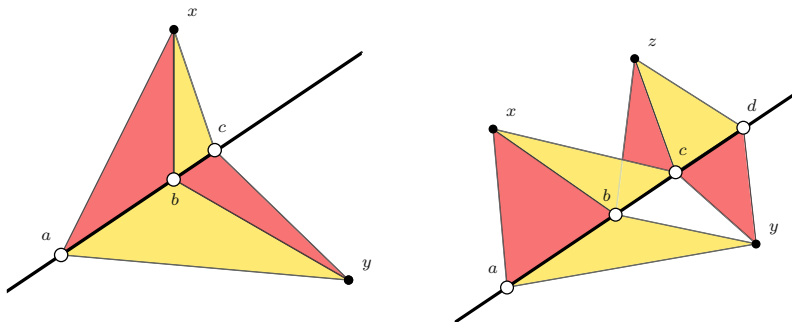
*John McLaughlin*

In this chapter we will focus on an important concept of projective geometry: *quadrilateral sets*. On the one hand, these configurations can be considered a generalization of harmonic points. On the other hand, they have a close relationship to the liftability of lower-dimensional point configurations to prescribed higher-dimensional scenarios. Their algebraic counterparts also expose highly symmetric (almost rhythmic) structures coming from the interplay of the *six* intersections of *four* lines.

## 8.1 Points on a Line

We will begin our studies with the algebraic consequences of having  $n$  points on a line. Any bracket formed by three collinear points will automatically be zero. Via Grassmann-Plücker relations, this vanishing bracket causes other relations among the other (nonzero) brackets. (From now on, we will freely omit the commas in brackets whenever no confusion can arise in order to make the formulas a bit more compact and readable; thus we may write  $[abc]$  instead of  $[a, b, c]$ ). Consider the three-summand Grassmann-Plücker relation

$$[abc][axy] - [abx][acy] + [aby][acx] = 0.$$



**Fig. 8.1** The product of the red areas equals the product of the yellow areas.

We know that this equation holds for arbitrary points  $a, b, c, x, y$  of  $\mathbb{RP}^2$ . If in a configuration we know in addition that  $a, b, c$  are collinear, then the first summand of this equation vanishes, and we obtain the equation

$$[abx][acy] = [aby][acx].$$

This relation generalizes to more general contexts:

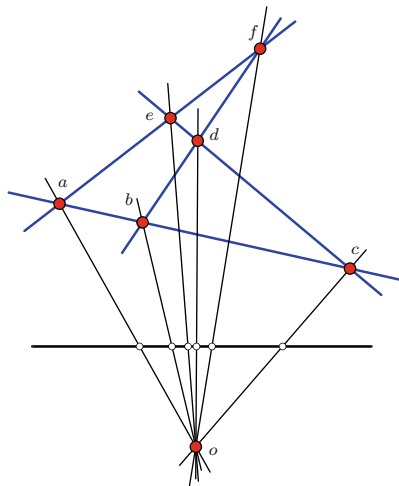
**Theorem 8.1.** *Let  $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{RP}^2$  be  $2n$  collinear points (not necessarily distinct), and let  $x_1, \dots, x_n \in \mathbb{RP}^2$  be  $n$  arbitrary additional points. Then the following bracket equation holds for any permutation  $\pi \in S_n$ :*

$$\prod_i [a_i, b_i, x_i] = \prod_i [a_i, b_i, x_{\pi(i)}].$$

*Proof.* In principle, one can prove this result by a suitable linear combination of Grassmann-Plücker relations. However, we will use this result to introduce the technique of “proof by specialization.” The argument goes as follows. The conclusion of the proof is obviously a projectively invariant property (every letter occurs as often on the left as on the right). Thus it is invariant under projective transformations and rescaling of homogeneous coordinates. Thus we may without loss of generality assume that all the last entries of the homogeneous coordinates are 1 and all points are in finite position. In this case, the determinant  $[a, b, c]$  equals twice the oriented area of the corresponding triangle. For this situation we can provide a very elementary proof (compare Figure 8.1).

The area of the triangle  $(a_i, b_i, x_i)$  can be calculated as  $|a_i, b_i| \cdot h(x_i)/2$ , where  $|a_i, b_i|$  is the oriented distance from  $a_i$  to  $b_i$  and  $h(x_i)$  is the altitude of point  $x$  over the line on which the  $a_i$  and  $b_i$  lie. Thus both sides of the





**Fig. 8.2** Projecting the six intersections of four lines.

expression  $\prod_i [a_i, b_i, x_i] = \prod_i [a_i, b_i, x_{\pi(i)}]$  must be equal, since they simply represent two different ways of ordering the factors of a product.  $\square$

## 8.2 Quadrilateral Sets

Our next example studies the conditions under which points on a line are the projection of a certain incidence configuration in  $\mathbb{RP}^2$ . Consider the picture in Figure 8.2. The four distinct blue lines intersect in six points. These six points are projected (with viewpoint  $o$ ) to the black line  $\ell$ . How can we characterize whether six points on  $\ell$  arise from such a projection? We will present several approaches to this question.

First of all, since all six points are collinear, the characterization must be expressible as a one-dimensional condition on the line. This condition in turn must be expressible purely on the level of determinant equations. Since the points are projected with projection center  $o$ , it must be possible to express the condition as a determinant expression in which each determinant involves point  $o$ . For deriving this equation we will now introduce a technique that is also applicable in many other contexts.

We consider the four collinearities  $[abc] = [aef] = [bdf] = [cde] = 0$  that hold in our picture. From these collinearities we obtain the following four equations:

$$\begin{aligned}
[abc] = 0 &\implies [abe][bcf][cad] = [abf][bcd][cae] \\
[ae f] = 0 &\implies \underline{[aeo]}[afb] = [aeb]\underline{[af o]} \\
[bfd] = 0 &\implies \underline{[bfo]}[bdc] = [bfc]\underline{[bdo]} \\
[cde] = 0 &\implies \underline{[cdo]}[cea] = [cda]\underline{[ceo]}
\end{aligned}$$

The last three are direct consequences of Grassmann-Plücker relations. The first equation is an application of Theorem 8.1. Multiplying all left sides and all right sides and canceling determinants that appear on both sides (those that are not underlined), we arrive at the equation

$$[aeo][bfo][cdo] = [af o][bdo][ceo].$$

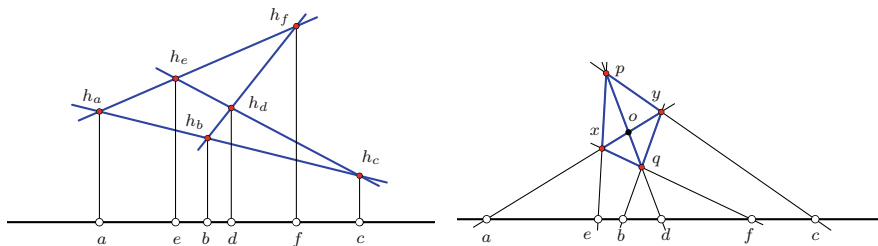
This is the desired characterization. Since in each bracket the point  $o$  is involved, we can also read this expression as a rank-2 expression of the corresponding projections on the line.

Before we study the symmetry of this bracket expression we will have a look at two other ways of deriving this formula. Consider the picture in Figure 8.3 left. There the point of projection has been moved to infinity. We want to give an affine argument from which we can generate the desired formula. This time we will directly head for the rank-2 formula: Under what conditions are six points  $a, b, c, d, e, f$  on a line liftable to the blue incidence configuration? We want a nontrivial lifting, in which not all lines are identical. We furthermore assume that points whose lifted images should be collinear do not coincide on the line. Similar questions arise in the field of scene analysis, a branch of projective geometry in which quadrilateral sets and bracket algebra play a fundamental role [29, 31].

We assume that the points that are involved have homogeneous (rank-2) coordinates  $(x_a, 1), \dots, (x_f, 1)$ . A lifting of the six points corresponds to an assignment of an altitude  $h_a, \dots, h_f$  to each of the points. Assume that we have such a lifting such that the lifted points  $a, b, c$  are collinear. The lifted points have homogeneous coordinates  $(x_a, 1, h_a), (x_b, 1, h_b), (x_c, 1, h_c)$ . Their collinearity is expressed as

$$0 = \det \begin{pmatrix} x_a & 1 & h_a \\ x_b & 1 & h_b \\ x_c & 1 & h_c \end{pmatrix} = [ab]h_c - [ac]h_b + [bc]h_a.$$

We get similar expressions for the other four lines. In a lifting these four conditions have to be satisfied simultaneously. Thus any lifting corresponds to a solution of the linear system of equations



**Fig. 8.3** Two ways of generating a quadrilateral set.

$$\begin{pmatrix} +[bc] & -[ac] & +[ab] & 0 & 0 & 0 \\ +[ef] & 0 & 0 & 0 & -[af] & +[ae] \\ 0 & +[df] & 0 & -[bf] & 0 & +[bd] \\ 0 & 0 & +[de] & -[ce] & +[cd] & 0 \end{pmatrix} \cdot \begin{pmatrix} h_a \\ h_b \\ h_c \\ h_d \\ h_e \\ h_f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order to get nontrivial liftings, the solution space of this system has to have at least dimension three (having all lifted points in trivial position on a single line already accounts for two dimensions). Thus if the points are liftable, the matrix must have rank at most 3. Thus any  $4 \times 4$  subdeterminant must vanish. Considering the last four columns, we get

$$\det \begin{pmatrix} +[ab] & 0 & 0 & 0 \\ 0 & 0 & -[af] & +[ae] \\ 0 & -[bf] & 0 & +[bd] \\ +[de] & -[ce] & +[cd] & 0 \end{pmatrix} = [ab] \cdot ([ae][bf][cd] - [af][bd][ce]).$$

Since  $[ab]$  was assumed to be nonzero, the vanishing of this determinant implies our desired characterization:

$$[ae][bf][cd] = [af][bd][ce].$$

Points on a line that satisfy this relation are called a *quadrilateral set* (or *quadset*, for short). By considering other columns of our matrix, we could derive similar-looking equivalent bracket expressions.

Here is another way of deriving the same formula from a different geometric situation. The right picture of Figure 8.3 shows the dual situation of the one we have considered so far. We draw the six lines through four points  $x, z, p, q$  and intersect these lines with another line. The points of intersection are called  $a, \dots, f$ . For this configuration we consider the equalities of cross-ratios

$$(a, c; e, d) = (a, y; x, o) = (a, b; f, d).$$

The first equality comes from projection through point  $p$ ; the second is a consequence of projection through  $q$ . Expanding the cross-ratios, we get

$$\frac{[ae][cd]}{[ad][ce]} = \frac{[af][bd]}{[ad][bf]}.$$

Canceling  $[ad]$  and multiplying by the denominators, we get the desired equation

$$[ae][bf][cd] = [ce][af][bd].$$

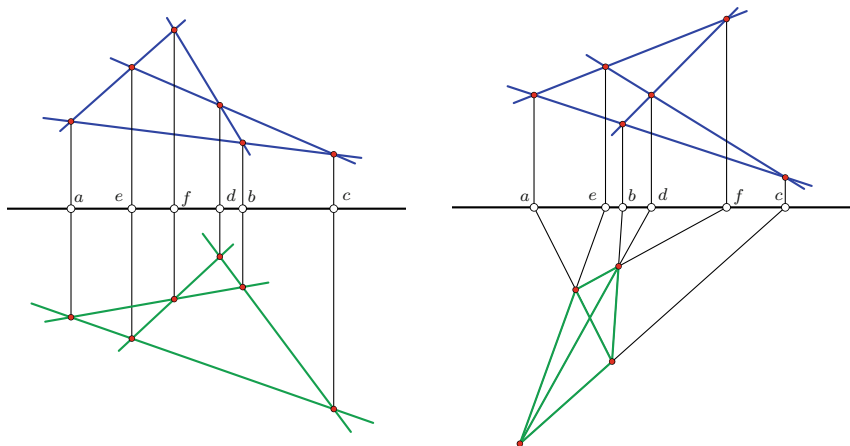
### 8.3 Symmetry and Generalizations of Quadrilateral Sets

The quadrilateral set configuration has interesting inner structures. In our labeling there are three pairs of points  $(a, d)$ ,  $(b, e)$ , and  $(c, f)$  such that the points of each pair do not share a line of the configuration. Every line of the complete quadrilateral is obtained by selecting exactly one point from each of the pairs. For instance, the line  $(a, b, c)$  takes the first point of each of the pairs. In our expression

$$[ae][bf][cd] = [ce][af][bd],$$

the line  $(a, b, c)$  plays a special role. The two monomials of the expression are formed by three brackets, and each of these brackets contains exactly one point of the line and one other point such that they do not form one of the three pairs above. For each line we get exactly one such characterization of the quadrilateral set.

Besides the four lines in our original quadrilateral set there are also four other lines that can be formed by taking exactly one point from each of the pairs. These lines  $(d, e, f)$ ,  $(a, b, f)$ ,  $(b, c, d)$ , and  $(a, c, e)$  describe the *associated complete quadrilateral* to our original one. The symmetry of  $[ae][bf][cd] = [ce][af][bd]$  also implies that this expression is as well a characterization of the quadrilateral set generated by the *associated complete quadrilateral*. The situation is illustrated in Figure 8.4 left. The blue part of the picture is our original quadrilateral set configuration, the green part is the complementary one. In particular, if we interchange the roles of two points in one of the pairs  $(a, d)$ ,  $(b, e)$ ,  $(c, f)$ , then we transfer the complete quadrilateral into its associated one. This implies that the triple of pairs characterizes the quadrilateral set. The right part of Figure 8.4 is an illustration of obtaining a quadrilateral set by projection or by intersection. Both pictures represent projective incidence theorems. If all coincidences except the last one are satisfied, then the last one is satisfied automatically.



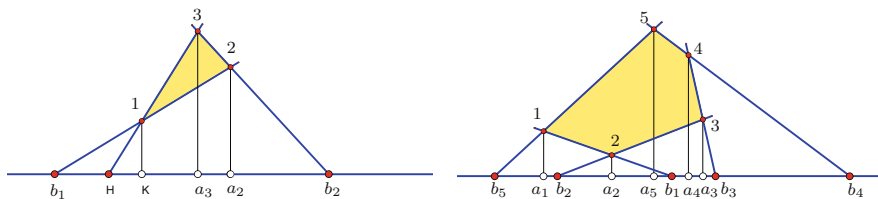
**Fig. 8.4** Incidence theorems from quadrilateral sets.

Finally, we want to explore how the notion of quadrilateral sets can be generalized. One way of doing this is to use one of the lines of the complete quadrilateral itself for the projection. Figure 8.5 left shows the situation for the usual quadrilateral set. The six points on the base line are the three projections  $a_1, a_2, a_3$  of the points of a triangle and the intersections  $b_1, b_2, b_3$  with its sides. We get the equation

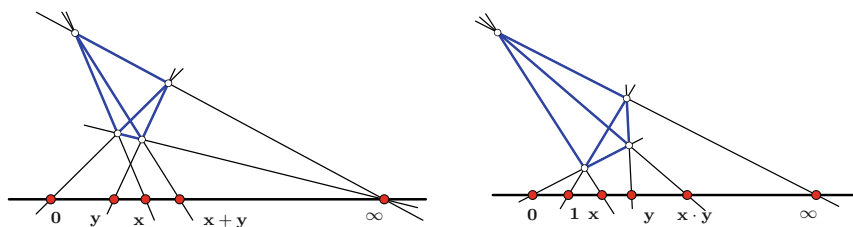
$$[a_1, b_1][a_2, b_2][a_3, b_3] = [a_1, b_3][a_2, b_1][a_3, b_2].$$

If we consider an arbitrary  $n$ -gon with vertices  $1, \dots, n$ , the projections of its vertices  $a_1, \dots, a_n$  to a line  $\ell$ , and the intersections  $b_i = \ell \cap (i \vee (i + 1))$  (indices modulo  $n$ ), then we get the equation

$$\prod_i [a_i, b_i] = \prod_i [a_i, b_{i-1}].$$



**Fig. 8.5** Sections of an  $n$ -gon.



**Fig. 8.6** Geometric addition and multiplication, with a finite point  $\infty$ .

This formula can be proven easily by the techniques we used for the characterization of quadrilateral sets. A nontrivial lifting of the segment  $(i, i + 1)$  implies the existence of nonzero altitudes  $h_i, h_{i+1}$  such that

$$\frac{[a_i, b_i]}{[a_{i+1}, b_i]} = \frac{h_i}{h_{i+1}}.$$

Forming the product over all  $i$  yields (indices modulo  $n$ )

$$\prod_{i=1}^n \frac{[a_i, b_i]}{[a_{i+1}, b_i]} = \prod_{i=1}^n \frac{h_i}{h_{i+1}} = 1,$$

which is equivalent to the desired formula.

## 8.4 Quadrilateral Sets and von Staudt

Let us reconsider the original von Staudt constructions we got to know in Section 5.6. The von Staudt constructions provided a tool for performing addition and multiplication with respect to a projective basis  $\mathbf{0}, \mathbf{1}, \infty$  on a line. Figure 8.6 shows the situation with the point  $\infty$  moved to a finite position (compare the drawing with Figure 5.5). We observe that the relevant points of the calculation are the intersections with the six lines through four other points. In other words, we see that the von Staudt constructions are based on a quadrilateral set construction.

What makes von Staudt constructions work is the fact that (in the usual identification of the projective line with  $\mathbb{R} \cup \{\infty\}$ ) the following triples of (ordered) pairs define quadrilateral sets:

$$((0, x + y); (x, y); (\infty, \infty)) \quad \text{and} \quad ((0, \infty); (x, y); (1, x \cdot y)).$$

The reader is invited to check this fact explicitly by hand calculation.

## 8.5 Slope Conditions

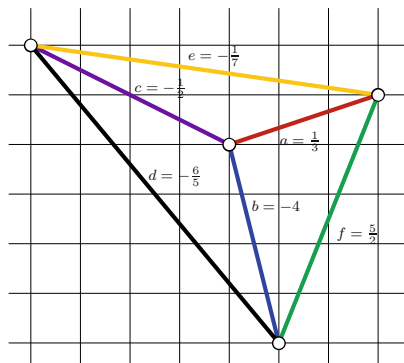
We continue to play with quadrilateral sets. What happens, for instance, if we intersect the six lines through four (finite) points with the line at infinity? In this case we get a quadrilateral set on the line at infinity. From a projective point of view this is not at all a special case. However, we want to interpret the quadrilateral set from the perspective of a special coordinatization. With respect to the usual standard embedding of the Euclidean plane at the affine  $z = 1$  plane, infinite points have coordinates of the form  $(x, y, 0)$ . A finite line  $\ell$  with equation  $ax + by + c = 0$  intersects the line at infinity in the point  $(b, -a, 0)$ . We could rewrite the line equation as  $y = -\frac{a}{b}x - \frac{c}{b}$ . Thus the intersection point has (after rescaling of homogeneous coordinates) the coordinates  $(1, s, 0)$ , where  $s$  is the slope of the line. If we choose the basis  $\mathbf{0} = (1, 0, 0)$ ,  $\infty = (0, 1, 0)$ , and  $\mathbf{1} = (1, 1, 0)$  for the line at infinity, the parameter of a point with respect to this basis is exactly the slope of the line bundle passing through it.

Taking these considerations into account, we can, after four finite points are given, read off a quadrilateral set condition for the slopes of six lines spanned by them. Figure 8.7 illustrates this fact. If  $a, \dots, f$  are the slopes of the three lines, the quadrilateral set condition reads

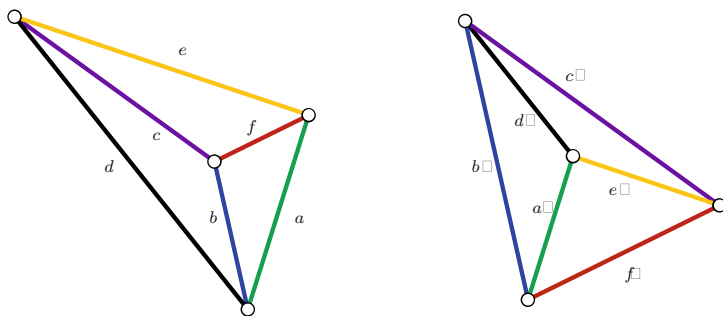
$$(a - e) \cdot (b - f) \cdot (c - d) = (a - f) \cdot (b - d) \cdot (c - e).$$

In the concrete example we get:

$$\left(\frac{1}{3} - \left(-\frac{1}{7}\right)\right) \cdot \left(-5 - \frac{5}{2}\right) \cdot \left(-\frac{1}{2} - \left(-\frac{6}{5}\right)\right) = \left(\frac{1}{3} - \frac{5}{2}\right) \cdot \left(-5 - \left(-\frac{6}{5}\right)\right) \cdot \left(-\frac{1}{2} - \left(-\frac{1}{7}\right)\right),$$



**Fig. 8.7** Line slopes between four points.

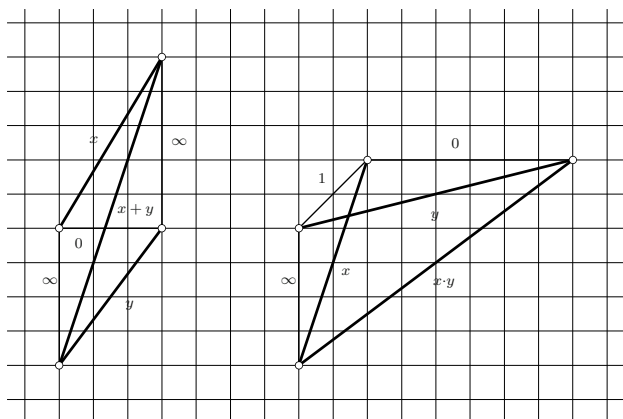


**Fig. 8.8** Two combinatorially different drawings with parallel slopes.

which is reduced to the identity

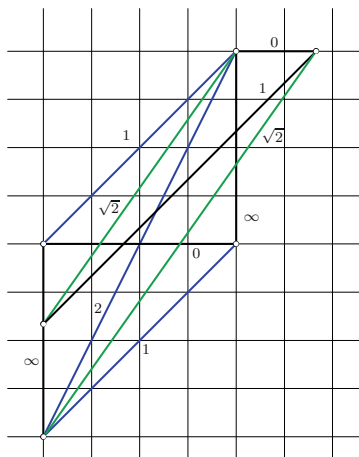
$$\frac{10}{21} \cdot \left(-\frac{13}{2}\right) \cdot \frac{7}{10} = -\frac{13}{6} \cdot \left(-\frac{14}{5}\right) \cdot \left(-\frac{5}{14}\right).$$

The relation of slopes and quadrilateral sets can be used to derive interesting theorems in affine geometry (in which parallels can be used as a primitive predicate). In Figure 8.4 (left) we illustrated the fact that quadrilateral sets arise in two combinatorially different ways as projections of the intersections of four lines. This translates to the fact illustrated in Figure 8.8. In this picture, lines with identical colors are parallel. If four points in the plane are given the slopes of the six lines spanned by them form a quadrilateral set. Thus we can find lines that are parallel to these lines that form a combinatorially different drawing of the slopes between four points. The corresponding



**Fig. 8.9** Addition and multiplication of line slopes.





**Fig. 8.10** Constructing a slope of  $\sqrt{2}$ .

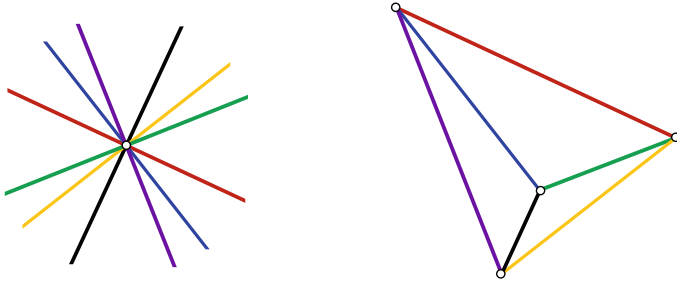
lines that pass through a point in the left picture form a triangle in the right picture.

Combining our considerations on slopes and the relationships between quadrilateral sets and von Staudt constructions we can also perform geometric addition and multiplication on the level of slopes. Figure 8.9 shows the two corresponding configurations. After fixing lines with slopes 0 and  $\infty$ , the left drawing demonstrates how to perform addition of two slopes  $x$  and  $y$ . In the right drawing, furthermore, a line with slope 1 is fixed. The construction forces multiplication of the slopes  $x$  and  $y$ . In both cases it is very easy to prove the relations of slopes by elementary considerations.

As an example of combining several slope addition and multiplication devices, the drawing in Figure 8.10 shows a configuration in which after fixing the slopes 0, 1, and  $\infty$ , some of the lines are forced to have slope  $\sqrt{2}$ . One subconfiguration is used to perform the operation  $1 + 1 = 2$ , and another configuration is used to calculate  $x \cdot x = 2$ . The slope  $x$  is then forced to be either  $\sqrt{2}$  or  $-\sqrt{2}$ .

## 8.6 Involutions and Quadrilateral Sets

There is also a very interesting connection of quadrilateral sets to projective transformations on a line. For this we have to consider projective involutions  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ . The defining property of an involution is that  $\tau(\tau(p)) = p$  for every point  $p$ . Every projective involution on  $\mathbb{RP}^1$  can be expressed by multiplication by a  $2 \times 2$  matrix  $T$  that satisfies  $T^2 = \lambda \cdot Id$ . Involutions are closely related to geometric reflections, since the characterizing property of a



**Fig. 8.11** Pairs of orthogonal slopes: The altitudes meet in a point.

reflection is that the mirror image of the mirror image is the original again. We now get the following result:

**Theorem 8.2.** *Let  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  be a projective involution that is not the identity and let  $a, b, c$  be arbitrary points in  $\mathbb{RP}^1$ . Then the pairs*

$$(a, \tau(a)), (b, \tau(b)), (c, \tau(c))$$

*form a quadrilateral set.*

*Proof.* Since  $T$  is an involution, we have  $T^2 = \lambda \cdot \text{Id}$ . This implies  $\det(T)^2 = \det(T^2) = \det((\lambda \cdot \text{Id})) = \lambda^2$ . Hence  $\det(T)$  is either  $+\lambda$  or  $-\lambda$ . We will first exclude the case  $\det(T) = \lambda$ . For this, the fact that  $T$  is a  $2 \times 2$  matrix is crucial. Let  $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . We then get

$$T^2 = \begin{pmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \alpha\gamma + \delta\gamma & \gamma\beta + \delta^2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

This can be satisfied only if either  $\alpha = -\delta$  or  $\gamma = \beta = 0$ . The first case leads to  $\det(T) = -\lambda$  (check it). The second case implies  $\alpha^2 = \delta^2$ . Since again  $\tau$  was assumed not to be the identity, this case implies  $\alpha = -\delta$ , which again implies (together with  $\beta = \gamma = 0$ ) the equation  $\det(T) = -\lambda$ . We now consider the monomial

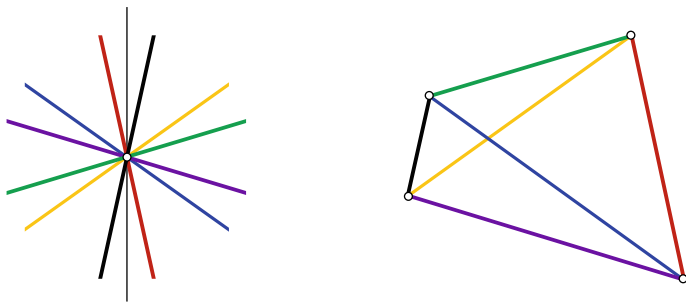
$$[a, Tb][b, Tc][c, Ta].$$

Applying the transformation  $T$  to each of the points of this monomial transfers this monomial to

$$[Ta, T^2b][Tb, T^2c][Tc, T^2a] = [Ta, \lambda b][Tb, \lambda c][Tc, \lambda a] = -\lambda^3 [b, Ta][c, Tb][a, Tc].$$

On the other hand, the transferred monomial must satisfy the equation

$$[Ta, T^2b][Tb, T^2c][Tc, T^2a] = (\det(T))^3 [a, Tb][b, Tc][c, Ta].$$



**Fig. 8.12** Mirroring slopes.

Comparing these two terms and using the fact  $\det(T) = -\lambda$ , we get

$$[a, Tb][b, Tc][c, Ta] = [a, Tc][b, Ta][c, Tb].$$

This is exactly the characterization of a quadrilateral set.  $\square$

Let us use this fact to derive immediate conclusions about configurations that concern the slopes of lines. For this, we consider involutions on the line at infinity. We will consider two natural types of involution that are related to operations in the *Euclidean* plane. The first is a rotation by  $90^\circ$  around an arbitrary point. Such a rotation is clearly an involution. It induces an involution on the line at infinity. Slopes are transferred by such a rotation to perpendicular slopes. For our incidence theorem we start with three arbitrary line slopes (compare Figure 8.11). In the picture they are black, yellow, and green. Then we construct line slopes that are perpendicular to these slopes. By the above theorem, the six slopes form a quadrilateral set. Thus one can draw a projection of a tetrahedron with exactly these line slopes. Lines that do not share a point in the tetrahedron are perpendicular to each other. Looking at the right part of Figure 8.11, one observes that this statement is nothing but a strange way to derive a well-known result: *the altitudes in a triangle meet in a point*.

There is another theorem that is not known so well that can be derived by the same argument. For this we consider an involution that arises from a line reflection. Figure 8.12 on the left shows six lines that are pairwise in a mirror relation to each other. The mirror axis is the thin black line. Since a mirror symmetry is an involution, the theorem implies that the six line slopes form a quadrilateral set. Hence again they can be used to form a drawing of a tetrahedron.

Theorem 8.2 showed that there is a close relationship between quadrilateral sets and projective involutions. Any three pairs of images and preimages of a projective involution on a line form a quadrilateral set. The converse is also

true. For this we first observe that if a projective map in  $\mathbb{RP}^1$  interchanges two points, then it will automatically be an involution.

**Lemma 8.1.** *Let  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  be a projective transformation with  $\tau(a) = a'$  and  $\tau(a') = a$  for distinct points  $a$  and  $a'$ . Then  $\tau$  is an involution.*

*Proof.* Without loss of generality we may (after a suitable projective transformation) assume that  $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $a' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Thus the matrix  $T$  that represents  $\tau$  must have the form  $T = \begin{pmatrix} \alpha & \beta \\ \beta & 0 \end{pmatrix}$  for nonzero parameters  $\alpha$  and  $\beta$ . Calculating  $T^2$ , we get

$$T^2 = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix} = \alpha\beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus  $T^2$  represents the identity map on  $\mathbb{RP}^1$ . □

With this lemma it is easy to prove that every quadrilateral set induces an involution.

**Theorem 8.3.** *Let  $a, b, c$  be three distinct points in  $\mathbb{RP}^1$ . Let  $a'b'c'$  be three additional points. If  $(a, a'); (b, b'); (c, c')$  forms a quadrilateral set, then the projective map  $\tau$  uniquely defined by  $\tau(a) = a'$ ,  $\tau(b) = b'$ ,  $\tau(c) = c'$  is an involution.*

*Proof.* If  $a = a'$ ,  $b = b'$ ,  $c = c'$ , then  $\tau$  must be the identity, which is trivially an involution. Thus at least one of the pairs consists of different points. Assume without loss of generality that  $a$  and  $a'$  are distinct. Consider the uniquely defined transformation  $\tau$  that satisfies  $\tau(a) = a'$ ,  $\tau(a') = a$ ,  $\tau(b) = b'$ . By Lemma 8.1 this transformation will be an involution. Thus it suffices to show that  $\tau(c) = c'$ . Theorem 8.2 implies that

$$(a, \tau(a)), (b, \tau(b)), (c, \tau(c))$$

form a quadrilateral set. Using our knowledge about  $\tau$ , we see that

$$(a, a'), (b, b'), (c, \tau(c))$$

is a quadrilateral set. Since five points of a quadrilateral set determine the sixth one uniquely, we must have that  $\tau(c) = c'$ . This proves the theorem. □

So to every quadrilateral set we can associate in a natural way an involution. It is a remarkable fact that the two fixed points of the involution are in harmonic position to all point pairs in the quadrilateral set.

If  $T$  represents a projective transformation, then the eigenvectors of  $T$  correspond to the fixed points of the transformation. For every eigenvector  $p$  of  $T$  we have  $Tp = \lambda p$ , and  $p$  is mapped to itself. If  $T$  is a  $2 \times 2$  matrix, then it may have either two real or two complex conjugate eigenvectors (up to

scalar multiples). If the eigenvectors are real, then they correspond to points in  $\mathbb{RP}^1$  that are invariant under  $T$ .

**Theorem 8.4.** *Let  $\tau$  be the involution associated to the points of a quadrilateral set  $(a, a'; b, b'; c, c')$ . Assume that  $\tau$  has two distinct real fixed points  $p$  and  $q$ . Then the point pairs  $(p, q)$  and  $(a, a')$  form a harmonic set.*

*Proof.* If  $\tau$  is an involution, then  $(a, \tau(a); p, \tau(p); q, \tau(q))$  is a quadrilateral set. Using our knowledge of the definition of  $\tau$  and the fixed point properties of  $p$  and  $q$ , we see that  $(a, a'; p, p; q, q)$  is a quadrilateral set. Thus we have  $[a, p][p, q][q, a'] = [a, q][p, a'][q, p]$ . Using the distinctness of  $p$  and  $q$  we can cancel  $[p, q]$  from this expression and are left with  $[a, p][q, a'] = -[a, q][p, a']$ , which is exactly the characterization for a harmonic pair of point pairs.  $\square$

Clearly, in the same way the pair of points  $(p, q)$  is also harmonic with respect to  $(b, b')$  and to  $(c, c')$ . This leads us to a nice characterization of fixed points of  $\tau$ . If  $\tau$  is an involution associated to  $(a, a'; b, b'; c, c')$ , then the fixed points of  $\tau$  are exactly those two points that are simultaneously harmonic to all three point pairs of the quadrilateral set. In fact, any two of the point pairs of the quadrilateral set determine the position of  $p$  and  $q$  uniquely. The fact that  $(p, q)$  are also harmonic with respect to the last pair is exactly the quadrilateral set condition.

There is a little subtlety concerning the existence of the fixed points that will become relevant later in this book: The fixed points need not be real. We saw that the fixed points correspond to the eigenvectors of the matrix  $T$  that represents  $\tau$ . If this matrix has complex eigenvalues (like the involution  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ), then the eigenvectors are also complex (and cannot become real by multiplication by a real scalar). In such a case the transformation has no real fixed points. This case geometrically is related to “rotation-like” transformations  $\tau$  such as the  $90^\circ$  rotation we used in Figure 8.11. The case in which real fixed points exist is related to “reflection-like” transformations, such as the mirror-image operation we used in Figure 8.12.



## Conics and Their Duals

*You always admire what you really don't understand.*

*Blaise Pascal*

So far, we have dealt almost exclusively with situations in which only points and lines were involved. Geometry would be quite a pure topic if these were the only objects to be treated. Large parts of classical elementary geometry deal with constructions involving *circles*. The most elementary drawing tools treated by Euclid (the *straightedge* and the *compass*) contain a tool for generating circles. In a sense, so far we have dealt with the straightedge alone. Unfortunately, circles are not a concept of projective geometry. This can easily be seen by observing that the shape of a circle is not invariant under projective transformations. If you look at a sheet of paper on which a circle is drawn from a skew angle, you will see an ellipse. In fact, projective transformations of circles include ellipses, hyperbolas, and parabolas. They are subsumed under the term *conic sections*, or *conics*, for short. Conics are the concept of projective geometry that comes closest to the concept of circles in Euclidean geometry. It is the purpose of this section to give a purely projective treatment of conics. Later on, we will see how certain specializations provide interesting insights into the geometry of circles.

### 9.1 The Equation of a Conic

Let us start with the unit circle in the Euclidean plane:

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

In this section we will investigate which shapes can arise if we transform this circle projectively. For this we again consider the Euclidean plane embedded at the  $z = 1$  plane of  $\mathbb{R}^3$  and represented by homogeneous coordinates (any other affine embedding not containing the origin would serve as well and would lead to the same result). Thus the points of the circle correspond to points with homogeneous coordinates  $(x, y, 1)$  with  $x^2 + y^2 = 1$ . Taking into account that homogeneous coordinates are specified only up to a scalar multiple, we may rewrite this condition in a more general way as points  $(x, y, z)$  with  $x^2 + y^2 = z^2$ . Setting  $z = 1$ , we obtain the original formula. According to the fact that every term of the expression  $x^2 + y^2 = z^2$  is quadratic, a vector  $(x, y, z)$  satisfies this expression if  $(\lambda x, \lambda y, \lambda z)$ ,  $\lambda \neq 0$ , satisfies it. We may rewrite the quadratic equation as

$$(x, y, z) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Thus the homogeneous coordinate vectors that satisfy this equation are exactly those that represent points of our circle. We now want to transform these points projectively. Applying a projective transformation to the points can be carried out by replacing both occurrences of the vector  $p = (x, y, z)$  by a transformed vector  $M \cdot p$ . Thus we obtain that a projectively transformed unit circle can be expressed algebraically as the solutions of the equation

$$((x, y, z) \cdot M^T) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot (M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}) = 0,$$

where  $M$  is a real  $3 \times 3$  matrix with nonzero determinant. Multiplying the three matrices in the middle of this expression, we are led to an equation

$$(x, y, z) \cdot \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \quad (*)$$

for suitable parameters  $a, b, c, d, e, f$ . Observe that the matrix in the middle is necessarily symmetric. Expanding the above product expression yields the following quadratic equation:

$$a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + 2b \cdot xy + 2d \cdot xz + 2e \cdot yz = 0.$$

We call such an expression of the form  $p^T A p$  a *quadratic form* (regardless of whether the matrix  $A$  is symmetric). The set of points that satisfies such an equation will be called a *conic*. In a sense we are working on three different levels when we speak about matrices, quadratic forms, and conics. Before we continue, we want to clarify this relationship. Let  $A$  be a  $3 \times 3$  matrix with entries in some field  $\mathbb{K}$ . The associated quadratic form is a homogeneous



quadratic function  $\mathcal{Q}_A : \mathbb{K}^3 \rightarrow \mathbb{K}$  defined by

$$\mathcal{Q}_A(p) = p^T A p.$$

It is important to notice that different matrices may lead to the same quadratic form. This can be seen by expanding

$$\mathcal{Q}_A = (x, y, z) \cdot \begin{pmatrix} a & b_1 & d_2 \\ b_2 & c & e_1 \\ d_1 & e_2 & f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to

$$\mathcal{Q}_A = a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + (b_1 + b_2) \cdot xy + (d_1 + d_2) \cdot xz + (e_1 + e_2) \cdot yz.$$

Observe that the quadratic form associated to a matrix depends only on the diagonal entries and the sums of matrix entries  $A_{ij} + A_{ji}$ . For every potentially nonsymmetric matrix  $A$ , the matrix  $(A + A^T)/2$  creates the same quadratic form:

$$\mathcal{Q}_A = \mathcal{Q}_{(A+A^T)/2}.$$

We call this process *symmetrization*. Furthermore, it is also clear that for every homogeneous quadratic polynomial  $f(x, y, z)$ , a (symmetric) matrix  $A$  with  $\mathcal{Q}_A(x, y, z) = f(x, y, z)$  also exists.

The conics themselves are the third level with which we have to deal. They consist of all points in  $\mathcal{P}_{\mathbb{K}}$  for which a quadratic form vanishes. The conic associated to a matrix (or quadratic form) is

$$\mathcal{C}_A = \{[p] \in \mathcal{P}_{\mathbb{K}} \mid p^T A p = 0\}.$$

Since the expression  $p^T A p = 0$  is stable under scalar multiplication of  $p$  by a nonzero scalar, we will, by common abuse of notation, work as usual on the level of homogeneous coordinates and omit the brackets  $[\dots]$ . How much information about the matrix  $A$  is still present in the conic in a sense depends on the underlying field. Over  $\mathbb{R}$  it may, for instance, happen that a conic (such as the solution set of  $x^2 + y^2 + z^2 = 0$ ) has no nonzero solutions at all. In such a case there are still complex solutions. We will deal with them later.

We return to the case of a projectively transformed circle and the quadratic form described in equation (\*). In this case the parameters  $a, \dots, f$  are not completely independent. *Sylvester's law of inertia* from linear algebra tells us that the signature of the eigenvalues must be the same as in the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and that this is the only restriction.<sup>1</sup>

If we allow general real parameters in equation (\*), we can classify the different cases by the signature of the eigenvalues. Since the quadratic forms of the matrices  $A$  and  $-A$  describe identical point sets, we may identify a signature vector with its negative. Thus we end up with five essentially different cases of signatures:

$$(+, +, -), \quad (+, +, +), \quad (+, -, 0), \quad (+, +, 0), \quad (+, 0, 0).$$

Each of these cases describes a geometric situation that cannot be transformed projectively into any of the other cases. The case of a circle corresponds to the first entry in the list. The last three cases correspond to *degenerate* conics (the determinant of  $A$  is zero), and the second case corresponds to a situation like  $x^2 + y^2 + z^2 = 0$ , in which we have no real solutions at all (but where complex solutions still exist). Before we clarify the geometric meaning of the other four cases in detail, we will have a closer look at the circular case. Projectively, all quadratic forms with eigenvalue signature  $(+, +, -)$  have to be considered as isomorphic (they are the projective images of a unit circle). However, if we fix a certain embedding of the Euclidean plane into  $\mathbb{R}^3$  (say the standard  $z = 1$  embedding) and by this single out a specific line at infinity, then we may classify them also with respect to Euclidean motions. In fact, there is an infinite variety of forms of conics that are inequivalent under Euclidean transformations. Still there is a very useful (and well-known) coarser classification if we just count the intersections of the conic with the line at infinity. Intersecting the quadratic form of a projectively transformed circle

$$a \cdot x^2 + c \cdot y^2 + f \cdot z^2 + 2b \cdot xy + 2d \cdot xz + 2e \cdot yz = 0$$

and the line at infinity (i.e., setting  $z = 0$ ) leaves us with the equation

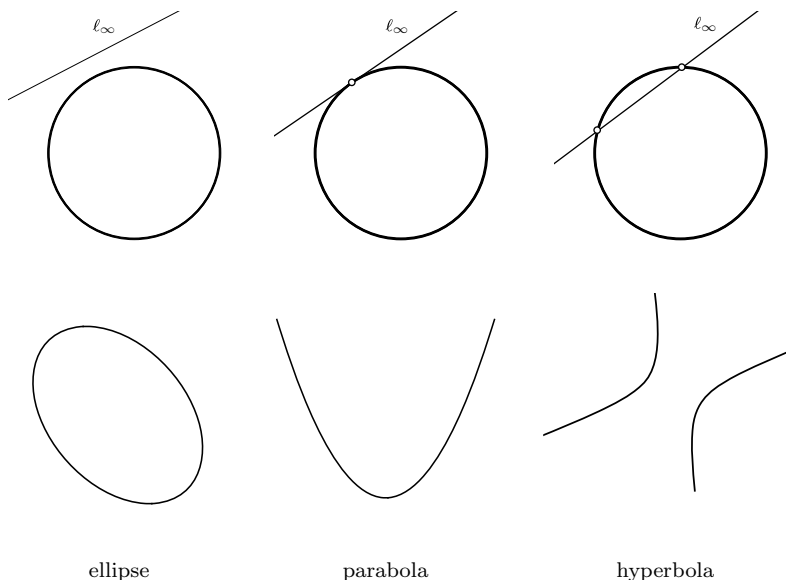
$$a \cdot x^2 + c \cdot y^2 + 2b \cdot xy = 0.$$

This homogeneous quadratic equation may lead to zero, one, or two solutions up to scalar multiples. The three cases are shown in Figure 9.1. They correspond to the well-known cases of ellipses, parabolas, and hyperbolas. Thus one could say that a parabola is a conic that *touches the line at infinity*, while a hyperbola has two points at infinity. The three cases can be algebraically distinguished by considering the discriminant of the equation  $a \cdot x^2 + c \cdot y^2 + 2b \cdot xy = 0$ . The sign of the discriminant turns out to be the sign of the determinant

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

---

<sup>1</sup> Sylvester's law of inertia states that if  $M$  is nonsingular, then  $A$  and  $M^T A M$  have the same signature of eigenvalues for any symmetric matrix  $A$ . Furthermore, this is the only restriction on the coefficients of the symmetric matrix  $M^T A M$ .



**Fig. 9.1** The possible images of a circle.

If this sign is negative we are in the hyperbolic case. If it is zero, we get a parabola, and if it is positive, we get an ellipse.

## 9.2 Polars and Tangents

For the moment we will stick to the case of nondegenerate conics that are projective images of a circle. Our next task will be to calculate a tangent to such a conic. For this we first need an algebraic characterization of a line being tangent to a conic. There are essentially two ways of doing this. The first is related to concepts of differential geometry: A line is tangent to a conic if at a point of intersection it has the same slope as the conic. The other approach is intersection-theoretic and uses the fact that we know that a conic is a quadratic curve: A line is tangent to a conic if it has exactly one point in common with it. The first approach is slightly more general, since it also covers the case of degenerated conics. Still we want to follow the second approach, since it fits smoothly into the concepts introduced so far, and we will generalize it later.

Before we will start investigating the intersection properties of lines and conics we have to recall a few facts concerning homogeneous quadratic equations. First consider the quadratic equation:

$$(\lambda, \mu) \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = a \cdot \lambda^2 + 2b \cdot \lambda\mu + c \cdot \mu^2 = 0.$$

Clearly if  $(\lambda, \mu)$  is a solution then any scalar multiple of it is a solution as well. If we as usual consider equivalence classes  $[(\lambda, \mu)]$  of solutions modulo nonzero scalars, this quadratic equation will have zero, one, or two solutions if at least one of the parameters  $a, b, c$  does not vanish. In this case the number of solutions depends on the sign of the determinant

$$\det \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

If the sign is zero we get exactly one solution; if it is negative we get two solutions; if it is positive we get no real solutions but two complex solutions. If all parameters  $a, b, c$  are zero, any  $(\lambda, \mu) \in \mathbb{R}^2$  will be a solution.

The second fact we need is that a quadratic form  $\mathcal{Q}_A(p)$  may factor into two linear terms. Then it has the form  $\langle p, l \rangle \cdot \langle p, g \rangle$  for two suitable vectors  $l$  and  $g$ . The quadratic form may then be written as  $\mathcal{Q}_{lg^T}(p)$ , since we have

$$p^T (lg^T) p = (p^T l) (g^T p) = \langle p, l \rangle \cdot \langle p, g \rangle.$$

Furthermore, if the corresponding conic contains a line with homogeneous coordinates  $l$ , then the quadratic form must necessarily factor to the above form for suitable  $g$ .

In order to approach the calculation of a tangent, we will study the possible types of intersections of a line given by a linear combination of two points  $a$  and  $b$  on it and a conic given by a symmetric matrix  $A$ . In what follows all coordinates of points and lines will be given by homogeneous coordinates, and we will frequently identify the points with their coordinate representations.

**Lemma 9.1.** *Let  $A$  be a symmetric real  $3 \times 3$  matrix and let  $l$  be a line  $\lambda a + \mu b$  given by two distinct points  $a$  and  $b$ . Then points  $q$  on the line that satisfy  $q^T A q = 0$  correspond to the solutions of the homogeneous system*

$$(\lambda, \mu) \cdot \begin{pmatrix} (a^T A a) & (b^T A a) \\ (b^T A a) & (b^T A b) \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0.$$

*In particular, we get exactly two (real or complex) solutions (up to scalar multiples) if and only if the determinant of the above matrix does not vanish.*

*Proof.* The proof goes simply by expansion of the formula  $(\lambda a + \mu b)^T A (\lambda a + \mu b) = 0$ , which describes the common points of the line and the quadratic form. Expanding it, we get

$$(\lambda, \mu) \cdot \begin{pmatrix} (a^T A a) & (b^T A a) \\ (a^T A b) & (b^T A b) \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = 0,$$

which is the desired formula except for one switch of  $a$  and  $b$  in the lower left entry. However, this can be done easily, since  $A$  was assumed to be symmetric. The second part of the theorem is just an application of the discriminant formula.  $\square$

Our next lemma states that matrices representing degenerate conics must have a vanishing determinant.

**Lemma 9.2.** *A quadratic equation  $q^T A q = 0$  with a symmetric  $3 \times 3$  matrix  $A$  with  $\det(A) \neq 0$  cannot be satisfied by all points of a projective line.*

*Proof.* If the solution space of the quadratic equation  $q^T A q = 0$  contains a whole line  $l$ , then one must be able to rewrite it as product of two linear factors  $\langle p, l \rangle \cdot \langle p, g \rangle = 0$ . One can rewrite this as

$$(q^T l) \cdot (g^T q) = 0.$$

This in turn can be interpreted as

$$q^T (l \cdot g^T) q = 0.$$

This is a quadratic form with a nonsymmetric rank-1 matrix  $M = l \cdot g^T$ . We can symmetrize this matrix by replacing it with  $M + M^T$ . That this matrix is singular will be shown by a simple case distinction. Case 1:  $g = \lambda l$  and the matrix  $M + M^T$  itself has rank 1. Case 2:  $g$  and  $l$  are linearly independent and

$$(M + M^T)(g \times l) = (l \cdot g^T + g \cdot l^T)(g \times l) = l \langle g, g \times l \rangle + g \langle l, g \times l \rangle = 0.$$

Thus in this case  $M + M^T$  must be degenerate, since it has the nontrivial element  $g \times l$  in its kernel. Since  $M + M^T$  must be a multiple of  $A$ , this proves the claim.  $\square$

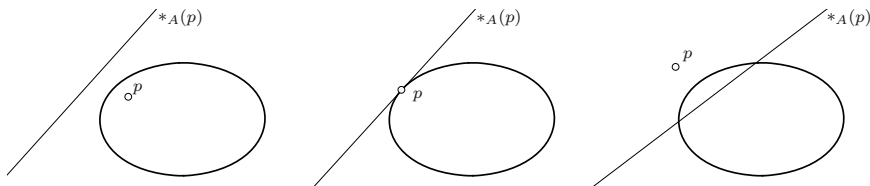
We now define tangency in terms of numbers of intersections for the non-degenerate case of  $\det(A) \neq 0$ . We postpone the definition of tangents in the degenerate case until later, since it is a little more subtle.

**Definition 9.1.** Let  $A$  be a symmetric real  $3 \times 3$  matrix with  $\det(A) \neq 0$ . A line  $l$  is *tangent* to the conic  $C_A$  if it has exactly one intersection with it.

The following theorem gives a recipe for calculating a tangent to a conic.

**Theorem 9.1.** *Let  $A$  be a symmetric real  $3 \times 3$  matrix with  $\det(A) \neq 0$  and let  $p$  be a point on the corresponding conic. Then  $A \cdot p$  gives homogeneous coordinates for the tangent at point  $p$  to the conic.*

*Proof.* We consider  $p \mapsto A \cdot p$  as a function of the space of points to the space of lines (represented by homogeneous coordinates). Since  $A$  is nondegenerate,



**Fig. 9.2** Images under polarity.

this map is bijective. First we show that the point  $p$  and the line  $l := A \cdot p$  coincide. For this, simply observe, that

$$\langle p, l \rangle = \langle p, A \cdot p \rangle = p^T A p = 0.$$

Now assume that there was a second point  $q$  that is on  $l$  as well as on the conic represented by  $A$ . If  $p$  and  $q$  represent different points, then  $\lambda p + \mu q$  must have more than one intersection with the conic. In this case we have the following three equations:  $p^T A p = 0$  (point  $p$  is on the conic),  $q^T A q = 0$  (point  $q$  is on the conic),  $q^T A p = 0$  (point  $q$  is on  $l$ ). Forming a linear combination of the first and the last equations yields  $(\lambda p + \mu q)^T A p = 0$  for arbitrary  $\lambda, \mu$ . Combining the second and the last yields  $q^T A(\lambda p + \mu q) = 0$ , which (by the symmetry of  $A$ ) gives  $(\lambda p + \mu q)^T A q = 0$ . Another combination shows that

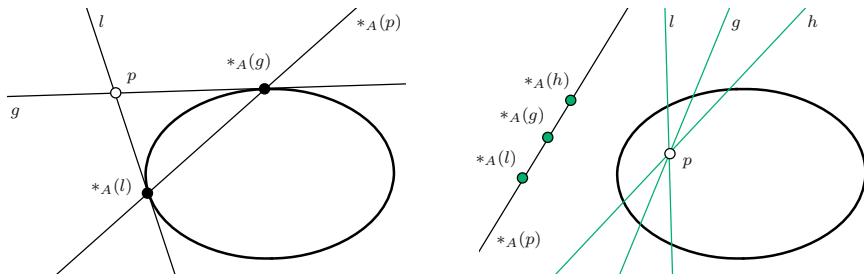
$$(\lambda p + \mu q)^T A(\lambda p + \mu q) = 0.$$

This proves that all linear combinations of  $p$  and  $q$  must be on the conic as well. Thus either  $p$  and  $q$  both represent the same point (and all linear combinations are also identical to this point) or the conic contains the entire line  $\lambda p + \mu q$ . The latter cannot happen by Lemma 9.2, since  $A$  was assumed to be nondegenerate.  $\square$

The last theorem gives a very simple recipe to calculate the homogeneous coordinates of a tangent  $l$  to a point  $p$  on a conic  $p^T A p = 0$ : simply calculate  $l = A p$ ! This function is also defined if  $p$  is not on the conic. The map  $p \mapsto A p$  is called a *polarity*, and it has various interesting properties. Figure 9.2 shows three drawings of a point and its polar line with respect to an ellipse. Observe that if the point is inside the ellipse, the corresponding polar lies entirely outside. If the point is outside the conic, then the polar intersects the conic in two points. In the limit situation when the point is on the ellipse, the polar is the tangent at this point. We want to investigate the properties of polarities in some detail.

**Definition 9.2.** Let  $A$  be a symmetric  $3 \times 3$  matrix with  $\det(A) \neq 0$ . The map

$$\begin{aligned} *A : \mathcal{P}_{\mathbb{R}} &\rightarrow \mathcal{L}_{\mathbb{R}}, \\ p &\mapsto A p, \end{aligned}$$



**Fig. 9.3** Properties of polarities.

is called a polarity. Since the domain and image space of  $*_A$  are disjoint, we extend  $*_A$  by its own inverse and define

$$\begin{aligned} *_A: \mathcal{L}_{\mathbb{R}} &\rightarrow \mathcal{P}_{\mathbb{R}}, \\ l &\mapsto A^{-1}l. \end{aligned}$$

Polarities are very closely related to projective transformations. However, they map points to lines instead of points to points.

**Theorem 9.2.** *Let  $A$  be a symmetric matrix with  $\det(A) \neq 0$  and let  $\mathcal{C}_A$  be the associated conic. The polarity  $*_A$  has the following properties:*

- (i) *For any element  $a \in \mathcal{P}_{\mathbb{R}} \cup \mathcal{L}_{\mathbb{R}}$  we have  $*_A(*_A(a)) = a$ .*
- (ii) *Three points  $a, b, c \in \mathcal{P}_{\mathbb{R}}$  are collinear if and only if the lines  $*_A(a), *_A(b), *_A(c) \in \mathcal{L}_{\mathbb{R}}$  are concurrent.*
- (iii) *Three lines  $a, b, c \in \mathcal{L}_{\mathbb{R}}$  are concurrent if and only if the points  $*_A(a), *_A(b), *_A(c) \in \mathcal{P}_{\mathbb{R}}$  are collinear.*
- (iv)  *$p \in \mathcal{P}_{\mathbb{R}}$  and  $l \in \mathcal{L}_{\mathbb{R}}$  are incident if and only if  $*_A(p)$  and  $*_A(l)$  are incident.*
- (v)  *$p$  and  $*_A(p)$  are incident if and only if  $p$  is on  $\mathcal{C}_A$ . Then  $*_A(p)$  is the tangent to  $p$  at the conic  $\mathcal{C}_A$ .*

*Proof.* (i) is clear, since  $A^{-1}A = AA^{-1} = E$ . (ii) and (iii) follow from the facts that collinearity/concurrence can be expressed by the condition  $\det(a, b, c) = 0$  and that this condition is invariant, since  $\det(A) \neq 0$  is equivalent to  $\det(Aa, Ab, Ac) = 0$ . (iv) is the equivalence of  $\langle p, l \rangle = 0$  and  $\langle Ap, A^{-1}l \rangle = 0$ . For (v) the incidence of  $p$  and  $Ap$  states simply that  $p^T Ap = 0$ , which means that  $p$  is on the conic. The property of being tangent is exactly the statement of Theorem 9.1.  $\square$

Polarities are in a very concrete sense a representation of the dual character of projective geometry. They give a concrete dictionary of how to translate

statements of projective geometry into their dual statements. For every non-degenerate matrix  $A$  we obtain such a dictionary. For a point  $p$  we will call  $l = Ap$  the *polar* of  $p$ . Likewise,  $p$  is called the *pole* of  $l$ .

Algebraically, multiplication by  $A$  (or  $A^{-1}$ ) is the simplest way to carry out a polarity. However, property (v) of the previous theorem gives us another possibility to construct a polar of a point  $p$  geometrically if we are able to construct a tangent to a conic. This is particularly easy if the point  $p$  is “outside” the conic (i.e., the polar of  $p$  intersects the conic in two points). Then we simply have to draw the two tangents of  $p$  to the conic. The points where they touch the conic are the poles of these lines. If we join these two points, then we get the polar of the original point. Figure 9.3 on the left shows this procedure. Algebraically, this construction is explained by the fact that if  $p$  is a point and  $l$  and  $g$  are two lines through this point, then by Theorem 9.2 (iv) the line  $*_A(p)$  is incident to the points  $*_A(l)$  and  $*_A(g)$ . Since  $l$  and  $g$  are the tangents, then by Theorem 9.2 (v) the polars are the corresponding touching points.

We can also reverse the construction in different ways. If the line  $*_A(p)$  is given, we can construct its pole  $p$  by intersection of the two tangents. Furthermore, if, for instance, in a computer geometry system a method for calculating the polar and pole is available (multiplication by  $A$  or  $A^{-1}$ ) and if it is possible to intersect a conic with a line, then this can be used to construct the tangents through a point  $p$  to a conic. Simply intersect the polar of the point with the conic and join the intersections with point  $p$ .

Figure 9.3 on the right demonstrates that concurrent lines lead to collinear poles. This property is also the key to constructing the polar of a point “inside” a conic. For this, one must simply draw two lines incident to the point, construct their poles, and join them.

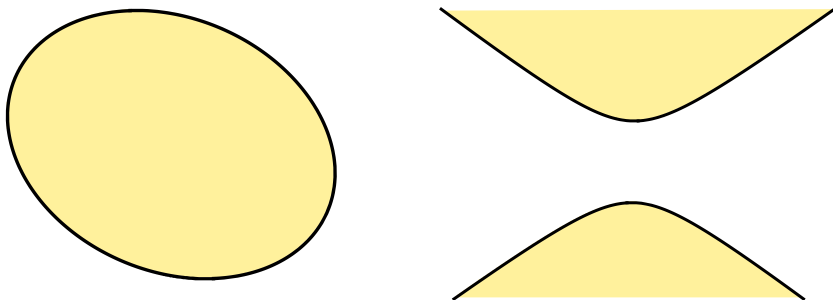
So far, we have used the terms “inside” and “outside” of a conic in a kind of informal way by always drawing images of an ellipse. In fact, one can formalize this concept by referring to the number of intersections of the conic with the corresponding polar.

**Definition 9.3.** For a  $3 \times 3$  symmetric matrix  $A$  with  $\det(A) \neq 0$ , a point  $a$  is *inside* the conic defined by  $p^T Ap = 0$  if the conic does not intersect the polar  $*_A(a)$ . If the polar intersects the conic in two (real) points, then the point  $a$  is *outside* the conic (compare Figure 9.4).

### 9.3 Dual Quadratic Forms

In Chapters 2 and 3 we learned that projective geometry is a dual theory. The roles of points and lines, meets and joins, etc. are interchangeable. In the previous section we saw how we can make the duality explicit using a





**Fig. 9.4** Inside an ellipse and inside a hyperbola.

polarity with respect to a conic. There must also be a dual counterpart of the concept of a conic. This will be defined in this section. Again we will deal with the nondegenerate case first.

A conic consists of all points  $p$  that satisfy an equation  $p^T A p = 0$ . The set of all tangents to this conic can be described as  $\{A p \mid p^T A p = 0\}$ . We can describe this set of homogeneous coordinates for lines directly as a quadratic form by the following observation:

$$p^T A p = p^T A A^{-1} A p = p^T A^T A^{-1} A p = (A p)^T A^{-1} (A p).$$

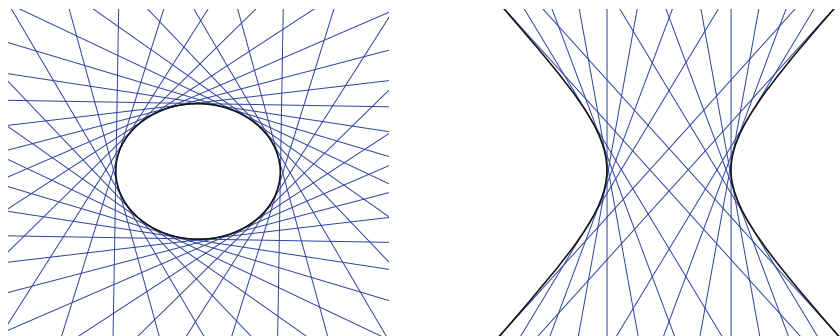
The right side of the equation explains how the set of tangent lines of a conic may be directly interpreted as the zero set of a quadratic form with matrix  $A^{-1}$ . Thus we obtain that the set of all tangents is described by

$$\mathcal{C}_A^* := \{[l] \in \mathcal{L}_{\mathbb{R}} \mid l^T A^{\Delta} l = 0\}.$$

In this expression we replaced the inverse  $A^{-1}$  by the adjoint  $A^{\Delta} = A^{-1} \cdot \det(A)$  of the matrix  $A$ , which is (for a symmetric matrix) defined by

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}^{\Delta} = \begin{pmatrix} + \begin{vmatrix} c & e \\ e & f \end{vmatrix} & - \begin{vmatrix} b & e \\ d & f \end{vmatrix} & + \begin{vmatrix} b & c \\ d & e \end{vmatrix} \\ - \begin{vmatrix} b & d \\ e & f \end{vmatrix} & + \begin{vmatrix} a & d \\ d & f \end{vmatrix} & - \begin{vmatrix} a & b \\ d & e \end{vmatrix} \\ + \begin{vmatrix} b & d \\ c & e \end{vmatrix} & - \begin{vmatrix} a & d \\ b & e \end{vmatrix} & + \begin{vmatrix} a & b \\ b & c \end{vmatrix} \end{pmatrix}.$$

Compared to the inverse, the adjoint has the advantage that it is also computable if  $A$  is not invertible, since it avoids division by the determinant.



**Fig. 9.5** Many tangents to conics.

**Definition 9.4.** The dual of the quadratic form  $p^T A p$  is the quadratic form  $l^T A^\Delta l$ . (The  $p$  and the  $l$  indicate that the former has to be interpreted in the world of points, while the latter has to be interpreted in the world of lines.)

Figure 9.5 gives a rough impression of how one could imagine a dual conic. A dual conic consists of all tangents to the original conic. While a conic is a set of points, a dual conic is a set of lines.

## 9.4 How Conics Transform

Before we continue studying the relationships between primal and dual conics, we first have to analyze how the algebraic representation of a conic is affected by a projective transformation  $\tau : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ . Since every projective transformation can be expressed as matrix multiplication, we may assume that a point  $p$  is transformed to  $\tau(p) = Tp$  with a regular  $3 \times 3$  matrix  $T$ .

Now let  $A$  be the symmetric matrix of a primal conic represented by the equation  $p^T A p = 0$ . Any point that satisfies this equation should be incident with the transformed conic represented by a suitable matrix  $\tau(A)$ . This implies that the equation of the transformed conic is  $(T^{-1}p)^T A (T^{-1}p)$ . Thus  $\tau$  acts on the matrix of the primal conic according to

$$A \mapsto T^{-1T} A T^{-1}.$$

Similarly, the equation of the dual conic  $l^T B l = 0$  transfers to  $(T^T l)^T B (T^T l)$ , and the matrix of the dual conic transforms according to

$$B \mapsto T B T^T.$$

To summarize: the primal matrix  $A$  is affected by left and right application of  $T^{-1}$ , while the dual matrix is affected by left and right application of  $T$ .

Since  $A$  (resp.  $B$ ) is a symmetric matrix, it is in particular possible to choose a suitable transformation matrix  $T$  such that  $\tau(A)$  (resp.  $\tau(B)$ ) is a diagonal matrix. For this, the matrix  $T$  may even be chosen to be orthogonal (i.e.,  $T^T T = E$ ). The diagonal entries are the eigenvalues of  $A$  (resp.  $B$ ). By choosing  $T$  appropriately, the diagonal entries of  $A' = \tau(A) = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$  may occur in any order.

We may even apply a further transformation by a diagonal matrix  $S$  by computing  $A'' = (S^{-1})^T A' S^{-1}$  if we set

$$S_{ii} := \begin{cases} \sqrt{|\alpha_i|} & \text{if } \alpha_i \neq 0, \\ 1 & \text{if } \alpha_i = 0. \end{cases}$$

Then  $A''$  is a diagonal matrix whose entries are in  $\{+1, -1, 0\}$ . As a direct consequence we obtain the following:

**Theorem 9.3.** *Every planar conic is projectively equivalent to a conic of the form  $\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = 0$  with  $\sigma_i \in \{+1, -1, 0\}$  for  $i \in \{1, 2, 3\}$ .*

## 9.5 Degenerate Conics

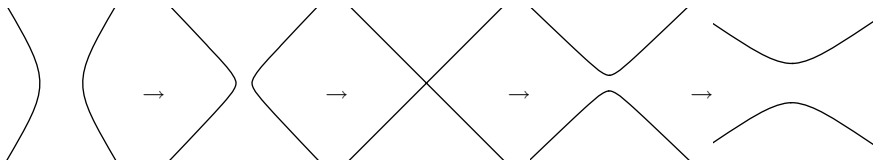
So far, we have considered mainly conics that came from invertible matrices  $A$ . In this case there is a one-to-one correspondence between conics and their duals. We will now study the degenerate case. We first study the possible cases in the world of *primal* conics which correspond to point sets. For a classification we have again to study the different signatures of eigenvalues that we met in Section 9.1:

$$(+, +, -), \quad (+, +, +), \quad , (+, -, 0) \quad (+, +, 0), \quad (+, 0, 0).$$

Theorem 9.3 tells us that these are up to projective equivalence all cases that can arise. The first case corresponds to the class of real nondegenerate conics, which we have studied previously. The second case is still not degenerate, but the corresponding conic consists entirely of complex points. The last three cases lead to situations with vanishing determinant and have to be considered degenerate conics. Up to projective equivalence they may be represented by the following quadratic forms:

$$x^2 - y^2 = 0, \quad x^2 + y^2 = 0, \quad x^2 = 0.$$

The first case consists of all points  $(x, y, z)$  for which  $|x| = |y|$ . Thus up to scalar multiple the points are of the form  $(1, 1, \alpha)$ ,  $(1, -1, \alpha)$ , or  $(0, 0, 1)$ . The first and second types of these vectors describe two lines each with one point missing, and the last vector describes the missing intersection point of the



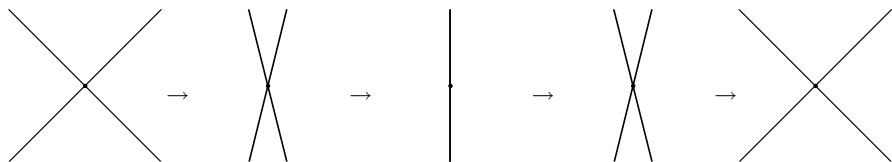
**Fig. 9.6** From a hyperbola to degenerate conic, and back.

two lines. Thus the conic  $x^2 - y^2 = 0$  consists of two intersecting lines. In this specific case with the usual  $z = 1$  embedding of the Euclidean plane these are the two lines with slope  $\pm 45^\circ$  through the origin.

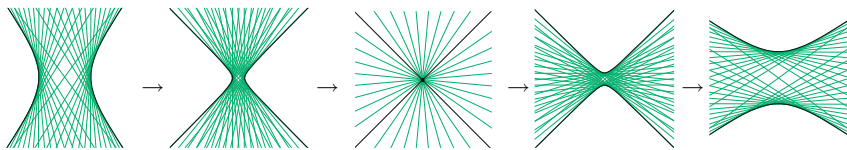
We may interpret this degenerate case by considering the limit case of a hyperbola with a very sharp bend as indicated in Figure 9.6. From a projective viewpoint we can say that the case with eigenvalue signature  $(+, -, 0)$  represents the situation in which the conics have degenerated into two real lines  $l$  and  $g$ . The corresponding quadratic form is then given by  $Q_{lg^T}$ . Here  $lg^T$  is a nonsymmetric rank-one matrix. Equivalently, we may consider the symmetrized matrix  $lg^T + gl^T$ .

The second case in our list corresponds to the equation  $x^2 + y^2 = 0$ . The only real point satisfying this condition is  $(0, 0, 1)$ . Still we get many complex solutions (and we will list them here for later use). Treating this case similarly to the one above, we see that the points on this conic are of the form  $(1, i, \alpha)$ ,  $(1, -i, \alpha)$ , or  $(0, 0, 1)$ . The first two cases correspond to complex lines. These lines still have a real intersection, namely the point  $(0, 0, 1)$ . In summary, the case with signature  $(+, +, 0)$  can be considered as consisting of two conjugate complex lines together with their real point of intersection.

The last case to be considered has signature  $(+, 0, 0)$  and corresponds to the equation  $x^2 = 0$ . This implies that  $x = 0$  and all points on the conic are of the form  $(0, \alpha, \beta)$ . This is exactly the line with homogeneous coordinates  $(1, 0, 0)$ . In other words, this case consists of a real line. In fact, it is reasonable to consider this line as having multiplicity *two* (a double line), since the situation arises as a limit case when the two lines of a conic with signature  $(+, -, 0)$  coincide. The situation is indicated in Figure 9.7.



**Fig. 9.7** From two single lines to a double line, and back.



**Fig. 9.8** Tangents of a hyperbola.

## 9.6 Primal-Dual Pairs

What happens to tangents in the process of degeneration? Figure 9.8 shows the situation as a kind of continuous process. A hyperbola is deformed such that the deformation passes through a situation in which the conic degenerates into two real lines. The green lines in the pictures are tangents of the conic. First note that the tangents in the first picture all are “outside” the conic. When the conic becomes more degenerate, the tangents seem to accumulate in the center of the conic. In the degenerate situation, in which the conic consists of two lines, there are indeed many lines that have only one intersection with the conic. They all pass through the singular point at which the two lines meet. After the degenerate situation, the hyperbola “switches its branches” (left/right to top/bottom). Again the tangents are outside the hyperbola, but they now occupy a different region of the projective plane. The degenerate situation is a kind of intermediate stage, in which tangents in both regions are possible. Our tangency concept of Definition 9.1 did not cover the degenerate case.

For the degenerate case of a conic consisting of two lines it is reasonable to consider *all* lines through the point of intersection as tangents, including the two lines of the conic themselves. Algebraically, this can be done quite elegantly and explicitly, again by considering the adjoint of the corresponding matrix.

**Theorem 9.4.** *Let  $A$  be the symmetric  $3 \times 3$  matrix corresponding to a degenerate conic  $\mathcal{C}_A$  consisting of two distinct lines  $g$  and  $h$ . Then the adjoint  $A^\Delta$  has the property that the lines passing through the intersection  $g \times h$  are exactly those that satisfy  $l^T A^\Delta l = 0$ .*

*Proof.* This theorem can be proved using the following formula, which holds for arbitrary three-dimensional vectors  $g = (g_1, g_2, g_3)^T$  and  $h = (h_1, h_2, h_3)^T$ :

$$(gh^T + hg^T)^\Delta = -(g \times h)(g \times h)^T.$$

First observe that this formula can be proved simply by expanding the terms on both sides of the expression and comparing the entries of the resulting  $3 \times 3$  matrices (this is left to the patient reader).

The left side of the equation is the adjoint of the symmetric matrix representing a degenerate conic consisting of the lines  $g$  and  $h$ . If  $g$  and  $h$  represent distinct vectors, then  $g \times h = p$  represents their point of intersection. Thus the right side has the form  $-pp^T$ . Now assume that a line  $l$  is incident to  $p$ . Then we have  $l^T(-pp^T)l = -(l^Tp)(p^Tl) = 0$ . Conversely, if a line satisfies  $l^T(-pp^T)l = 0$ , then we have  $(l^Tp)^2 = 0$ . Thus  $l$  must be incident to  $p$  as claimed in the theorem.  $\square$

The previous theorem together with the fact that for a nondegenerate matrix  $A$  the inverse is a multiple of the adjoint allows us to nicely combine the description of polars for both cases. In both cases we can consider the dual as the following set of lines:

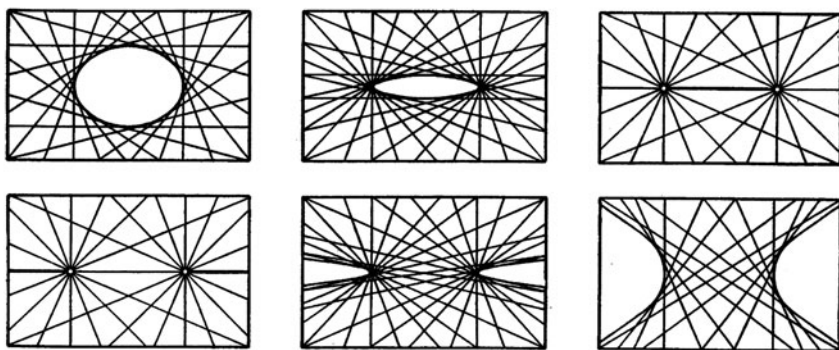
$$\{[l] \in \mathcal{L}_{\mathbb{R}} \mid l^T A^{\triangle} l = 0\}.$$

In the case of a degenerate primal conic, the dual describes all points that pass through the *double point* in which the two lines intersect.

So far, we have not treated the case in which the conic degenerates into just *one* line (we may consider this a double line). Algebraically, this case is characterized by the property that the symmetric matrix  $A$  has rank 1 and is of the form  $A = gg^T$ . It also corresponds to the eigenvalue signature  $(+, 0, 0)$ . In this case the adjoint is simply the zero matrix and no longer carries any geometric information. In fact, in this case it is reasonable (and as we will later see, highly useful) to *blow up* the situation and assign information by attaching a suitable matrix to the dual.

Since the situation is a little subtle, we will approach it from several perspectives. First of all, we consider a deformation of a conic that passes through the situation of a double line. For this, consider the drawing in Figure 9.9, which has been reproduced from the brilliant book by Felix Klein *Vorlesungen über nicht-euklidische Geometrie* (1925) [68]. There we see a sequence of pictures. The first shows an ellipse. This ellipse is squeezed horizontally until it becomes extremely thin. We can model this situation by considering the equation  $\alpha x^2 + y^2 - \alpha z^2 = 0$  with  $\alpha$  moving from 0.5 to 0. In the limit situation all tangents seem to pass through the two “endpoints” of the squeezed ellipse. If we deform  $\alpha$  further (from 0 to  $-0.5$ ), the conic becomes a hyperbola. Again close to the limit case all tangents seem to pass through the two special points. In the limit case  $\alpha = 0$  the conic consists of just the doubly covered  $x$ -axis. On this axis two points play a special role.

We can reinterpret this situation in terms of dual conics. What is the dual of a conic that consists of *two distinct lines* and a *double point* where they meet? Dualizing this description word by word, we see that the dual of such a conic must consist of *two distinct points* and a *double line* that joins them. This is exactly what we see in Klein’s drawing. Algebraically, the dual conic may be represented by a single quadratic form  $l^T B l = 0$  with a rank-2 symmetric matrix  $B$ . Its dual is described by the matrix  $B^{\triangle}$ . The quadratic form  $p^T A^{\triangle} p = 0$  describes exactly the double line through the two points.



**Fig. 9.9** Deformation via a double line (reproduction of a drawing by Felix Klein [68]).

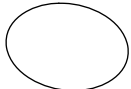
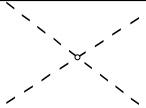
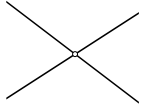

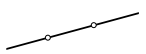
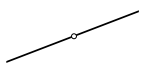
The considerations of the last few paragraphs suggest that it is reasonable to represent a conic by a *pair* of matrices, one of them representing the primal object and the other representing the dual object. If one of them is too degenerate, the other may still carry geometric information. In Section 9.1 we saw that we may classify the primal conics (up to real projective transformations) by their eigenvalue signatures

$$(+, +, -), \quad (+, +, +), \quad , (+, -, 0) \quad (+, +, 0), \quad (+, 0, 0),$$

which geometrically correspond to *real nondegenerate*, *complex nondegenerate*, *two distinct real lines (meeting in a real point)*, *two distinct complex conjugate lines (meeting in a real point)*, *a real double line*, respectively. Similarly, the same signatures of eigenvalues characterize the possible dual conics. We get (for the same order of signatures) the following geometric types of dual conics: *real nondegenerate*, *complex nondegenerate*, *two distinct real points (spanning a real line)*, *two distinct complex conjugate points (spanning a real line)*, *a real double point*. Now, a suitable description for primal/dual pairs of conics must define which primal conics (described by a real matrix  $A$ ) can be combined with which dual conics (described by a matrix  $B$ ). Not only must such a description specify the possible eigenvalue signatures of such pairs. It must as well specify the relative positions of the primal and the dual conics (for instance, the two points of a degenerate dual conic must coincide with the corresponding double line of the primal conic)—preferably encoded by a simple algebraic condition.

If at least one of the matrices has a nonzero adjoint, then the situation is clear, since the other matrix has (up to a scalar multiple) to be the adjoint of the other. In very highly degenerate situations it may, however, also happen that the conic degenerates into a real double line with a real double point on it. This may happen, for instance, by the movement indicated in Figure 9.7.

The degenerate situation corresponds to the middle picture of this sequence. Dually, this situation may also arise if for a conic consisting of a primal double line whose dual consists of two points these two points approach each other and in the limit case coincide. In such a situation both matrices  $A$  and  $B$  have rank 1, and their adjoint is the zero matrix. On the other hand, not every combination of two rank-1 matrices may occur as a primal/dual pair of a conic, since generically in such pairs the double line encoded by  $A$  does not coincide with the double point encoded by  $B$ . The following table collects all possible cases that may arise geometrically as pairs of primal and dual conics.

$A$	$B$	type	
$(+, +, +)$	$(+, +, +)$	complex nondegenerate conic	
$(+, +, -)$	$(+, +, -)$	real nondegenerate conic	
$(+, +, 0)$	$(+, 0, 0)$	two complex lines and a double real point on them	
$(+, -, 0)$	$(+, 0, 0)$	two real lines and a double real point on them	
$(+, 0, 0)$	$(+, +, 0)$	two complex points and a real double line through them	
$(+, 0, 0)$	$(+, -, 0)$	two real points and a real double line through them	
$(+, 0, 0)$	$(+, 0, 0)$	a real double line and a real double point through it	

*Remark 9.1.* In fact, all degenerate cases indicated by the pairs in the table (the last five cases) may occur as limit cases of nondegenerate conics. Conversely, all possible limit situations of nondegenerate conics correspond to a type given in the table. We will not prove this here formally.

We will now aim for a simple algebraic characterization of the geometrically meaningful pairs listed in the table. The following definition covers



also the above-mentioned highly degenerate case, in which the primal conic degenerates into a double line and the dual conic degenerates into a double point on this line.<sup>2</sup> Compared to the formula in Theorem 9.4, it is an implicit rather than an explicit description.

**Definition 9.5.** A primal/dual pair of conics is given by a pair  $(A, B)$  of real symmetric nonzero  $3 \times 3$  matrices such that there exists a factor  $\lambda \in \mathbb{R}$  with  $AB = \lambda E$  (here  $E$  is the usual unit matrix).

By looking at this definition it is not clear at all that it exactly characterizes all possible cases. To see this, we have to do a little work. We break the argument into a sequence of small lemmas that cover all important properties.

**Lemma 9.3.** *Being a primal/dual pair is a projectively invariant property.*

*Proof.* Assume that  $(A, B)$  is a primal/dual pair. Thus there is a  $\lambda$  with  $AB = \lambda E$ . Let  $\tau$  be a projective transformation that transforms points by  $p \mapsto Tp$ . This transformation maps the primal conic  $A$  according to  $A \mapsto T^{-1T}AT^{-1}$ . The matrix of the dual conic  $B$  is mapped according to  $B \mapsto TBT^T$ . Applying  $\tau$  to both objects in the expression  $A \cdot B$ , we get  $\tau(A) \cdot \tau(B) = T^{-1T}AT^{-1}TBT^T = T^{-1T}ABT^T = T^{-1T}\lambda ET^T = \lambda E$ .  $\square$

**Lemma 9.4.** *If  $\det(A) \neq 0$ , then  $(A, B)$  is a primal/dual pair if and only if  $B$  is a nonzero multiple of  $A^{-1}$ .*

*Proof.* Let  $\det(A) \neq 0$ . Then for every nonzero  $3 \times 3$  matrix  $B$ , the product  $AB$  is not the zero matrix. If  $(A, B)$  is a primal pair, then  $AB = \lambda E$  with  $\lambda \neq 0$ . Hence  $B$  must be a multiple of  $A^{-1}$ . Conversely, if  $B$  is a multiple of  $A^{-1}$ , then  $AB = \lambda E$ .  $\square$

The previous lemma tells us that at least in the nondegenerate case, the primal/dual pairs correspond to a pair of corresponding primal and dual conics. We now show that all other primal/dual pairs characterize exactly those pairs of primal and dual conics that arise as limit cases of nondegenerate situations. For the proper treatment of the nondegenerate cases we have to be aware of the following two subtleties related to the question of real vs. complex coordinate entries and the fact that matrices and vectors differing by nonzero scalar multiples represent the same geometric object. We will always assume that the matrices  $A$  and  $B$  have *real* entries. Still, it may happen that, for instance, a primal matrix  $A$  decomposes into two complex conjugate lines, since  $\bar{l}l^T + \bar{l}l^T$  is real. We will also have to express real rank-1 matrices  $A$  by expressions of the type  $A = ll^T$ . For this it may happen that although  $A$  is real,  $l$  must be chosen to be complex. For instance, consider the

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<sup>2</sup> The concept of primal/dual pairs given in Definition 9.5 nicely generalizes to an appropriate concept even in higher dimensions. The reader is invited to figure out the corresponding theory for three-dimensional or even higher-dimensional cases by analogy.

matrix  $A$  with the only nonzero element  $-1$  in the upper left corner. For a decomposition  $A = ll^T$  we must choose either  $l = (i, 0, 0)^T$  or  $l = (-i, 0, 0)^T$ . Still,  $l$  represents a real point, since it is a scalar multiple of  $(1, 0, 0)^T$ . Similar problems may arise by decomposing a rank-two matrix with no negative eigenvalues into a pair of real lines. Since matrices differing only by a nonzero scalar multiple represent the same geometric object, we may easily bypass this problem by the following general assumption, which we make from now on whenever necessary:

*Both matrices in a primal/dual pair  $(A, B)$  should have at least one positive eigenvalue.*

Since  $(A, B)$  is a primal/dual pair if and only if  $(-A, B)$  or  $(A, -B)$  is, this assumption can be made without any loss of generality.

**Lemma 9.5.** *If  $A$  describes a conic consisting of two distinct lines  $g$  and  $l$  then  $(A, B)$  is a primal/dual pair if and only if  $B$  is a multiple of  $pp^T$ , with  $p = g \times l$ .*

*Proof.* If  $A$  encodes a primal conic consisting of two distinct lines  $l$  and  $g$ , then up to a scalar multiple, the matrix  $A$  equals  $gl^T + lg^T$ . This matrix has rank 2, so the product with another matrix  $B$  can never be the unit matrix  $E$ . Thus for the primal/dual pair  $(A, B)$  we have  $AB = 0$  (here 0 denotes the zero matrix). In this case  $B$  has rank 1, since every column of  $B$  must lie in the (one-dimensional) kernel of  $A$ . Thus  $B$  is of the form  $pp^T$  for a suitable  $p \in \mathbb{C}^3$ . We get

$$0 = AB = (gl^T + lg^T)(pp^T) = gl^T pp^T + lg^T pp^T = (l^T p) gp^T + (g^T p) lp^T.$$

To derive the desired result we have to inspect the expression above. Since  $l$  and  $g$  represent distinct lines, the matrices  $gp^T$  and  $lp^T$  do not differ just by a scalar multiple. They also are nonzero, since none of the vectors  $p, g, l$  is zero. Hence this expression can be the zero matrix only if  $l^T p = g^T p = 0$ . Thus  $p$  must be incident to both  $l$  and  $m$ . Hence it is a multiple of  $g \times l$ .

If, conversely,  $B = pp^T$  for a point  $p$  incident to  $l$  and  $g$ , then the expression  $AB = (l^T p)gp^T + (g^T p)lp^T$  vanishes.  $\square$

*Remark 9.2.* As said before, we may assume that the point  $B$  has a single positive eigenvalue and thus  $p$  in the above lemma can be chosen to be completely real. We also want to point out that the above argument also covers the case that  $l$  and  $g$  have complex conjugate coordinates.

Obviously this lemma also has also a corresponding dual statement that covers the case of a matrix  $B$  describing a dual conic consisting of two distinct (perhaps complex conjugate) points. The dual statement of the lemma then claims that the matrix  $A$  describes a doubly covered real line  $l$  through these two points.

We now analyze what the property of being a primal/dual pair tells us if we know that the matrix  $A$  describes a conic that consists of a single (doubly counted) line  $l$ . In this case,  $A$  is a rank-1 matrix of the form  $ll^T$ .

**Lemma 9.6.** *Let  $A = ll^T$  describe a conic consisting of a real double line  $l$ .*

- (i) *If  $(A, B)$  is a primal/dual pair, then  $B$  is of the form  $pq^T + qp^T$ , with  $l^T p = 0$  and  $l^T q = 0$  for suitable  $p, q \in \mathbb{C}^3$ . If further,  $B$  has at least one positive eigenvalue, then  $p$  and  $q$  may be chosen either to be both real or as complex conjugates (in particular,  $p$  and  $q$  may coincide).*
- (ii) *If a real symmetric matrix  $B$  is of the form  $pq^T + qp^T$ , with  $l^T q = 0$  and  $l^T p = 0$ , then  $(A, B)$  is a primal/dual pair.*

*Proof.* We start with statement (i). If  $A = ll^T$  for a line  $l$ , then  $A$  has rank 1. Let us assume that  $(A, B)$  is a primal/dual pair. Then  $AB$  can never be the unit matrix. Thus if  $(A, B)$  is a primal/dual pair, then we must have  $AB = 0$ . This implies that  $B$  has at most rank 2 but is still nonzero. Hence at least one eigenvalue of  $B$  (say  $\lambda_3$ ) is zero and at least one eigenvalue (say  $\lambda_1$ ) is nonzero. Since  $B$  is by definition real and symmetric, it has an orthonormal basis  $a_1, a_2, a_3$  of eigenvectors. Thus we have  $Ba_i = \lambda_i a_i$  and

$$a_i^T a_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (9.2)$$

We now consider the two points

$$p = \frac{\sqrt{\lambda_1} a_1 + \sqrt{-\lambda_2} a_2}{2} \quad \text{and} \quad q = \frac{\sqrt{\lambda_1} a_1 - \sqrt{-\lambda_2} a_2}{2}.$$

A simple calculation (expand and cancel) shows, that

$$X := pq^T + qp^T = \lambda_1 a_1 a_1^T + \lambda_2 a_2 a_2^T.$$

Using (9.2), we get

$$X a_1 = \lambda_1 a_1 a_1^T a_1 + \lambda_2 a_2 a_2^T a_1 = 1 \cdot \lambda_1 a_1 + 0 \cdot \lambda_2 a_1 = \lambda_1 a_1,$$

$$X a_2 = \lambda_1 a_1 a_1^T a_2 + \lambda_2 a_2 a_2^T a_2 = 0 \cdot \lambda_1 a_1 + 1 \cdot \lambda_2 a_1 = \lambda_2 a_2,$$

$$X a_3 = \lambda_1 a_1 a_1^T a_3 + \lambda_2 a_2 a_2^T a_3 = 0 \cdot \lambda_1 a_1 + 0 \cdot \lambda_2 a_1 = 0.$$

Thus  $X$  acts on  $a_1, a_2, a_3$  exactly as  $B$  does. Hence  $B = X = pq^T + qp^T$ .

In order to see that  $l^T p = 0$  and  $l^T q = 0$ , consider the expression  $0 = AB = ll^T(pq^T + qp^T)$ . If  $p$  and  $q$  represent two different points, then we are in the situation of the dual of Lemma 9.6, and  $l^T p = l^T q = 0$  follows from this statement. Otherwise,  $p$  is a multiple of  $q$  and then  $B = \alpha pp^T$  for suitable nonzero  $\alpha$ . Then  $0 = AB = \alpha ll^T pp^T = (l^T p)lp^T$ . Since both  $l$  and  $p$  are nonzero, this can happen only if  $l^T p = 0$ . If we furthermore assume

that without loss of generality  $\lambda_1$  is positive then we get the following three possible geometric situations:

- If  $\lambda_2 > 0$  then  $p$  and  $q$  are complex conjugates,
- If  $\lambda_2 < 0$  then  $p$  and  $q$  are real and distinct,
- If  $\lambda_2 = 0$  then  $p = q$  is real.

To prove (ii) assume that  $B = pq^T + qp^T$  with  $l^T p = l^T q = 0$ . Expanding  $AB$  and using  $l^T p = l^T q = 0$  immediately proves  $AB = 0$ ; hence  $(A, B)$  is a primal/dual pair.  $\square$

Summarizing the results of the previous lemmas we yields the following theorem:

**Theorem 9.5.** *If  $(A, B)$  is a primal/dual pair, then we have one of the geometric situations listed in the table on page 162. Conversely, if  $A$  and  $B$  describe any of the cases in this table, then  $(A, B)$  is a primal/dual pair.*

*Proof.* The two nondegenerate cases are covered by Lemma 9.4. The five degenerate cases are covered by Lemma 9.5 and Lemma 9.6.  $\square$

*Remark 9.3.* The reader might wonder why we have devoted such a long section to the proper treatment of the degenerate cases of conics and the admissible primal/dual pairs of matrices. We will see that they become of crucial importance in Chapters 20 to 24, when we deal with different *Cayley-Klein geometries*. We will see that each of the cases in our table of primal/dual pairs leads to a different type of “metric geometry” with quite specific and very interesting properties.

# Conics and Perspectivity

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*Some people have a mental horizon of radius zero and call it their point of view.*

*Attributed variously to David Hilbert,  
Albert Einstein and Leonhard Euler*

In the previous chapter we treated conics more or less as isolated objects. We defined points on them and lines tangent to them. Now we want to investigate various geometric and algebraic properties of conics. In particular, we will see how we can treat conics on the level of bracket algebra.

## 10.1 Conic through Five Points

We begin by calculating a conic through a given set of points. For this, consider the quadratic equation that defines a conic:

$$a \cdot x^2 + b \cdot y^2 + c \cdot z^2 + d \cdot xy + e \cdot xz + f \cdot yz = 0.$$

This equation has six parameters  $a, \dots, f$ .<sup>1</sup> Multiplying all of them simultaneously by the same nonzero scalar leads to the same conic. Thus the parameter vector  $(a, \dots, f)$  behaves like a vector of homogeneous coordinates. Counting degrees of freedom shows that in general it will take five points to uniquely determine a conic. To find the parameters for a conic through five points  $p_i = (x_i, y_i, z_i)$ ,  $i = 1, \dots, 5$ , we simply have to solve the following linear system of equations:

---

<sup>1</sup> Compared to Section 9.1 we have relabeled the parameters and put the factor 2 of the mixed terms into the parameters.

$$\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1 y_1 & x_1 z_1 & y_1 z_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2 y_2 & x_2 z_2 & y_2 z_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3 y_3 & x_3 z_3 & y_3 z_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4 y_4 & x_4 z_4 & y_4 z_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5 y_5 & x_5 z_5 & y_5 z_5 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If this system has a full rank of 5, then there is a (up to scalar multiple) unique solution  $(a, \dots, f)$  that defines the corresponding conic. If more than three points are simultaneously collinear or if two points coincide, the rank of the system may be lower than 5. This corresponds to the situation that there is more than one conic passing through the given set of points. This method of determining the parameter vector  $(a, \dots, f)$  is mathematically elegant. However it is computationally expensive. We first have to calculate the squared parameters and then have to solve a  $5 \times 6$  system of equations.

There is also another way to calculate such a conic more or less directly. This way will also give us additional structural insight into the geometry and underlying algebra of a conic. In preparation we have to understand how to calculate a degenerate conic that consists of two lines with homogeneous coordinates  $g$  and  $h$ . A degenerate conic must be represented by a quadratic form  $p^T A p = 0$  that vanishes if  $p$  is on either of the lines. The (nonsymmetrized) matrix of such a quadratic form is simply given by

$$A = gh^T.$$

This can be easily seen, since the quadratic form

$$p^T A p = p^T (gh^T) p = (p^T g)(h^T p) = \langle p, g \rangle \langle p, h \rangle$$

vanishes if one of the two scalar products on right side vanishes. This in turn corresponds geometrically to the situation in which  $p$  is on  $g$  or on  $h$ .

Assume that the line  $g$  is spanned by two points labeled 1 and 2 and that line  $h$  is spanned by two points labeled 3 and 4. Then we have  $g = 1 \times 2$  and  $h = 3 \times 4$ . The quadratic form becomes

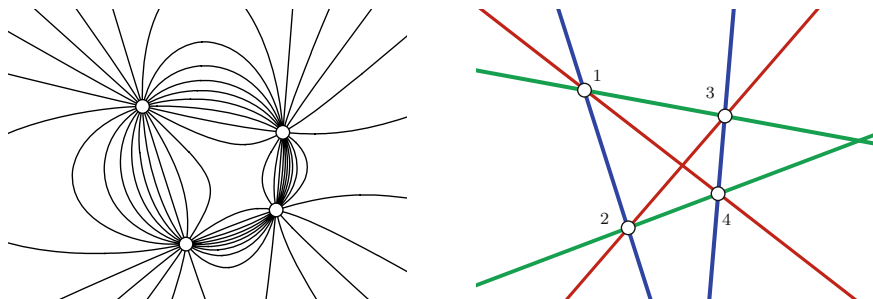
$$\langle p, 1 \times 2 \rangle \langle p, 3 \times 4 \rangle = 0.$$

We may as well express this term as the product of two determinants

$$[p, 1, 2][p, 3, 4] = 0.$$

Each factor describes a linear condition on the point  $p$ . The product calculates the conjunction between the two expressions.

Now assume that we want to describe the set of conics that pass through for points 1,  $\dots$ , 4 in general position. Clearly, there are many conics that satisfy this condition. The corresponding system of linear equations consists of four equations in six variables. Hence the solution space will be two-dimensional.



**Fig. 10.1** Bundle of conics through four points. Three degenerate special cases.

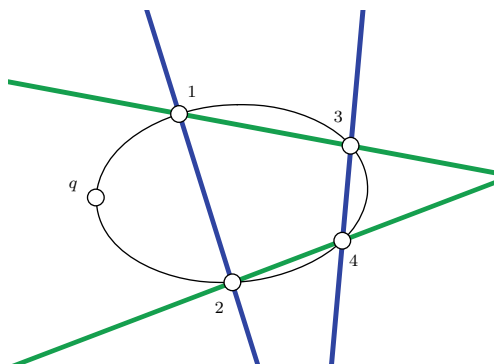
One of these two degrees of freedom goes into the homogeneity of the conic parameters. Therefore we have a bundle of geometric solutions with one degree of freedom. Figure 10.1 (left) illustrates such a bundle of conics. Among these conics there are three degenerate conics, each of them equal to a pair of lines spanned by the four points. In Figure 10.1 (right) these pairs of lines are marked by identical colors. They correspond to the following three quadratic forms:

$$[p, 1, 2][p, 3, 4] = 0, \quad [p, 1, 3][p, 2, 4] = 0, \quad [p, 1, 4][p, 2, 3] = 0.$$

A linear combination of two of these forms (say the last two)

$$\lambda[p, 1, 3][p, 2, 4] + \mu[p, 1, 4][p, 2, 3] = 0$$

again generates a quadratic form. The set of points  $p$  satisfying this equation again forms a conic. This conic passes through all four points  $1, \dots, 4$ , since both summands vanish on these points. If  $\lambda$  and  $\mu$  run through all possible



**Fig. 10.2** Constructing a conic through five points.

values we obtain all the conics in the bundle through the four points. Applying the technique of Plücker's  $\mu$  (compare Section 6.3), we can adjust these values such that the resulting conic passes through another given point  $q$ . For this we simply have to choose

$$\lambda = [q, 1, 4][q, 2, 3]; \quad \mu = -[q, 1, 3][q, 2, 4].$$

The resulting conic equation can be written as

$$[q, 1, 4][q, 2, 3][p, 1, 3][p, 2, 4] - [q, 1, 3][q, 2, 4][p, 1, 4][p, 2, 3] = 0.$$

Observe that this equation is a multihomogeneous bracket polynomial that is quadratic in each of the six involved points. Figure 10.2 illustrates the situation. We can also interpret it as a bracket condition encoding the (projectively invariant) property that six points  $1, \dots, 4, p, q$  are on a conic (compare Section 6.4 and Section 7.3). We will return to this interpretation in the next section.

But first we will give the procedure for calculating the symmetric matrix for the conic through the five points  $1, 2, 3, 4, q$ . We give it as a kind of simple computer program:

```

1:  $g_1 := 1 \times 3$ ;
2:  $g_2 := 2 \times 4$ ;
3:  $h_1 := 1 \times 4$ ;
4:  $h_2 := 2 \times 3$ ;
5:  $G := g_1 g_2^T$ ;
6:  $H := h_1 h_2^T$ ;
7:  $M := q^T H q G - q^T G q H$ ;
8:  $A := M + M^T$ .
```

The matrix  $A$  assigned in the last line of the program contains the symmetrized matrix.

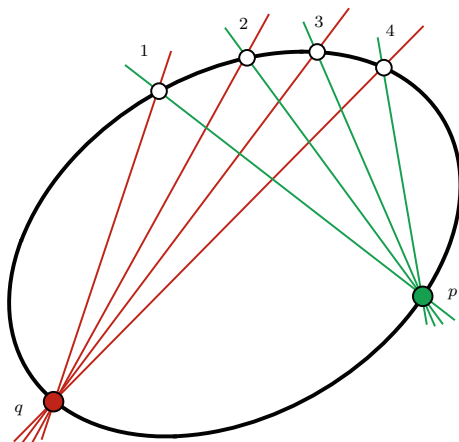
## 10.2 Conics and Cross-Ratios

Let us return to the equation

$$[q, 1, 4][q, 2, 3][p, 1, 3][p, 2, 4] - [q, 1, 3][q, 2, 4][p, 1, 4][p, 2, 3] = 0, \quad (*)$$

which characterizes whether six points are on a conic. First observe that this equation is highly symmetric. For each bracket in one term, its complement (the bracket consisting of the other three letters) is in the other term. The





**Fig. 10.3** Four points on a conic seen from two other points of a conic.

symmetry becomes a bit more transparent if we rewrite the equation with new point labels:

$$[A, B, C][A, Y, Z][X, B, Z][X, Y, C] - [A, B, Z][A, Y, C][X, B, C][X, Y, Z] = 0.$$

There is another important observation that we can make by rewriting equation (\*). We assume that the conic is nondegenerate and that none of the determinants vanishes. In this case we can rewrite (\*) in the form

$$\frac{[q, 1, 3][q, 2, 4]}{[q, 1, 4][q, 2, 3]} = \frac{[p, 1, 3][p, 2, 4]}{[p, 1, 4][p, 2, 3]}.$$

Both sides of the equation represent cross-ratios. The right side is the cross-ratio of the lines  $\overline{p1}, \overline{p2}, \overline{p3}, \overline{p4}$ ; the left side of the equation is the cross-ratio of the lines  $\overline{q1}, \overline{q2}, \overline{q3}, \overline{q4}$ . We abbreviate

$$(1, 2; 3, 4)_q := \frac{[q, 1, 3][q, 2, 4]}{[q, 1, 4][q, 2, 3]}.$$

This is the cross-ratio of 1, 2, 3, 4 as “seen from” point  $q$ . Thus equation (\*) may be restated as

$$(1, 2; 3, 4)_q = (1, 2; 3, 4)_p.$$

*Point  $p$  and point  $q$  see the points 1, 2, 3, 4 under the same cross-ratio.* The situation is shown in Figure 10.3. We summarize this in a theorem:

**Theorem 10.1.** *Let 1, 2, 3, 4,  $p$  be five points on a conic such that  $p$  is distinct from the other four points. Then the cross-ratio  $(1, 2; 3, 4)_p$  is independent of the special choice of  $p$ .*

We will later on see that this theorem is very closely related to the *exterior angle theorem* for circles, which states that in a circle a fixed secant is seen from an arbitrary point on the circle under the same angle (modulo  $\pi$ ).

The previous theorem enables us to speak of *the* cross-ratio of four points on a fixed conic as long as no more than two of the points  $(1, 2, 3, 4)$  coincide and we can speak of a cross-ratio at all. For this we simply choose an arbitrary point  $p$  that does not coincide with  $1, 2, 3, 4$  and take the cross-ratio  $(1, 2; 3, 4)_p$ .

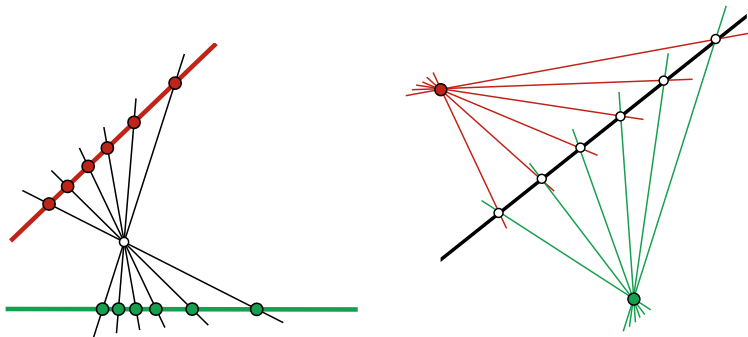
The theorem is useful under many aspects. In particular, it is useful to parameterize classes of objects. We will investigate two of these applications. First assume that the points  $1, 2, 3, 4$  are fixed. The previous theorem states that for a fixed conic  $\mathcal{C}$ , the value of  $(1, 2; 3, 4)_p$  is independent of the choice of  $p$ . Thus it can be considered as a characteristic number that singles out the specific conic  $\mathcal{C}$  from all other conics through the four points. Thus we can take this number as a kind of coordinate for the conic within the one-dimensional bundle of conics through  $1, 2, 3, 4$ . In fact, if we do so the three special degenerate conics in this bundle (compare Figure 10.1) correspond to the values  $0$ ,  $1$ , and  $\infty$ .

In the second application we fix the conic itself as well as the position of the points  $1, 2, 3$ . The point  $p$  may be an arbitrary point on the conic whose exact position is not relevant for the calculations as long as it does not coincide with the other points. If point  $4$  takes all possible positions on the conic, then the value of  $(1, 2; 3, 4)_p$  takes all possible values of  $\mathbb{R} \cup \{\infty\}$ , since the line  $\overline{p, 4}$  takes all possible positions through  $p$ . Thus we can use the cross-ratio  $(1, 2; 3, 4)_p$  to characterize the position of  $4$  with respect to  $1, 2, 3$  on the conic. The three special values  $0$ ,  $1$ , and  $\infty$  are assumed when  $4$  is identical to  $1, 3, 2$ , respectively. In this setup we may consider the conic itself as a model of the *real projective line*. The three points  $1, 2, 3$  above play the role of a projective basis on this line with respect to which we measure the cross-ratio. In this model it is obvious that the topological structure of the real projective line is a circle.

### 10.3 Perspective Generation of Conics

The considerations of the last section can be reversed in order to create conics by perspective bundles of lines. For this we consider the points  $p$  and  $q$  as centers of two bundles of lines that are projectively related to each other. Forming the intersections of corresponding lines from each bundle creates a locus of points that all have to lie on a single conic.

To formalize this fact (in particular to deal with the special cases) we have to refine our notion of projective transformations slightly.



**Fig. 10.4** A line perspectivity and a point perspectivity.

**Definition 10.1.** Let  $l_1$  and  $l_2$  be two distinct lines in  $\mathcal{L}_{\mathbb{R}}$  and  $o \in \mathcal{P}_{\mathbb{R}}$  not incident to  $l_1$  or  $l_2$ . Furthermore, let  $\mathcal{P}_{l_1}$  and  $\mathcal{P}_{l_2}$  be the sets of points on the two lines, respectively. The map  $\tau: \mathcal{P}_{l_1} \rightarrow \mathcal{P}_{l_2}$  defined by  $\tau(p) = \text{meet}(l_2, \text{join}(o, p))$  is called a (point) perspectivity.

We furthermore use the term *projective transformation* from  $\mathcal{P}_{l_1}$  to  $\mathcal{P}_{l_2}$  in the following sense. We represent the points on  $\mathcal{P}_{l_i}$ ,  $i = 1, 2$ , by suitable linear combinations  $\alpha_i a_i + \beta_i b_i$ . If  $\tau: \mathcal{P}_{l_1} \rightarrow \mathcal{P}_{l_2}$  can be expressed as

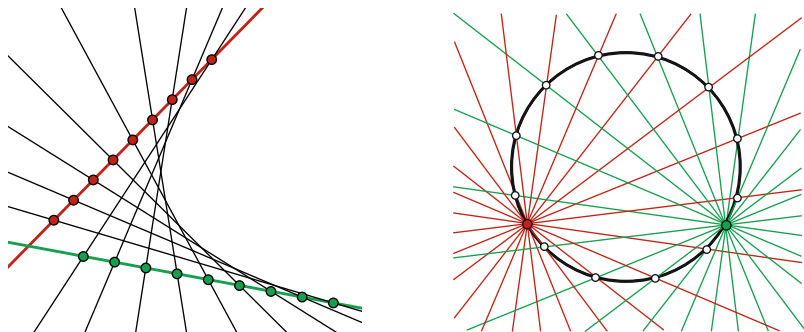
$$\tau \left( \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix},$$

then we call  $\tau$  a projective transformation. Theorem 5.1 established that harmonic maps are projective transformations. In Lemma 4.3 we proved that perspectivities are particular projective transformations. Dually, we can also speak about perspectivities of bundles of lines.

**Definition 10.2.** Let  $p_1$  and  $p_2$  be two distinct points in  $\mathcal{P}_{\mathbb{R}}$  and  $o \in \mathcal{L}_{\mathbb{R}}$  not incident to  $p_1$  or  $p_2$ . Furthermore, let  $\mathcal{L}_{p_1}$  and  $\mathcal{L}_{p_2}$  be the sets of lines through the two points, respectively. The map  $\tau: \mathcal{L}_{p_1} \rightarrow \mathcal{L}_{p_2}$  defined by  $\tau(l) = \text{join}(p_2, \text{meet}(o, l))$  is called a (line) perspectivity.

Figure 10.4 shows images for both types of perspectivities. We will also consider projective transformations  $\tau: \mathcal{L}_{p_1} \rightarrow \mathcal{L}_{p_2}$  in the corresponding dual sense to point transformations. Again, line perspectivities are special projective transformations. Now we will use Theorem 10.1 to prove the following fact.

**Theorem 10.2.** Let  $p$  and  $q$  be two distinct points in  $\mathbb{RP}^2$ . Let  $\mathcal{L}_p$  and  $\mathcal{L}_q$  be the sets of all lines that pass through  $p$  and  $q$ , respectively. Let  $\tau: \mathcal{L}_p \rightarrow \mathcal{L}_q$  be a projective transformation that is not a perspectivity. Then the points  $\text{meet}(l, \tau(l))$  are all points of a certain conic  $\mathcal{C}$ .



**Fig. 10.5** Generation of a conic by projective bundles.

*Proof.* Let  $l_1, l_2, l_3, l_4$  be four arbitrary lines from  $\mathcal{L}_p$  not through  $q$ . Consider the points  $a_i = \mathbf{meet}(l_i, \tau(l_i))$ ,  $i = 1, \dots, 4$ . Since the two bundles of lines were related by a projective transformation, the cross-ratio  $(l_1, l_2; l_3, l_4)$  equals the cross-ratio  $(\tau(l_1), \tau(l_2); \tau(l_3), \tau(l_4))$ . This relation can be written as  $(a_1, a_2; a_3, a_4)_p = (a_1, a_2; a_3, a_4)_q$ . Hence the six points  $a_1, a_2, a_3, a_4, p, q$  lie on a conic. Since  $\tau$  is not a perspectivity, the points  $a_1, a_2, a_3$  cannot be collinear. (Assume, to the contrary, that they lie on a line  $\ell$ . Then the image of an arbitrary fourth line  $l_4$  must satisfy the relation  $(l_1, l_2; l_3, l_4) = (\tau(l_1), \tau(l_2); \tau(l_3), \tau(l_4))$ . Hence the intersections of  $l_4$  with  $\ell$  and  $\tau(l_4)$  with  $\ell$  must coincide. This means that  $\tau$  is a perspectivity.) Thus the conic  $\mathcal{C}$  uniquely defined by  $p, q, a_1, a_2, a_3$  is nondegenerate. Since  $l_4$  was chosen to be arbitrary, all other intersections  $a = \mathbf{meet}(l, \tau(l))$  must lie on  $\mathcal{C}$  as well. Conversely, for any point  $a$  on  $\mathcal{C} \setminus \{p, q\}$  there is a line  $l$  that joins  $p$  and  $a$ . The intersection of  $l$  and  $\tau(l)$  must be on the conic. Thus this intersection must be point  $a$ .  $\square$

The previous theorem gives us a nice procedure to explicitly generate a conic as a locus of points. The conic is determined by two points  $p$  and  $q$  and a projective transformation between the line bundles through these two points. For generation of the conic we take a free line from the bundle  $\mathcal{L}_p$  and let it sweep through the bundle. All intersections of  $l$  and  $\tau(l)$  form the points of the conic. Dually, if we have two lines  $l_1$  and  $l_2$  whose point sets are connected by a projective transformation  $\tau$ , we can consider a point  $p$  freely movable on  $l_1$ . The lines  $\mathbf{join}(p, \tau(p))$  form the set of tangents to a particular conic.

Figure 10.5 shows two particularly simple (but still interesting) examples of this generation principle. On the right, two bundles of lines are shown, where the second one simply arises from shifting and rotating the first one (this is a particularly simple projective transformation). The resulting conic that comes from intersecting corresponding lines is a circle. This result could

also be derived elementarily using the *exterior angle theorem* for circles. We will see later on that this theorem is highly related to our conic constructions. The second example shows two sets of equidistant points on two different lines (they are again related by a projective transformation). Joining corresponding points yields the envelope of a conic. One should compare these two pictures with Figure 10.4, in which pairs of objects are shown that are related by a perspectivity. This case is the degenerate limit case of the above construction.

*Remark 10.1.* The construction underlying Theorem 10.2 also demonstrates that the set of points on a nondegenerate conic can be polynomially parameterized (in homogeneous coordinates). For this consider two points  $p, q$  on the conic and the two corresponding bundles of lines together with the corresponding projective transformation  $\tau$ . We introduce a projective basis on each of the two bundles together with a suitable homogeneous coordinatization (say we represent lines from the first bundle by  $\lambda_p l_1 + \mu_p l_2$  and points from the second bundle by  $\lambda_q m_1 + \mu_q m_2$ ). The projective transformation  $\tau$  can be written as  $\begin{pmatrix} \lambda_q \\ \mu_q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_p \\ \mu_p \end{pmatrix}$  for a suitable matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Thus the points on the conic have homogeneous coordinates

$$(tl_1 + (1-t)l_2) \times ((at + b(t-1))m_1 + (ct + d(t-1))m_2).$$

Here  $t$  is a parameter that runs through all elements of  $\mathbb{R}$  from  $-\infty$  to  $+\infty$ . By this we get all points of the conic except for the one corresponding to  $t = \infty$ . The above formula is simply a polynomial function.

A similar statement is no longer true for curves of higher degree. In general, they cannot be parameterized by rational or even polynomial functions.

## 10.4 Transformations and Conics

Let us turn to the interesting task of studying all projective transformations that leave a given fixed conic  $\mathcal{C}$  invariant. Such a transformation must map points on  $\mathcal{C}$  to points on  $\mathcal{C}$ . We here discuss the nondegenerate case only and postpone the degenerate case to later chapters. The key to the classification of such transformations is Theorem 10.1, which allows us to identify the points on a conic with the points on a projective line and to associate a cross-ratio to quadruples of such points. Our aim is to prove that a projective transformation  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  that leaves  $\mathcal{C}$  invariant induces a projective transformation on  $\mathcal{C}$  (considered as a projective line). For the following considerations we fix a nondegenerate conic  $\mathcal{C}$  and identify it with the projective line. As indicated in Section 10.2, we will speak of *the* cross-ratio  $(1, 2; 3, 4)_{\mathcal{C}}$  of four points on  $\mathcal{C}$ , which is  $(1, 2; 3, 4)_p$  for a suitably nondegenerate choice of  $p$ .

**Theorem 10.3.** *Let  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  be a projective transformation that leaves  $\mathcal{C}$  invariant. Then the restriction of  $\tau$  to  $\mathcal{C}$  is a projective transformation on  $\mathcal{C}$ .*

*Proof.* Let  $\mathbf{0}$ ,  $\mathbf{1}$ , and  $\infty$  be three distinct points on  $\mathcal{C}$ . The position of an arbitrary point  $\mathbf{x}$  on  $\mathcal{C}$  is uniquely determined by the value of the cross-ratio  $(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})_{\mathcal{C}}$ . Let  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  be a projective transformation that leaves  $\mathcal{C}$  invariant. We will prove that the position of  $\tau(\mathbf{x})$  is already defined by the positions of  $\tau(\mathbf{0})$ ,  $\tau(\mathbf{1})$ , and  $\tau(\infty)$  and that we have in particular

$$(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})_{\mathcal{C}} = (\tau(\mathbf{0}), \tau(\infty); \tau(\mathbf{x}), \tau(\mathbf{1}))_{\mathcal{C}}.$$

For this let  $p$  on  $\mathcal{C}$  be chosen such that  $p$  does not coincide with the point  $\mathbf{0}$ ,  $\mathbf{1}$ ,  $\infty$ , or  $\mathbf{x}$ . Then  $\tau(p)$  will automatically not coincide with  $\tau(\mathbf{0})$ ,  $\tau(\mathbf{1})$ ,  $\tau(\infty)$ , or  $\tau(\mathbf{x})$ . Since  $\tau$  is a projective transformation, we have

$$(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})_p = (\tau(\mathbf{0}), \tau(\infty); \tau(\mathbf{x}), \tau(\mathbf{1}))_{\tau(p)}.$$

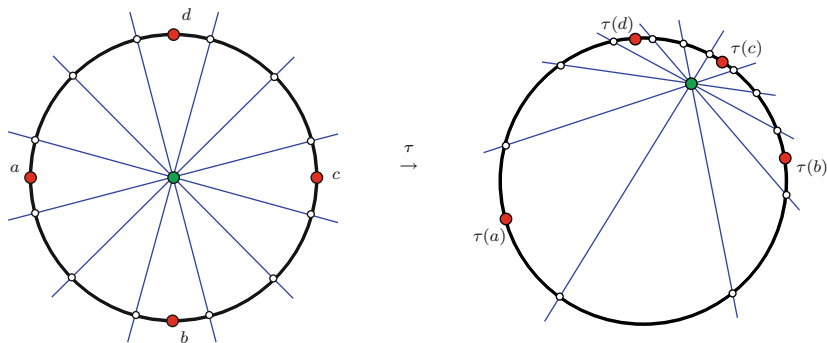
The special choice of the position of  $p$  guarantees that the cross-ratios are well defined. Now  $p$  as well as  $\tau(p)$  are on  $\mathcal{C}$ . The other four image points are also on  $\mathcal{C}$ . Thus the above two cross-ratios are the cross-ratios  $(\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})_{\mathcal{C}}$  on  $\mathcal{C}$  and  $(\tau(\mathbf{0}), \tau(\infty); \tau(\mathbf{x}), \tau(\mathbf{1}))_{\mathcal{C}}$  on  $\mathcal{C}$ . Thus these two cross-ratios must be equal, as claimed. This implies that the restriction of  $\tau$  to  $\mathcal{C}$  must be a projective transformation.  $\square$

The proof of the last theorem was algebraically simple but conceptually interesting. It relates a projective transformation on  $\mathbb{RP}^2$  that leaves  $\mathcal{C}$  invariant to its action on  $\mathcal{C}$  itself. With our concept of  $\mathcal{C}$  representing the projective line, we see that in this world  $\tau$  induces nothing but a one-dimensional projective transformation. In the theorem it was crucial that the value of the cross-ratio of four points *seen from* a fifth point  $p$  is independent of the choice of  $p$ . This allowed us to relate the image seen from  $p$  to the image seen from  $\tau(p)$ .

We can also do the opposite: define a projective transformation that leaves  $\mathcal{C}$  invariant by explicitly giving the images of four suitably chosen points on  $\mathcal{C}$ .

**Theorem 10.4.** *Let  $a, b, c, d$ , and  $a', b', c', d'$  be two quadruples of distinct points on a nondegenerate conic  $\mathcal{C}$  such that  $(a, b; c, d)_{\mathcal{C}} = (a', b'; c', d')_{\mathcal{C}}$ . Then there exists a unique projective transformation  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  with  $\tau(a) = a'$ ,  $\tau(b) = b'$ ,  $\tau(c) = c'$ ,  $\tau(d) = d'$  that furthermore leaves  $\mathcal{C}$  invariant.*

*Proof.* The transformation  $\tau$  is uniquely determined by the preimage points  $a, b, c, d$  and the image points  $a', b', c', d'$ . Thus we have only to show that  $\tau$  indeed leaves  $\mathcal{C}$  invariant. Since a nondegenerate conic is uniquely determined by five points on it it suffices to prove that there exists one more point  $p$  on  $\mathcal{C}$  whose image  $\tau(p)$  is also on  $\mathcal{C}$ . For this, let  $p$  be an arbitrary point distinct from the points  $a, b, c, d$ . Thus we have



**Fig. 10.6** A transformation that leaves a conic invariant.

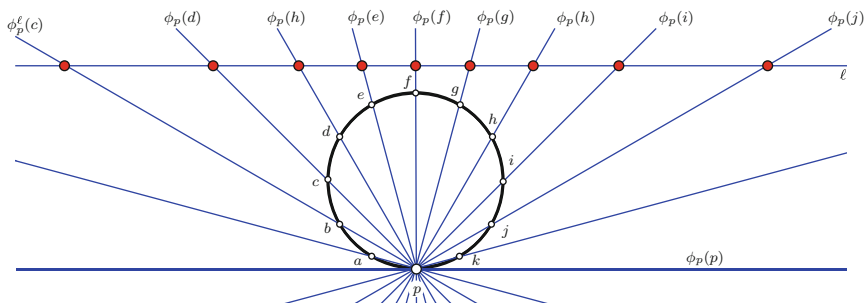
$$\begin{aligned}
 (a', b'; c', d')_{\mathcal{C}} &= (a, b; c, d)_{\mathcal{C}} = (a, b; c, d)_p \\
 &= (\tau(a), \tau(b); \tau(c), \tau(d))_{\tau(p)} = (a', b'; c', d')_{\tau(p)}.
 \end{aligned}$$

The third equation holds since  $\tau$  is a projective transformation. The fact that  $(a', b'; c', d')_{\mathcal{C}} = (a', b'; c', d')_{\tau(p)}$  shows that  $\tau(p)$  also must lie on the conic  $\mathcal{C}$ .  $\square$

Figure 10.6 shows a circle before and after a projective transformation that leaves the circle invariant. The transformation  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  is determined by the image of the four red points. The position of the four image points cannot be chosen arbitrarily. They must have the same cross-ratio with respect to the circle as the four preimage points. In the situation shown in the picture the four points are in harmonic position with respect to the circle. The white points in the preimage circle (left) map to the white points in the image circle (right). The lines and the central point indicate how the interior of the circle is distorted by  $\tau$ .

We can also make the relation of  $\mathcal{C}$  to the projective line  $\mathbb{RP}^1$  more explicit and relate the points of  $\mathcal{C}$  to the line bundle  $\mathcal{L}_p$  of lines through a point  $p \in \mathcal{C}$ . Such a line bundle considered as a set of lines is by duality a representation of the projective line. We can explicitly relate every point on  $\mathcal{C}$  to a line in  $\mathcal{L}_p$ : Each line is associated to its intersection with  $\mathcal{C}$  different from  $p$ . There is one line in the bundle that has to be treated separately. The tangent through  $p$  is associated to  $p$  itself. (This reflects the limit situation as the point on  $\mathcal{C}$  approaches  $p$ .) We can express this relation by a bijective map  $\phi_p: \mathcal{C} \rightarrow \mathcal{L}_p$  from  $\mathcal{C}$  to the bundle of lines through  $p$ . Now the previous theorem states that the projective transformation  $\tau$  induces a projective transformation  $\tau_p: \mathcal{L}_p \rightarrow \mathcal{L}_p$  in this line bundle via

$$\tau_p(l) := \phi_p(\tau(\phi_p^{-1}(l))).$$



**Fig. 10.7** Projection of a conic onto  $\mathbb{RP}^1$ .

The reader is invited to convince himself that the limit case of the tangent through  $p$  fits seamlessly into this picture.

Figure 10.7 illustrates the relation of the points on the conic to the line bundle. In addition to the line bundle, the picture also shows an additional line  $\ell$  that is intersected with every line of the bundle. So the points on the conic are also in one-to-one correspondence to the points on  $\ell$ . The picture exemplifies also how this relation of points on the conic to points on the line is closely related to the classical stereographic projection, a relation that will become much more important later. It is remarkable how important it is that the point  $p$  is really placed *on* the conic. If it were inside the conic we would get a two-to-one relation between points on the conic and lines in the bundle  $\mathcal{L}_p$ . If the point  $p$  were outside the conic, not all lines of the bundle would intersect the conic at all. An intersection of a line and a conic corresponds to solving a quadratic equation. The fact that we consider a bundle at a point *on* the conic implies that we already know one of the two solutions of this quadratic equation. Thus solving the quadratic equation in principle can be reduced to a linear problem by factoring out the already known solution. The linearity is the deeper reason why there is a one-to-one correspondence between  $\mathcal{C}$  and the lines in the bundle.

Let us close this section with a remark on the group structure of the set of those transformations that leave  $\mathcal{C}$  invariant. Theorem 10.3 can be interpreted in the following way: *The group of all projective transformations that leaves a nondegenerate conic invariant is isomorphic to the group of projective transformations of  $\mathbb{RP}^1$ .*

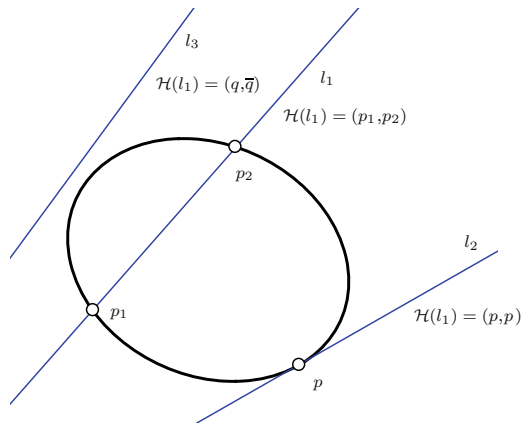


## 10.5 Hesse's "Übertragungsprinzip"

The last sections have made it clear that we can identify a nondegenerate conic with a projective line. In this section we will go one step further. We will demonstrate a way how one can interpret arbitrary lines and points of  $\mathbb{RP}^2$  as suitable objects on the projective line. This allows us to represent statements in the two-dimensional world of  $\mathbb{RP}^2$  by corresponding statements of certain objects on the projective line. The idea of this translation goes back to an article by Otto Hesse from 1866 [57]. Hesse was mainly interested in questions of invariant theory and studied several ways to linearize objects of higher degree. In his works around 1866 he was interested in generalizing the concept of duality. Duality allows us to derive for every theorem of projective geometry a corresponding dual theorem just by applying a dictionary that translates "point" by "line," "line" by "point," "intersection" by "meet," and so forth. In the same spirit Hesse formulated a principle that allowed him to derive a one-dimensional theorem from any two-dimensional theorem of projective geometry. He gave his principle the name "Übertragungsprinzip." A reasonable translation of this term could be "transfer principle."

His work had far-reaching consequences. It was used by Klein in his famous Erlanger Programm [65] to demonstrate the concept of equivalent geometries. It inspired further work and many interesting generalizations. Some of these generalizations had an important impact on the classification of Lie algebras and even on quantum theory. For a more elaborate treatment of this fascinating topic see [56]. In this book we will use the transfer principle for deriving elegant bracket expressions for geometric configurations involving conics and lines.

In his original work Hesse related points in  $\mathbb{RP}^2$  to solutions of one-dimensional quadratic forms. We will take a slightly more visual approach that allows us to represent the solutions of the quadratic forms directly as intersections of a conic with a line. As before, we consider a nondegenerate conic  $\mathcal{C}$  as an image of a projective line. Now to a line  $l$  in  $\mathbb{RP}^2$  we associate its two points of intersection with  $\mathcal{C}$ . A word of caution is necessary. First of all, not all lines will have two intersections with  $\mathcal{C}$ . This corresponds to the situation that Hesse studied, solutions of arbitrary quadratic forms with real coefficients. There may be two real solutions, two complex solutions (which are conjugates), or one (double) real solution. The three cases correspond to the situations in which the line intersects in two, in no, or in one point, respectively. To state Hesse's ideas in full generality we have also to deal with the complex solutions. This will be our first careful investigation of complex situations in projective geometry. Thus to treat Hesse's transfer principle properly, we must talk about  $\mathbb{CP}^1$  instead of  $\mathbb{RP}^1$ . However, the only objects we have to consider are pairs of points  $(p, q)$  that are either both real or complex conjugates ( $p = \bar{q}$ ) or coincide ( $p = q$ ). Figure 10.8 illustrates the three cases. For the following considerations one may either consider these



**Fig. 10.8** Hesse's transfer principle for lines. Each line is associated to a pair of points. In case the line does not intersect the conic, the points are complex and conjugates.

complex elements (all algebraic considerations work straightforwardly) or assume (for convenience) that the conic is large enough such that all lines under consideration intersect it in at least one point.

A line  $l$  that intersects the conic  $\mathcal{C}$  in two (real or complex) points  $p_1$  and  $p_2$  is represented by the pair  $\mathcal{H}_{\mathcal{C}}(l) := (p_1, p_2)$ . If  $l$  is tangent to the conic at point  $p$ , we represent it by the pair  $\mathcal{H}_{\mathcal{C}}(l) := (p, p)$ . In all our considerations related to Hesse's transfer principle the order of the points within such a pair will be irrelevant. Nevertheless, it is important to speak of pairs rather than sets to cover also the situation of a double point  $(p, p)$ .

If lines are represented by pairs of points, what is the corresponding representation of a point of  $\mathbb{RP}^2$ ? In Hesse's transfer principle points would be represented by projective transformations on the projective line that are furthermore involutions (i.e.,  $\tau^2 = \text{Id}$ ). Such a transformation is derived in the following way. For a point  $p$  not on  $\mathcal{C}$  we take two arbitrary distinct lines  $l$  and  $m$  through  $p$  that intersect  $\mathcal{C}$  and consider the pairs of points  $\mathcal{H}_{\mathcal{C}}(l) = (a_1, a_2)$  and  $\mathcal{H}_{\mathcal{C}}(m) = (b_1, b_2)$ . These four points are distinct, and since they lie on a nondegenerate conic, no three of them are collinear. Thus there is a unique projective transformation  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  that simultaneously interchanges  $a_1$  with  $a_2$  and  $b_1$  with  $b_2$ . In particular, this transformation leaves  $l$ ,  $m$ , and  $p$  invariant. Furthermore, we have (by Theorem 4.2)  $(a_1, a_2; b_1, b_2)c = (a_2, a_1; b_2, b_1)c$ . This in turn implies by Theorem 10.3 that  $\tau$  leaves the conic  $\mathcal{C}$  invariant. Such a projective transformation induces by Theorem 10.2 a corresponding transformation  $\tau_p$  on  $\mathcal{C}$  considered as  $\mathbb{RP}^1$ . This is the object to which  $p$  is translated. The crucial fact in the definition of  $\tau_p$  is that it depends only on the choice of  $p$  but is independent of the particular choice of  $l$  and  $m$ . We will not follow this line of thought here.

The reason for this is that we want to bypass a certain technical problem related to expressing a point  $p$  by a projective transformation  $\tau_p$ . If the point  $p$  is on the conic  $\mathcal{C}$ , then the above construction does not lead to a proper projective transformation, since  $a_1$  and  $b_1$  (or  $b_2$ ) become identical. Instead of introducing a concrete object that represents a point, we will characterize concurrence of lines  $k, l, m$  directly by a relation between the corresponding point pairs  $\mathcal{H}_{\mathcal{C}}(k)$ ,  $\mathcal{H}_{\mathcal{C}}(l)$ , and  $\mathcal{H}_{\mathcal{C}}(m)$ . This characterization also covers the degenerate cases in which the coincident point lies on  $\mathcal{C}$ .

**Theorem 10.5.** *Let  $\mathcal{C}$  be a conic and let  $k, l, m$  be lines in  $\mathbb{RP}^2$ . To exclude the complex case we assume that they intersect or touch the conic. If  $k, l, m$  are concurrent, then  $(\mathcal{H}_{\mathcal{C}}(k); \mathcal{H}_{\mathcal{C}}(l); \mathcal{H}_{\mathcal{C}}(m))$  form a quadrilateral set.*

We will prove this theorem by restriction to a remarkable special case by a suitable projective transformation. This special case was communicated by Yuri Matiyasevich (private communication), who discovered this remarkable configuration as a high-school student. Matiyasevich's configuration is a kind of geometric gadget for performing multiplications. He used this gadget to give a geometric construction for the prime numbers. We formulate it in the real Euclidean plane:

**Lemma 10.1.** *Let  $x$  and  $y$  be two real numbers. The join of the points  $(-x, x^2)$  and  $(y, y^2)$  crosses the  $y$ -axis at the point  $(0, x \cdot y)$ .*

*Proof.* We can prove this by direct calculation when we show that the three points are collinear.

$$\det \begin{pmatrix} -x & y & 0 \\ x^2 & y^2 & xy \\ 1 & 1 & 1 \end{pmatrix} = -x \cdot y^2 + y \cdot xy - (-x) \cdot xy - y \cdot x^2 = 0.$$

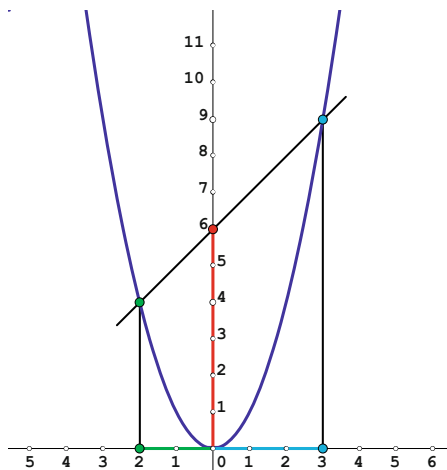
□

Figure 10.9 gives an impression of how the parabola-multiplication device works. For our purposes we must also cover the degenerate case  $y = -x$ . Then the join becomes a tangent and we obtain the following:

**Lemma 10.2.** *Let  $x$  be a real number. Then the tangent at  $(-x, x^2)$  to the parabola  $y = x^2$  crosses the  $y$ -axis at the point  $(0, -x^2)$ .*

*Proof.* This can also easily be checked by direct calculation. The tangent has slope  $-2x$ ; hence the tangent has equation  $f(t) = a - 2xt$ . Resolving for  $a$  gives  $x^2 = a - 2x(-x)$ . Thus  $a$  must be  $-x^2$ . □

Now, what has Matiyasevich's gadget to do with Hesse's transfer principle? The parabola plays the role of the conic. The points on the conic are vertically projected onto the  $x$ -axis. Thus the  $x$ -axis is the representation of  $\mathbb{RP}^1$  that is isomorphic to the points on the conic (the unique infinite point of the parabola corresponds to the point at infinity of the  $x$ -axis). The line shown



**Fig. 10.9** Multiplication of two real numbers using a parabola.

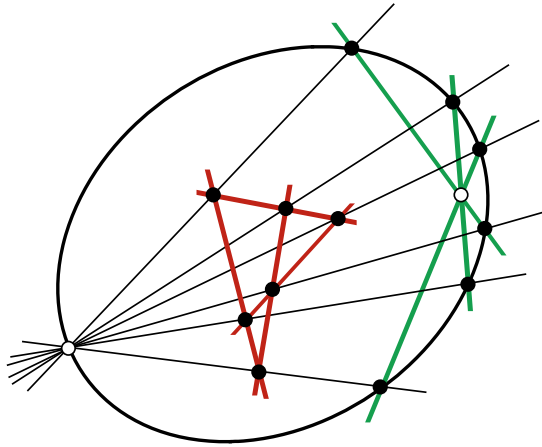
in Figure 10.9 intersects the conic in two points (the green and the blue ones). They are associated to their  $x$ -value by the projection. Thus the green and blue points on the  $x$ -axis correspond to the Hesse pair that represents the line. Now we are ready to prove Theorem 10.5 (which is essentially Hesse's transfer principle) as a simple application of Matiyasevich's construction.

*Proof of Theorem 10.5:* Since three tangents of a conic  $\mathcal{C}$  never intersect in one point, at least one of the lines must meet the conic in two points. After a suitable projective transformation we may assume that the conic  $p$  is the parabola  $y = x^2$  (in Euclidean coordinates) and that one of the lines (say  $k$ ) is the  $y$ -axis. We identify the  $x$ -axis together with its point at infinity  $\infty$  with the  $\mathbb{RP}^1$  associated to the conic. The corresponding mapping goes via vertical projection. Thus  $k$  is mapped to  $\mathcal{H}_{\mathcal{C}}(k) = (0, \infty)$ . Now assume that the other two lines  $l$  and  $m$  intersect the  $y$ -axis at the same point as required by the theorem. Let the corresponding point pairs on the  $x$ -axis be  $\mathcal{H}_{\mathcal{C}}(l) = (l_x, l_y)$  and  $\mathcal{H}_{\mathcal{C}}(m) = (m_x, m_y)$ . Since the two lines in the theorem intersect the  $y$ -axis in the same point, we can consider them as two instances of Matiyasevich's construction, and we get

$$(-l_x) \cdot (l_y) = (-m_x) \cdot (m_y).$$

This expression can be used to prove the corresponding quadset relation. For this we introduce homogeneous coordinates

$$\begin{pmatrix} l_x \\ 1 \end{pmatrix}, \begin{pmatrix} l_y \\ 1 \end{pmatrix}, \begin{pmatrix} m_x \\ 1 \end{pmatrix}, \begin{pmatrix} m_y \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



**Fig. 10.10** Hesse's transfer principle as an incidence theorem.

and calculate the characteristic quadset equation of Section 8.2. For the six points  $(l_x, l_y; m_x, m_y; 0, \infty)$  being a quadset we must show that

$$[l_x, \infty][m_x, l_y][0, m_y] = [l_x, m_y][m_x, \infty][0, l_y].$$

This expands to

$$\begin{vmatrix} l_x & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} m_x & l_y \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 0 & m_y \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} l_x & m_y \\ 1 & 1 \end{vmatrix} \begin{vmatrix} m_x & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & l_y \\ 1 & 1 \end{vmatrix}.$$

Expanding the determinants yields

$$(-1)(m_x - l_y)(-m_y) = (l_x - m_y)(-1)(-l_y),$$

which reduces to

$$-m_x m_y + l_y m_y = -l_x l_y + m_y l_y.$$

Subtracting  $l_y m_y$  on both sides leaves us exactly with the identity proven by Matiyasevich's equation.  $\square$

With Theorem 10.5 we have reduced the essence of Hesse's transfer principle to an incidence theorem in the projective plane. *Lines are represented by pairs of points. Three lines intersect if the corresponding three pairs of points form a quadrilateral set.* Figure 10.10 summarizes the essence of Hesse's transfer principle as an incidence theorem. The green lines are the three lines that intersect. The six points of intersection seen from one point on the boundary of the conic generate a line bundle that must form a quadrilateral set. The red part of the figure witnesses the quadset relation by the construction given in Figure 8.2.

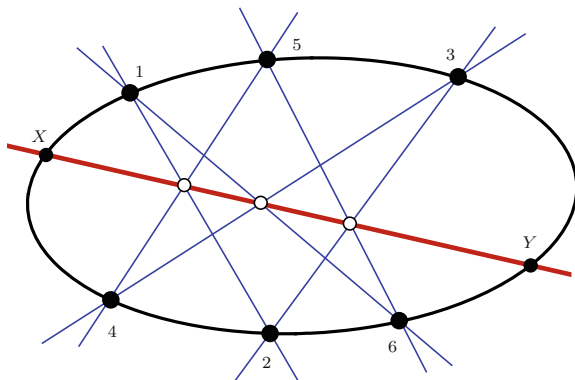


Fig. 10.11 Pascal's theorem.

## 10.6 Pascal's and Brianchon's Theorems

No exposition on conics would be complete without a treatment of Pascal's theorem. This theorem was discovered in 1640 by the famous Blaise Pascal and can be considered as a generalization of Pappos's theorem. Figure 10.11 shows an instance of this theorem.

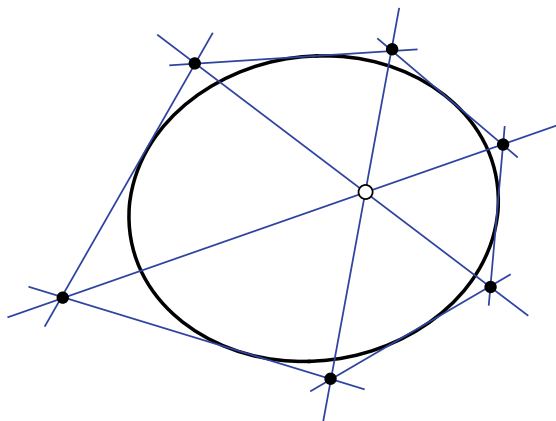
**Theorem 10.6 (Pascal's theorem).** *If  $1, \dots, 6$  are six points on a conic, then the three intersections of opposite sides of the hexagon  $(1, 2, 3, 4, 5, 6)$  are collinear.*

*Proof.* We already presented proofs of this theorem in Chapter 1. However, this time we want to add another proof, which is a simple application of Hesse's transfer principle. We may assume that the three intersection points are distinct, since otherwise they are trivially collinear. For the labeling refer to Figure 10.11. In order to apply the transfer principle we will simply express the three inner intersections of Pascal's theorem as quadrilateral set conditions. Since the six points  $1, \dots, 6$  all lie on the conic, we can identify them (applying the transfer principle) with points in  $\mathbb{RP}^1$ . We will also need two more points on the conic, namely its intersections  $X$  and  $Y$  with the central conclusion line (red). Also they are considered as points in  $\mathbb{RP}^1$ . Now, the fact that  $\overline{12}, \overline{45}, \overline{XY}$  meet in a point is equivalent to the condition that  $(1, 2; 4, 5; X, Y)$  forms a quadrilateral set. This corresponds to the algebraic condition

$$[1Y][52][X4] = [14][5Y][X2].$$

Similarly, the fact that  $\overline{34}, \overline{16}, \overline{XY}$  meet in a point can be encoded by the equation

$$[3Y][14][X6] = [36][1Y][X4].$$



**Fig. 10.12** Brianchon's theorem

Multiplying both left and right sides and canceling brackets that appear on both sides (the distinctness of the intersection points implies that they are nonzero) leaves us with

$$[52][3Y][X6] = [5Y][36][X2],$$

which implies that  $\overline{32}$ ,  $\overline{56}$ , and  $\overline{XY}$  meet in a point and thus proves the theorem.  $\square$

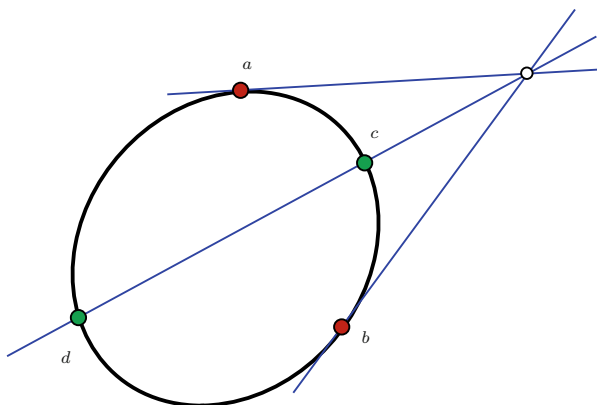
For reasons of completeness (and for later use) we also mention the dual of Pascal's theorem. It is named after Charles Julien Brianchon and was discovered in 1804 (more than 150 years after Pascal's theorem!).

**Theorem 10.7 (Brianchon's theorem).** *Let  $1, \dots, 6$  be six tangents to a conic (considered as the sides of a hexagon). Then the joins of opposite hexagon vertices meet in a point (see Figure 10.12).*

Pascal's theorem also holds in limit cases in which up to three consecutive points of the hexagon  $(1, \dots, 6)$  coincide. The join of two such consecutive points then becomes a tangent to the conic. We refer the reader to Section 1.4 for examples of such limit situations.

## 10.7 Harmonic points on a conic

As a (for now) final application of Hesse's transfer principle we want to show that it is extremely simple to construct a harmonic point on a nondegenerate conic. For this we again identify the conic  $\mathcal{C}$  with the projective line. If three



**Fig. 10.13** Construction of a harmonic quadruple  $(a, b; c, d) = -1$ .

points  $a, b, c$  on  $\mathcal{C}$  are given, we want to construct a fourth point  $d$  such that  $(a, b; c, d)_{\mathcal{C}} = -1$ . The construction is shown in Figure 10.13 and just consists of two tangents at  $a$  and  $b$  and a join of their intersection to  $c$ . By Hesse's transfer principle applied to this situation we get that  $(a, a; b, b; c, d)$  is a quadrilateral set (the tangents correspond to the double points  $(a, a)$  and  $(b, b)$ ). This means that

$$[ab][bd][ca] = [ad][ba][cb].$$

Dividing one term by the other and canceling the bracket  $[a, b]$  gives

$$\frac{[bd][ca]}{[ad][cb]} = -1,$$

which is after a slight reordering of the letters easily recognized as the condition for  $(a, b; c, d)$  to be harmonic. The reasoning also works in the other direction. Thus we have just proved the following theorem:

**Theorem 10.8.** *Let  $a, b, c, d$  be four points on a conic  $\mathcal{C}$ . Then the cross-ratio  $(a, b; c, d)_{\mathcal{C}}$  equals  $-1$  if and only if the tangents through  $a$  and  $b$  to  $\mathcal{C}$  meet on the line  $\text{join}(c, d)$ .*

It is an amazing fact that the construction of a harmonic point on a conic turns out to be even simpler than the corresponding task on a line. This reflects on the one hand the fundamental importance of conics and on the other hand the fact that conics are closely related to involutions, and involutions are closely related to harmonic sets. In particular, if we fix  $a$  and  $b$  and consider the construction of point  $d$  as a function  $\tau: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  with



$\tau(c) = d$ , then this map  $\tau$  turns out to be a projective involution with fixed points  $a$  and  $b$ .



## Calculating with Conics

*We [Kaplansky and Halmos] share a love of linear algebra. I think it is our conviction that we'll never understand infinite-dimensional operators until we have a decent mastery of finite matrices. And we share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.*

*Irving Kaplansky  
in Paul Halmos: Celebrating 50 Years of Mathematics*

A major part of classical elementary geometry was concerned with the question of which constructions can be carried out with a straightedge and compass. The decisive primitive operations here are connecting two points by a line, drawing a circle with a radius given by two other points, and marking intersections of objects as new points. In our projective framework we do not have circles, but still can consider elementary constructions with the objects we have studied so far (points, lines, and conics). Since we are interested in particular in *calculating* with geometric objects, we in particular want to know how we can *compute* the results of geometric primitive operations if the parameters of the involved objects are given. So far, we can roughly associate the classical straightedge/compass operations with our projective operations in the following way:

- connecting two points by a line corresponds to the *join* operation and can be carried out by a cross product,
- intersecting two lines corresponds to the *meet* operation and is also carried out by the cross product,
- constructing a circle from two points can be associated to our construction of a *conic by five points* as described in Section 10.1.

Furthermore, we had additional operations for

- constructing polars of points and lines with respect to a conic (this included the calculation of a tangent),
- transforming points/lines/conics by projective transformations,
- calculating the matrix of the dual of a given conic.

So far we do not have an algebraic equivalent of the classical operations of intersecting a line and a circle and of intersecting two circles. This chapter is (among others) dedicated to this task. We will develop algebraic methods for intersecting conics with lines and conics with conics. Clearly, these operations can be carried out by solving corresponding systems of polynomial equations. However, we will try to make the calculations for these operations as natural as possible in the framework of homogeneous coordinates and matrix representations for conics. This chapter is meant as a collection of recipes for such kinds of primitive operations. Many of these recipes are used in the implementation of the dynamic geometry program Cinderella [112].

## 11.1 Splitting a Degenerate Conic

Before we turn our attention to the problem of intersecting a conic with other objects we will study how it is possible to derive homogeneous coordinates for the two lines of a degenerate conic from the matrix of a conic. This will turn out to be a useful operation later on.

Assume that a conic  $\mathcal{C}_A$  is given by a symmetric matrix  $A$  such that as usual,  $\mathcal{C}_A = \{p \mid p^T A p = 0\}$ . If the conic is degenerate and consists of two lines or of one double line, then  $A$  will not have full rank. Thus we can determine a degenerate situation by testing  $\det(A) = 0$ . If a conic consists of two distinct lines with homogeneous coordinates  $g$  and  $h$ , then its symmetric (rank-2) matrix  $A$  can be written as  $A = gh^T + hg^T$  (up to a scalar multiple). The matrices  $gh^T$  and  $hg^T$  would in principle generate the same conic, but they are not symmetric. However, knowing one of these matrices (for instance  $gh^T$ ) would be equivalent to knowing the homogeneous coordinates of the two lines, since the columns of this matrix are just scalar multiples of  $g$  and the rows are just scalar multiples of  $h$ . Any nonzero column (resp. row) could serve as homogeneous coordinates for  $g$  (resp.  $h$ ). This can be seen easily by observing that the disjunction of the conditions  $\langle p, h \rangle = 0$  and  $\langle p, g \rangle = 0$  can be written as

$$0 = \langle p, g \rangle \cdot \langle p, h \rangle = (p^T g)(h^T p) = p^T (gh^T) p.$$

So, splitting a degenerate conic into its two lines essentially corresponds to finding a rank-1 matrix  $B$  that generates the same conic as  $A$ . The quadratic form is linear in the corresponding matrix (i.e., we have  $p^T(A + B)p = p^T A p + p^T B p$ ). Furthermore, those matrices for which the quadratic form

is identically zero are exactly the skew-symmetric matrices (those with  $A^T = -A$ ). All in all, for decomposing a symmetric degenerate matrix  $A$  into two lines we have to find a skew-symmetric matrix  $B$  such that  $A + B$  has rank 1. Thus in our case we have to find parameters  $\lambda$ ,  $\mu$ , and  $\tau$  such that the following matrix sum has rank 1:

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} + \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix}.$$

The determinant of every  $2 \times 2$  submatrix of a rank-1 matrix must vanish. Thus necessary conditions for the parameters are

$$\begin{vmatrix} a & b + \tau \\ b - \tau & c \end{vmatrix} = 0; \quad \begin{vmatrix} a & d - \mu \\ d + \mu & f \end{vmatrix} = 0; \quad \begin{vmatrix} c & e + \lambda \\ e - \lambda & f \end{vmatrix} = 0.$$

Resolving for  $\lambda$ ,  $\tau$ , and  $\mu$  gives

$$\tau^2 = -\begin{vmatrix} a & b \\ b & c \end{vmatrix}; \quad \mu^2 = -\begin{vmatrix} a & d \\ d & f \end{vmatrix}; \quad \lambda^2 = -\begin{vmatrix} c & e \\ e & f \end{vmatrix}.$$

This determines the parameters  $\lambda$ ,  $\mu$ , and  $\tau$  up to their sign. In principle one could test all eight possibilities to find a suitable choice that makes the entire matrix a rank-1 matrix. However, there is also a more direct way of calculating the values with their correct sign. For this we associate to the parameter vector  $p = (\lambda, \mu, \tau)^T$  the skew-symmetric matrix

$$\mathcal{M}_p := \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix}.$$

Left multiplication by the matrix  $\mathcal{M}_p$  encodes performing a cross product with the vector  $p$ . A simple calculation shows that for any three-dimensional vector  $q$  we have

$$\mathcal{M}_p \cdot q = p \times q.$$

**Lemma 11.1.** *Let  $A$  be a rank-2 symmetric  $3 \times 3$  matrix that defines a conic consisting of two distinct lines. Let  $p$  be the point of intersection of these lines. Then for a suitable factor  $\alpha$  the matrix  $A + \alpha \mathcal{M}_p$  has rank 1.*

*Proof.* Let  $g$  and  $h$  be homogeneous coordinates for the two lines. We may assume that these coordinates are scaled such that  $A$  has the form  $gh^T + hg^T$ . The intersection of these lines is  $g \times h$ . We have for a suitable factor  $\alpha$  the equation  $g \times h = \alpha p$ . If we consider the difference  $gh^T - hg^T$ , we obtain the skew-symmetric matrix

$$gh^T - hg^T = \begin{pmatrix} 0 & g_1h_2 - g_2h_1 & g_1h_3 - g_3h_1 \\ g_2h_1 - g_1h_2 & 0 & g_2h_3 - g_3h_2 \\ g_3h_1 - g_1h_3 & g_3h_2 - g_2h_3 & 0 \end{pmatrix}.$$

Comparison of coefficients shows that this matrix is nothing other than  $\mathcal{M}_{g \times h}$ . Thus we obtain

$$gh^T - hg^T = \mathcal{M}_{g \times h} = \mathcal{M}_{\alpha p} = \alpha \mathcal{M}_p.$$

With this we obtain

$$A + \alpha \mathcal{M}_p = (gh^T + hg^T) + (gh^T - hg^T) = 2gh^T.$$

Thus the result must have the desired rank-1 form, and the theorem is proven.  $\square$

In particular, if for specific coordinates  $g$  and  $h$  we have  $A = gh^T + hg^T$  and  $p = g \times h$ , we can choose the factor  $\alpha = 1$  and obtain

$$A - \mathcal{M}_p = 2hg^T.$$

If we instead add the matrices, we obtain  $A + \mathcal{M}_p = 2gh^T$ . The previous lemma allows us to calculate the corresponding rank-1 matrix if a symmetric matrix of a degenerate conic is given. However, it has one big disadvantage: We know neither  $p$  nor  $\alpha$  in advance. There are several circumstances (we will encounter them later) in which we, for instance, know  $p$  in advance. However, it is also possible to calculate  $p$  from the matrix  $A$  without too much effort. For this we have to use the formula

$$(gh^T + hg^T)^\Delta = -(g \times h)(g \times h)^T,$$

which we already used in Section 9.5. The matrix  $B = (g \times h)(g \times h)^T$  is a rank-1 matrix  $pp^T$  with  $p = g \times h$ . Thus each row/column of this matrix is a scalar multiple of  $p$ . Furthermore, the diagonal entries of this matrix correspond to the squared coordinate values of  $g \times h$ . Thus one can extract the coordinates of  $g \times h$  by searching for a nonzero diagonal entry of  $B$ , say  $B_{i,i}$ , and set  $p = B_i / \sqrt{B_{i,i}}$ , where  $B_i$  denotes the  $i$ th column of  $B$ . With this assignment we can calculate the matrix  $A^\Delta + \mathcal{M}_p$ , which gives either  $2gh^T$  or  $2hg^T$  depending on the sign of the square root. Altogether, we can summarize the procedure of splitting a matrix  $A$  that describes a conic consisting of two distinct lines.:

- 1:  $B := A^\Delta$ ;
- 2: Let  $i$  be the index of a nonzero diagonal entry of  $B$ ;
- 3:  $\beta = \sqrt{B_{i,i}}$ ;
- 4:  $p = B_i / \beta$ , where  $B_i$  is the  $i$ th column of  $B$ ;

- 5:  $C = A + \mathcal{M}_p$ ;
- 6: Let  $(i, j)$  be the index of a nonzero element  $C_{i,j}$  of  $C$ ;
- 7:  $g$  is the  $i$ th row of  $C$ ,  $h$  is the  $j$ th column of  $C$ .

After this calculation  $g$  and  $h$  contain the coordinates of the two lines. If  $A$  describes a conic consisting of a double line, then this procedure does not apply, since  $B$  will already be the zero matrix. Then one can directly split the matrix  $A$  by searching a nonzero row and a nonzero column.

## 11.2 The Necessity of “If” Operations

Compared to all our other geometric calculations (such as computing the *meet* of two lines, the *join* of two points, or a *conic through five* given points), the last computation for splitting a conic is considerably different. During the computation we had to inspect a  $3 \times 3$  matrix for a nonzero entry in order to extract a nonzero row (or a nonzero column). One might ask whether it is possible to perform such a computation without such an inspection of the matrix, which intrinsically requires *branching* if one implements such a computation on a computer. Indeed, it is not possible to do the splitting operation for all instances without any branches. In other words, *there is no closed formula for extracting homogeneous coordinates for the two lines of a degenerate conic*. No matter which calculation is performed, there will always be sporadic special cases that are not covered by the concrete formula. The reason for this is essentially of a topological nature. No continuous formula can be used for performing the splitting operation without any exceptions. This is already the case for extracting the double line of a conic that consists of a double line, as the next theorem shows.

**Theorem 11.1.** *Let  $\text{Deg} = \{ll^T \mid l \in \mathbb{R}^3 - \{(0, 0, 0)^T\}\}$  be the set of all  $3 \times 3$  symmetric rank-1 matrices. Let  $\phi: \text{Deg} \rightarrow \mathbb{R}^3$  be a continuous function that associates to each matrix  $pp^T$  a real scalar multiple  $\lambda p$  of the vector  $p$ . Then there is a matrix  $A \in \text{Deg}$  for which  $\phi(A) = (0, 0, 0)^T$ .*

*Proof.* Assume that  $\phi$  is a continuous function that associates to each matrix  $pp^T$  a scalar multiple  $\lambda p$  of  $p$ . We consider the path  $\tau: [0, \pi] \rightarrow \mathbb{R}^3$  with  $\tau(t) = (\cos(t), \sin(t), 0)^T$ . By our assumption, the function  $\phi(\tau(t)\tau(t)^T)$  must be continuous on the interval  $[0, \pi]$ . We have  $\tau(0) = (1, 0, 0)^T =: a$  and  $\tau(\pi) = (-1, 0, 0)^T = -a$  and therefore  $\phi(\tau(0)\tau(0)^T) = \phi(aa^T) = \phi((-a)(-a)^T) = \phi(\tau(\pi)\tau(\pi)^T)$ . By the definition of  $\phi$  we must have  $\phi(\tau(t)\tau(t)^T) = \lambda(\tau(t))$ . While  $t$  moves from 0 to  $\pi$ , the factor  $\lambda$  must itself behave continuously, since in every sufficiently small interval at least one of the coordinates of  $\tau(t)$  is constantly nonzero. However, we have that  $\phi(\tau(0)\tau(0)^T) = \tau(0)$  and

$\phi(\tau(\pi)\tau(\pi)^T) = \phi(\tau(0)) = -\tau(\pi)$ . Since  $\lambda$  was assumed to be real this implies by the intermediate value theorem that for at least one parameter  $t_0 \in [0, \pi]$  we must have  $\lambda = 0$ .  $\square$

The matrix  $ll^T$  that corresponds to the point  $p$  for which we have  $\phi(ll^T) = (0, 0, 0)^T = 0 \cdot p$  does not lead to a meaningful nondegenerate evaluation of  $\phi$ . The interpretation of this fact is a little subtle. It means that on the coordinate level there is no continuous way of extracting the double line of a conic  $\mathcal{C}_A$  from a symmetric rank-1 matrix  $A$ . On the other hand, the considerations of the last sections show that on the level of geometric objects there is a way to extract the coordinates of the line  $l$  from the matrix  $ll^T$  which must be necessarily continuous in the topology of our geometric objects. There is just no way of doing these computations without using branching within the calculations. This was reflected by the fact that we explicitly had to search for nonzero entries in our  $3 \times 3$  matrices.

In fact, the effect treated in this section is just the beginning of a long story that leads to the conclusion that elementary geometry and effects from complex function theory (such as monodromy, multivalued functions) are intimately interwoven. We will return to these issues in the very final section of this book.

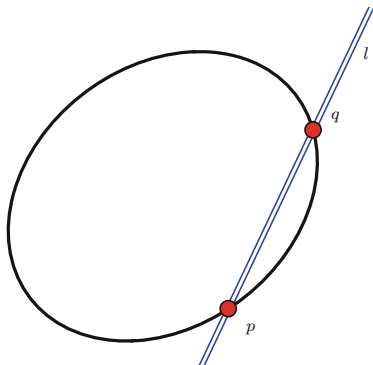
### 11.3 Intersecting a Conic and a Line

After all this preparatory work the final task of this chapter turns out to be relatively simple. We want to calculate the intersection of a conic given by a symmetric  $3 \times 3$  matrix  $A$  and a line given by its homogeneous coordinates  $l$ . Clearly the task is in essence nothing but solving a quadratic equation. However, we want to perform the operation that is as closely as possible related to the coordinate representation. For this we will use the operation of splitting a matrix that represents a degenerate conic as introduced in Section 11.1 as a basic building block. Essentially, the square root needed to solve a quadratic equation will be the one required for this operation.

Our aim will be to derive a closed formula for the degenerate conic that consists of the line  $l$  as double line and whose dual consists of the two points of intersection. Such a conic is given by a pair of matrices  $(A, B)$ , where  $A$  is a rank-1 symmetric matrix and describes the double line  $l$  and where  $B$  is a symmetric rank-2 matrix with  $B^\Delta = A$  that describes the dual conic and with this the position of the two points of intersection. Splitting the matrix  $B$  yields the two points of intersection.

So, how do we derive the matrix  $B$ ? We will characterize the matrix via its properties. Let  $p$  and  $q$  be the two (not necessarily distinct) intersection points. The quadratic form  $m^T B m$  must have the property that it vanishes for exactly those lines  $m$  that pass through (at least) one of the points  $p$  and  $q$ . A matrix with these properties is given by





**Fig. 11.1** Intersecting a conic and a line: Consider the line as a conic consisting of a double line and two points on it.

$$B = \mathcal{M}_l^T A \mathcal{M}_l.$$

To see this, we calculate the quadratic form  $m^T B m$ . Using the property  $\mathcal{M}_l m = l \times m$ , we get

$$m^T B m = m^T \mathcal{M}_l^T A \mathcal{M}_l m = (\mathcal{M}_l m)^T A (\mathcal{M}_l m) = (l \times m)^T A (l \times m).$$

The right side of this chain of equations can be interpreted as follows:  $l \times m$  calculates the intersection of  $l$  and  $m$ . The condition  $(l \times m)^T A (l \times m)$  tests whether this intersection is also on the conic  $\mathcal{C}_A$ . Thus as claimed,  $\mathcal{M}_l^T A \mathcal{M}_l$  is the desired matrix  $B$ . It is the matrix of a dual conic describing two points on  $l$ . Splitting this matrix finally gives the intersections in question.

Compared to Section 11.1 we are this time in the dual situation. We want to split a matrix of a dual conic into two *points*. For this we have to transform it into an equivalent rank-1 matrix (one that defines the same conic) by adding a skew-symmetric matrix. For the splitting procedure we do not have to apply the full machinery of Section 11.1. This time we are in the good situation that we already know the skew-symmetric matrix up to a multiple. It must be the matrix  $\mathcal{M}_l$ , since  $l$  was by definition the join of the two intersection points. Thus the desired rank-1 matrix has the form

$$\mathcal{M}_l^T A \mathcal{M}_l + \alpha \mathcal{M}_l$$

for a suitably chosen factor  $\alpha$ . This factor must be chosen in a way that the resulting matrix has rank 1. The parameter  $\alpha$  can be simply determined by considering a suitable  $2 \times 2$  submatrix of the resulting matrix.

All in all, the procedure of calculating the intersections of a line  $l$  and a conic given by  $A$  can be described as follows (without loss of generality we assume that the last coordinate entry of  $l = (\lambda, \mu, \tau)^T$  is nonzero):

- 1:  $B = \mathcal{M}_l^T A \mathcal{M}_l$ ;
- 2:  $\alpha = \frac{1}{\tau} \sqrt{-\begin{vmatrix} B_{1,1} & B_{1,2} \\ B_{1,2} & B_{2,2} \end{vmatrix}}$ ;
- 3:  $C = B + \alpha \mathcal{M}_l$ ;
- 4: Let  $(i, j)$  be the index of a nonzero element  $C_{i,j}$  of  $C$ ;
- 5:  $p$  is the  $i$ th row of  $C$ ,  $q$  is the  $j$ th column of  $C$ .

The choice of  $\alpha$  in the second row ensures that the matrix  $C$  will have rank 1. The particular sign of  $\alpha$  is irrelevant, since a sign change would result in interchanging the points  $p$  and  $q$ . If the entry  $\tau$  of  $l$  were zero, one would have to take a different  $2 \times 2$  matrix for determining the value of  $\alpha$ . It is also a remarkable fact that if for some reason one is not interested in the individual coordinates of  $p$  and  $q$  but is interested in treating them as a pair, then all necessary information is already encoded in the matrix  $B$ . Furthermore, notice that for the calculation of the individual coordinates it is necessary to use a square-root operation just once. This is unavoidable, since intersecting a conic and a line can be used to solve a quadratic equation.

## 11.4 Intersecting Two Conics

Now, we want to intersect two conics. For this we will use the considerations of the last sections as auxiliary primitives and will reduce the problem of intersecting two conics to the problem of intersecting a conic with a line. Intersecting a conic and a line as considered in the previous section was essentially equivalent to solving a quadratic equation. Therefore it was necessary to use at least one square root operation. This will no longer be the case for the intersection of two conics. There the situation will be worse. Generally, two conics will have *four* more or less independent intersections. This indicates that it is necessary to solve a polynomial equation of degree four for the intersection operation. However, we will present a method that requires us to solve only a cubic (degree-3) equation. This results from the fact that in a certain sense the *algebraic difficulty* of solving degree-three and degree-four equations is essentially the same.

The idea for calculating the intersection of two conics is very simple. We assume that the two conics  $\mathcal{C}_A$  and  $\mathcal{C}_B$  are represented by matrices  $A$  and  $B$ . It is helpful for the following considerations to assume that the two conics have four real intersections. However, all calculations presented here can be carried out as well over the complex numbers. All linear combinations  $\lambda A + \mu B$  of the matrices represent conics that pass through the same four points of intersection as the original matrices. In the *bundle of conics*  $\{\lambda A + \mu B \mid \lambda, \mu \in \mathbb{R}\}$  we now search for suitable parameters  $\lambda$  and  $\mu$  such that the matrix  $\lambda A +$

$\mu B$  is degenerate. After this we split the degenerate conic by the procedure described in Section 11.1. Then we just have to intersect the two resulting lines with one of the conics  $\mathcal{C}_A, \mathcal{C}_B$  by the procedure described in Section 11.2.

In order to get a degenerate conic of the form  $\lambda A + \mu B$  we must find  $\lambda, \mu$ , such that

$$\det(\lambda A + \mu B) = 0.$$

At least one of the parameters  $\lambda, \mu$  must be nonzero in order to get a proper conic. The problem of finding such parameters leads essentially to the problem of solving a cubic equation. To see this, one can simply expand the above determinant and observe that each summand contains a factor of the form  $\lambda^i \mu^{3-i}$  with  $i \in \{0, 1, 2, 3\}$ . Collecting all these factors leads to a polynomial equation of the form

$$\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0.$$

We can easily calculate the parameters  $\alpha, \beta, \gamma, \delta$  using the multilinearity of the determinant function. Assume that the matrix  $A$  consists of column vectors  $A_1, A_2, A_3$  and that the matrix  $B$  consists of column vectors  $B_1, B_2, B_3$ . Expanding  $\det(\lambda A + \mu B)$  yields

$$\begin{aligned} \det(\lambda A + \mu B) = & \lambda^3 [A_1, A_2, A_3] \\ & + \lambda^2 \mu ([A_1, A_2, B_3] + [A_1, B_2, A_3] + [B_1, A_2, A_3]) \\ & + \lambda \mu^2 ([A_1, B_2, B_3] + [B_1, A_2, B_3] + [B_1, B_2, A_3]) \\ & + \mu^3 [B_1, B_2, B_3]. \end{aligned}$$

Thus we get

$$\begin{aligned} \alpha &= [A_1, A_2, A_3], \\ \beta &= [A_1, A_2, B_3] + [A_1, B_2, A_3] + [B_1, A_2, A_3], \\ \gamma &= [A_1, B_2, B_3] + [B_1, A_2, B_3] + [B_1, B_2, A_3], \\ \delta &= [B_1, B_2, B_3]. \end{aligned}$$

If we find suitable  $\lambda, \mu$  that solve  $\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0$ , then  $\lambda A + \mu B$  will represent a degenerate conic. From this degenerate conic it is easy to calculate the intersections of the original conics. Thus the problem of intersecting two conics ultimately leads to the problem of solving a cubic polynomial equation (and this is unavoidable). The story of solving cubic equations goes back to the sixteenth century and has a long and exciting history. (This is one of *the* legends in mathematics, including human tragedies, challenges, vanity, competition. The main actors in this play were Scipione del Ferro, Anton Maria Fior, Nicolo Tartaglia, Girolamo Cardano, and Lodovico Ferrari, all of whom lived between 1465 and 1569. In Chapter 15 we will give a brief historical overview. We refer readers interested in elaborate details to the book by Yaglom [136], the novel “Der Rechenmeister” by Jörgensen [61], and the numerous articles on this topic on the Internet.)

For our purposes we will confine ourselves to a direct way for solving a cubic equation. Our procedure has three nice features compared to usual solutions of cubic equations. It works in the homogeneous setting where we ask for values of  $\lambda$  and  $\mu$  instead of just a one-variable version that does not handle “infinite” cases properly. It needs to calculate exactly one square root and exactly one cube root, and we will not have to take care which specific roots of all complex possibilities we take. It works on the original cubic equation (most solutions presented in the literature work only for a reduced equation where  $\beta = 0$ ).

All calculations in our procedure have to be carried out over the complex numbers, since it may happen that intermediate results are no longer real. For the computation we will need one of the cube roots of unity as a constant. We abbreviate it by

$$\omega = -\frac{1}{2} + i \cdot \sqrt{\frac{3}{4}}.$$

A reasonable procedure for solving the equation

$$\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0$$

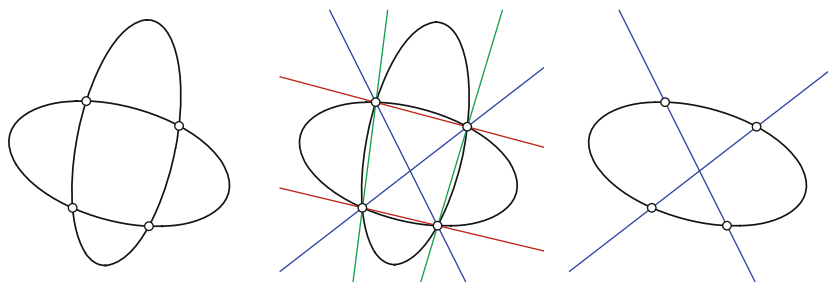
is given by the following sequence of operations (don’t ask why the parameters and formulas work; it’s a long story):

- 1:  $W = -2\beta^3 + 9\alpha\beta\gamma - 27\alpha^2\delta$ ;
- 2:  $D = -\beta^2\gamma^2 + 4\alpha\gamma^3 + 4\beta^3\delta - 18\alpha\beta\gamma\delta + 27\alpha^2\delta^3$ ;
- 3:  $Q = W - \alpha\sqrt{27D}$ ;
- 4:  $R = \sqrt[3]{4Q}$ ;
- 5:  $L = (2\beta^2 - 6\alpha\gamma, -\beta, R)^T$ ;
- 6:  $M = 3\alpha(R, 1, 2)^T$ ;

The two vectors  $L$  and  $M$  are the key to finding the three solutions for the final computation of  $\lambda$  and  $\mu$ . For this we have to compute

$$\begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \quad \begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}.$$

The pairs  $(\lambda_1, \mu_1)$ ,  $(\lambda_2, \mu_2)$ , and  $(\lambda_3, \mu_3)$  are the three solutions of the cubic equation. In Step 4 of the above procedure we have to choose a specific cube root. If  $R$  is one cube root, the other two cube roots are  $\omega R$  and  $\omega^2 R$ . The reader is invited to convince himself that no matter which of these cube roots we take, we obtain the same set of solutions (in fact they are permuted). Similarly, if we change the sign of the square root in Step 3, two of the solutions are interchanged.



**Fig. 11.2** Intersecting two conics: The original problem—the three degenerate conics—the reduced problem.

After collecting all these pieces it is easy to use them to give a procedure for intersecting two conics. We just have to put the pieces together. If  $A$  and  $B$  are the two matrices representing the conics, we can proceed in the following way (this time we just give rough explanations of the steps instead of detailed formulas).

- 1: Calculate  $\alpha, \beta, \gamma, \delta$  as described before.
- 2: Find a solution  $(\lambda, \mu)$  of the cubic equation  $\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0$ .
- 3: Let  $C = \lambda A + \mu B$ .
- 4: Split the conic  $C$  into two lines  $g$  and  $h$ .
- 5: Intersect both lines  $g$  and  $h$  with the conic  $C_A$ .

All in all, we obtain four intersections, two for each of the two lines  $g$  and  $h$ . Figure 11.2 illustrates the process. We start with the two conics and calculate one degenerate conic that may be obtained as a linear combination of them. Then we split this degenerate conic into two lines and intersect each of these lines with one of the original conics. There are a few subtleties concerning which intermediate results are real and which are complex. We will discuss them in the next section.

## 11.5 The Role of Complex Numbers

Let us take a step back and look at what we did in the previous section. At the beginning of our discussions on intersecting conics with conics we said that all calculations should be carried out over the complex numbers. In all previous chapters we focused on *real* projective geometry. However, solving the cubic equation may make it inevitable to have complex numbers at least

as an intermediate result (if the value of  $D$  calculated in step two of our cubic equation procedure becomes negative).

We first discuss what it means to have a *real* object in the framework of homogeneous coordinates. In all our considerations so far we have agreed to identify coordinate vectors that differ only by a scalar multiple. If we work with complex coordinates we will do essentially the same. However, this time we will also allow complex multiples. We will call a coordinate vector  $p$  “real” if there is a (perhaps complex) scalar  $s$  such that  $s \cdot p$  has only real coordinates. In this sense the vector  $(1 + 2i, 3 + 6i, -2 - 4i)^T$  represents a real object, since we can divide by  $1 + 2i$  and obtain the real vector  $(1, 3, -2)^T$ . Similarly, the vector  $(1, i, 0)$  is a proper complex vector, since no matter by which nonzero number we multiply it, at least one of the entries will be complex. It is an amazing fact that we may get real objects from calculations with proper complex objects. Consider the following calculation, which could be considered the join of two proper complex points:

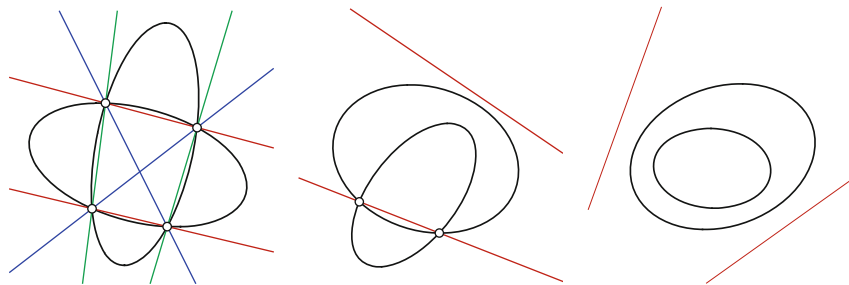
$$\begin{pmatrix} 1 + 2i \\ 3 + i \\ 1 - i \end{pmatrix} \times \begin{pmatrix} 1 - 2i \\ 3 - i \\ 1 + i \end{pmatrix} = \begin{pmatrix} 8i \\ -6i \\ 10i \end{pmatrix} = 2i \cdot \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}.$$

The coordinates of the result are complex, but they still represent the real object  $(4, -3, 5)^T$ . The reason for this is that in this example we took the join of two complex conjugate points. More generally, we get for a point  $p + iq$ , with real vectors  $p$  and  $q$ ,

$$(p + iq) \times (p - iq) = p \times p + (iq) \times p + p \times (-iq) + (iq) \times (-iq) = 2i(q \times p).$$

The join (meet) of two conjugate complex points (lines) is a real geometric object. This perfectly fits in our geometric intuition. Imagine you intersect a line  $l$  with a conic  $\mathcal{C}_A$ . If the line is entirely outside the conic, they do not have real intersections. As algebraic solution we get two complex conjugate points. Joining these two points, we get the real line  $l$  again. Generally, when we deal with homogeneous coordinates we will call an object real if there is a scalar multiple that simultaneously makes all entries real. We will apply this definition to points, lines, conics, transformations and also to bundles of objects parameterized by homogeneous coordinates.

Now let us return to the problem of intersecting conics. If we have two real conics  $\mathcal{C}_A$  and  $\mathcal{C}_B$ , then the parameters  $\alpha, \dots, \delta$  for the cubic equation will be real as well. A cubic equation in  $\mathbb{C}$  has three solutions (if necessary counted with multiplicity). Except for degenerate cases in which two or all three of these solutions coincide, we will have one of the following two cases: Either all these three solutions are real or just one of the solutions is real and the other two are complex conjugates. In any case we will have at least one real solution. The term “real solution” in the homogeneous setup in which we identify scalar multiples of  $(\lambda, \mu)$  means that there is a scalar  $s$  that makes



**Fig. 11.3** Real degenerate conics from a pair of conics.

both coordinates of  $s \cdot (\lambda, \mu)$  simultaneously real. In other words, the quotient  $\lambda/\mu$  is real. Such a real solution corresponds to the fact that the degenerate conic  $C = \lambda A + \mu B$  is real again. If the conics have four points in common, then we will have indeed three real solutions corresponding to the three real solutions of the cubic equation. If the two conics have only two points in common, then there will be only one real solution. This solution will consist of two lines. One of these lines is the join of the intersection points; the other is another line not hitting the conics in any real points, but still it passes through the other two intersections of the conics, which are properly complex and conjugates of each other. If the two conics do not have any real intersection, then there is still a real solution to the cubic equation. We still get one real degenerate conic in the bundle generated by the two conics. This degenerate conic consists of two lines each of them passing through a pair of complex conjugate intersection points. The three situations are shown in Figure 11.3.

It is also very illuminating to study the case in which we pass from the “three real solutions” situation to the “one real two complex conjugates” situation. In this case the cubic equation will have a real double root and a real single root. This means that two of the three real degenerate conics in the bundle  $\lambda A + \mu B$  will coincide. Geometrically, this corresponds to the case in which the two conics meet tangentially at one point and have two real intersections elsewhere. Figure 11.4 shows a situation an “epsilon” before the tangent situation. In the picture we still have four real intersections. However, two of them approach each other tightly. It can be seen how in this case two of the degenerate conics almost coincide as well, and how one of the lines of the other conic almost becomes a tangent. In the limit case the two first degenerate conics will really coincide and the tangent line of the third conic will really be a tangent at the point of tangency of the two conics.

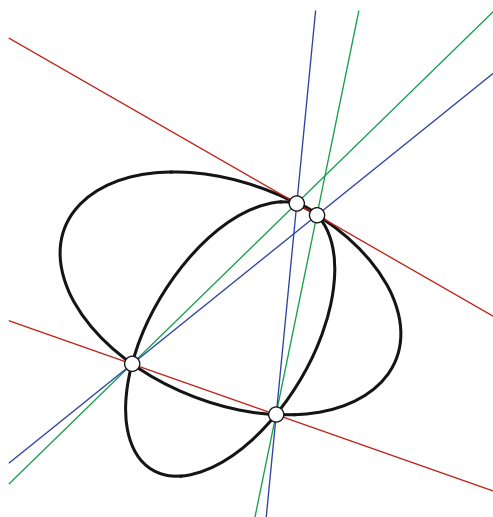
What does all this mean for our problem of calculating the intersections of two conics? If we want to restrict ourselves to real calculations whenever possible, then it might be reasonable to pick the real solution of the cubic and

proceed with this one. Then we finally have to intersect two real lines with a conic. Still it may happen that one or both of the lines do not intersect the conic in real points. However, whenever we have real intersection points we will find them by this procedure by intersecting a real line and a real conic.

If we think of implementing such an operation in a computer program, it may also be the case that the underlying math library can properly deal with complex numbers (it should anyway for solving the cubic equation). In such a case we do not have to explicitly pick a real solution. We can take any solution we want. If we by accident pick a complex solution, then we will get a complex degenerate conic, which splits into two complex lines. However, intersecting the lines with one of the original conics will result in the correct intersections. If the correct intersections turn out to be real points, then it just may be necessary to extract a common complex factor from the homogeneous coordinates.

## 11.6 One Tangent and Four Points

As a final example of a computation we want to derive a procedure that calculates a conic that passes through four points and at the same time is tangent to a line (see Figure 11.5 for the two possible solutions for an instance of such a problem). Based on the methods we have developed so far, there are several ways to approach this construction problem. Perhaps the most straightforward is to construct a fifth point on the conic and then



**Fig. 11.4** An almost degenerate situation.



calculate the conic through these five points. This fifth point may be chosen on the line itself. Then it must be at the position where the resulting conic touches the line. Generally, there are two possible positions for such a point, corresponding to the two possible conics that satisfy the tangency conditions.

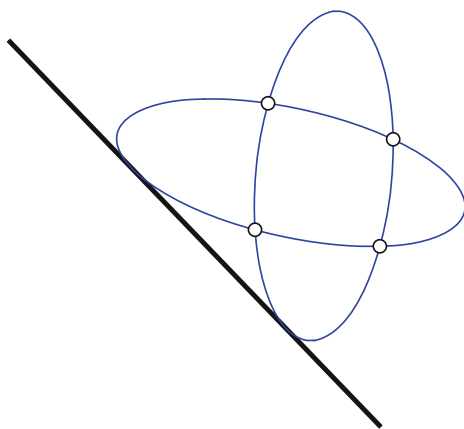
The crucial observation that allows us to calculate these two points is the fact that pairs of points that are generated by intersecting all possible conics through four given points with a line  $l$  are the pairs of points of a projective involution on the line. For this, recall that the conics through four points  $A, B, C, D$  have the form

$$\mathcal{C}_\lambda := \{p \mid (A, B; C, D)_p = \lambda\},$$

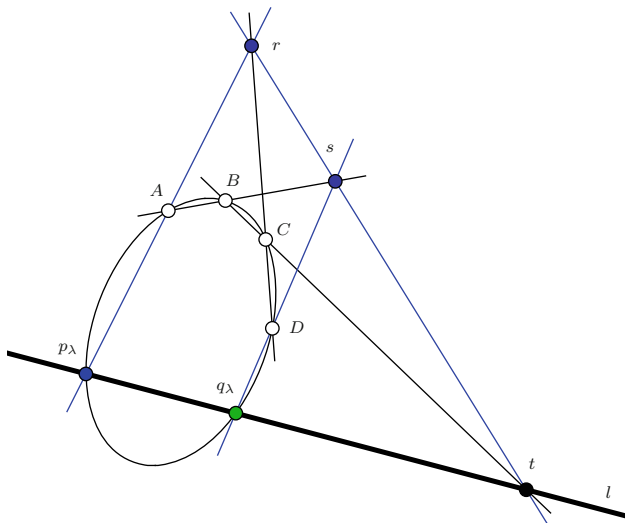
where  $\lambda$  may take an arbitrary value in  $\mathbb{R} \cup \{\infty\}$ . For each such conic there exists a pair of points on  $l$  that are as well on  $\mathcal{C}_\lambda$ . These pairs of points may be both real, both complex, or coinciding. If the points coincide, then we are exactly in the tangent situation. Let  $\{p_\lambda, q_\lambda\}$  be the pair of such points on the conic  $\mathcal{C}_\lambda$ . Then we have the following:

**Lemma 11.2.** *There is a projective involution  $\tau$  on  $l$  such that for any  $\lambda \in \mathbb{R} \cup \{\infty\}$  we have  $\tau(p_\lambda) = q_\lambda$ .*

*Proof.* Instead of giving an algebraic proof we will directly use a geometric argument. With the help of Pascal's theorem we can construct the point  $q_\lambda$  from  $A, B, C, D$ , and  $p_\lambda$  without explicit knowledge of the conic  $\mathcal{C}_\lambda$ . Figure 11.6 shows the construction. The black and white elements of the picture depend only on  $A, B, C, D$  and  $p_\lambda$ . Starting with the point  $p_\lambda$  we have to



**Fig. 11.5** Conics through four points and tangent to a line.



**Fig. 11.6** Using Pascal's theorem to construct the second intersection.

construct first point  $r$  by intersecting  $\overline{p_\lambda A}$  with  $\overline{CD}$ , then we construct  $s$  by intersecting  $\overline{rt}$  with  $\overline{AB}$ . Finally, we derive point  $q_\lambda$  by intersecting  $\overline{sD}$  with  $l$ . In homogeneous coordinates the whole sequence of construction steps can be expressed as a sequence of cross-product operations that use the coordinates of point  $p_\lambda$  exactly once. Thus if  $a$  and  $b$  are homogeneous coordinates of two arbitrary distinct points on  $l$  and we express a point  $p$  on  $l$  by  $p = \alpha a + \beta b$ , then a sequence of operations

$$x_k \times (x_{k-1} \dots \times (x_2 \times (x_1 \times (\alpha a + \beta b))) \dots) =: \alpha' a + \beta' b$$

can be expressed as a single matrix multiplication

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.$$

Thus it is a projective transformation by the fundamental theorem of projective geometry. It is clearly an involution, since the same construction could be used to derive  $p_\lambda$  from  $q_\lambda$ .  $\square$

*Remark 11.1.* Alternatively, the proof could also be carried out in purely algebraic terms. We briefly sketch this. Let again  $a$  and  $b$  be two homogeneous coordinates of two arbitrary distinct points on  $l$  and express a point  $p$  on  $l$  as  $p = \alpha a + \beta b$ . The two points on  $l$  on the conic  $\mathcal{C}_\lambda$  satisfy the relation

$$[\alpha a + \beta b, A, C][\alpha a + \beta b, B, D] = \lambda[\alpha a + \beta b, A, D][\alpha a + \beta b, B, C].$$

Resolving for  $\alpha$  and  $\beta$  yields the equation

$$(\alpha, \beta)(X + \lambda Y) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0,$$

where  $X$  and  $Y$  are suitable symmetric  $2 \times 2$  matrices in which all parameters of the first equation have been encoded. Just knowing  $X$  and  $Y$  is enough information to derive the matrix  $A$  with the property  $Ap_\lambda = q_\lambda$ . The matrix  $A$  can be calculated by the amazingly simple formula

$$A = X^\Delta Y - Y^\Delta X.$$

The reader is invited to check algebraically that  $A$  is an involution and that it converts one solution of the quadratic equation into the other.

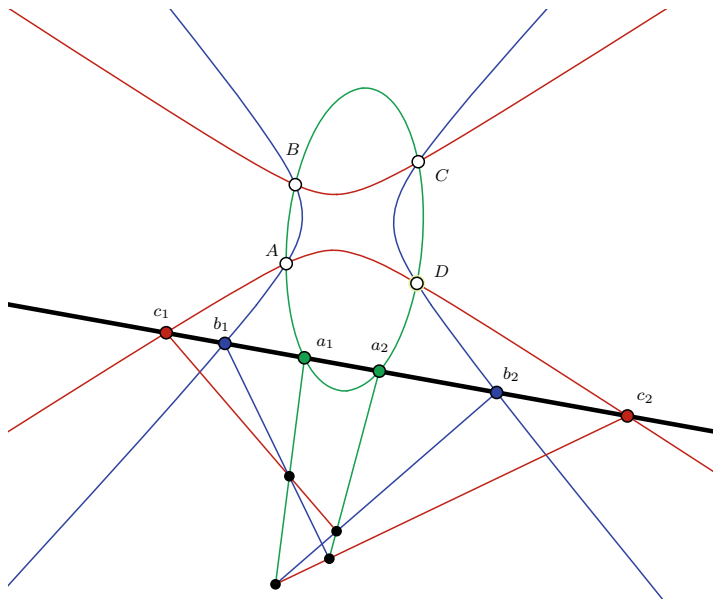
Lemma 11.2 reveals another remarkable connection between conics and quadrilateral sets. If we consider three different conics through four points  $A, B, C, D$  and consider the three point pairs that arise from intersecting these conics with line  $l$ , then these three pairs of points form a quadrilateral set. This is a direct consequence of Lemma 11.2 and Theorem 8.4 that connects projective involutions to quadrilateral sets. A corresponding picture that illustrates this fact is given in Figure 11.7. The incidence structure that is supported by the four black points and their joins is a witness that the six points on the black line form a quadrilateral set. In fact, the six lines of this witness construction can themselves be considered three degenerate conics that intersect the black line in a quadrilateral set.

Now we have collected all necessary pieces to construct the two tangent conics to  $l$  through  $A, B, C, D$ . The four points induce a projective involution  $\tau$  on  $l$  that associates the pairs of points that arise by intersection with a conic through  $A, B, C, D$ . What we are looking for in order to construct a tangent conic are the two fixed points of the involution  $\tau$ . Our considerations after Theorem 8.4 in Section 8.6 showed that these two fixed points  $x$  and  $y$  are simultaneously harmonic to all point pairs  $(p, \tau(p))$ . Thus we can reconstruct the position of these points if we know two such point pairs by solving a quadratic equation. We can construct even three such point pairs by considering the three degenerate conics through  $A, B, C, D$ . The situation is illustrated in Figure 11.8. Considering the line  $l$  as  $\mathbb{RP}^1$  and working with homogeneous coordinates on this space,  $x$  and  $y$  must satisfy the equations

$$[a_1, x][a_2, y] = -[a_1, y][a_2, x] \text{ and } [b_1, x][b_2, y] = -[b_1, y][b_2, x].$$

These equations are solved by the two solutions

$$\begin{aligned} x &= \sqrt{[a_2, b_1][a_2, b_2]}a_1 + \sqrt{[a_1, b_1][a_1, b_2]}a_2, \\ y &= \sqrt{[a_2, b_1][a_2, b_2]}a_1 - \sqrt{[a_1, b_1][a_1, b_2]}a_2 \end{aligned}$$

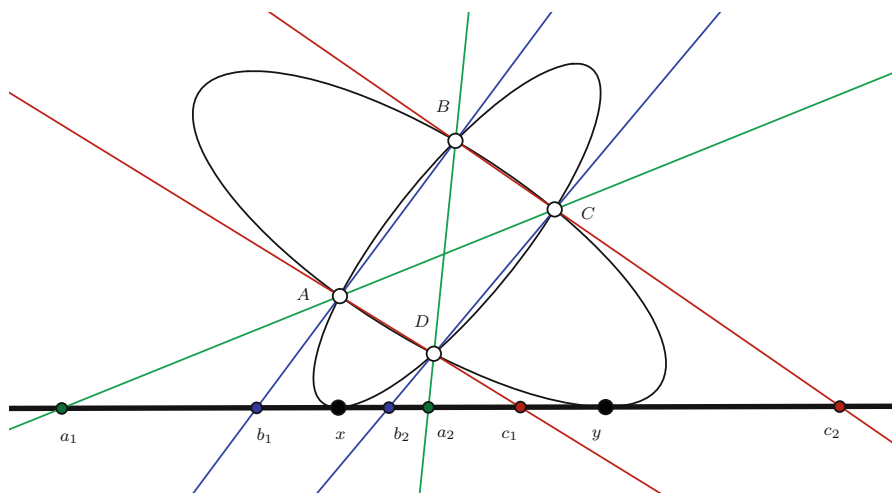


**Fig. 11.7** Quadrilateral sets from bundles of conics.

(observe the beautiful symmetry of the solution). This solution can be derived as a variant of Plücker's  $\mu$  technique if we try to express the solution as a linear combination  $\lambda a_1 + \mu a_2$ . The solution can be easily verified by plugging the expressions for  $x$  and  $y$  into the two equations and expanding the terms.

All in all, the procedure of calculating a conic through  $A, B, C, D$  tangent to  $l$  can be summarized as follows (we formulate the procedure so that the transition to the coordinates on  $l$  is only implicitly used):

- 1: Construct the four intersections  $a_1 = \overline{AC} \wedge l$ ,  $a_2 = \overline{BD} \wedge l$ ,  $b_1 = \overline{AB} \wedge l$ ,  $b_2 = \overline{CD} \wedge l$ .
- 2: Choose an arbitrary point  $o$  not on  $l$ .
- 3: Let  $x = \sqrt{[o, a_2, b_1][o, a_2, b_2]}a_1 + \sqrt{[o, a_1, b_1][o, a_1, b_2]}a_2$ .
- 4: Let  $y = \sqrt{[o, a_2, b_1][o, a_2, b_2]}a_1 - \sqrt{[o, a_1, b_1][o, a_1, b_2]}a_2$ .
- 5: Return the two conics through  $A, B, C, D, x$  and through  $A, B, C, D, y$ .



**Fig. 11.8** The final construction.



## Projective $d$ -space

*Es ist wünschenswert, daß neben der Euklidischen Methode neuere Methoden der Geometrie in den Unterricht auf Gymnasien eingeführt werden.*

*Felix Klein, 1868  
One of the “Theses” of his thesis defense*

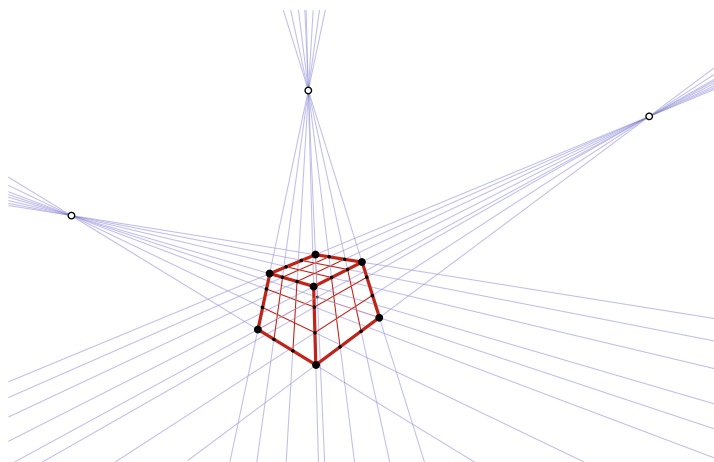
*Mathematics is a game played according to certain simple rules with meaningless marks on paper.*

*David Hilbert*

Different topic! So far, we have dealt almost exclusively with projective geometry of the line and of the plane. We explored the tight and very often elegant relationships between geometric objects and their algebraic representations. Our central issues were:

- Introducing elements at infinity to bypass many special cases of ordinary Euclidean geometry,
- representing geometric objects by homogeneous coordinates,
- performing algebraic operations directly on geometric objects (via homogeneous coordinates),
- performing transformations by matrix multiplication,
- duality,
- expressing geometric relations by bracket expressions.

The close interplay of homogeneous coordinates, finite and infinite elements, and linear algebra made it possible to express geometric relations by very elegant algebraic expressions. This chapter deals with generalizations of these concepts to higher dimensions.



**Fig. 12.1** A projectively correct cube.

Here we will just scratch the surface—in the hope that the reader might get a rough impression of the beauty and richness of the entire theory. It would be easy to fill another 600 pages with an in-depth study of projective geometry in higher dimensions. We will restrict ourselves here to representations of basic objects (*points*, *lines*, *planes*, and *transformations*) and elementary operations (*join* and *meet*). Most often we will not give explicit proofs and confine ourselves with the general concepts.

## 12.1 Elements at Infinity

The three-dimensional projective *space* carries many similarities to the two-dimensional projective plane. Similarly to our treatment of the two-dimensional case we will start with our investigations by considering the usual Euclidean space  $\mathbb{R}^3$ . Like the Euclidean plane, this space is full of special cases. Planes may, for instance, be parallel or meet in a line. Similarly to the treatment of the projective plane we will extend the usual space  $\mathbb{R}^3$  by elements at infinity to get rid of many of these special cases. For every bundle of parallel lines of  $\mathbb{R}^3$  we introduce *one point at infinity*. The totality of all points in  $\mathbb{R}^3$  together with these infinite points forms the set of points of the three-dimensional projective space.

As usual, certain subsets of these points will be considered (projective) lines, and (since we are in the three-dimensional case) there will also be subsets that are considered projective planes. For every finite line  $l$  of  $\mathbb{R}^3$  there is a unique point at infinity corresponding to the parallel bundle of this line. The finite part of  $l$  together with this point is considered a line



in projective space. In addition, there are many lines that lie entirely at an infinite position.

For this consider a usual finite plane  $h$ . This plane is extended by all infinite points of lines that are contained in  $h$ . The extended plane is nothing but a usual projective plane. All infinite points of this plane form a *line at infinity*. This line consists entirely of infinite points. Every ordinary plane of  $\mathbb{R}^3$  is extended by such a unique line at infinity.

There is one object we have not yet covered in our collection of points, lines, and planes. All infinite points taken together again form a projective plane: the *plane at infinity*. The lines of this plane are all the infinite lines. We could say that the real projective three-space may be considered to be  $\mathbb{R}^3$  extended by a projective plane at infinity in the same way as we may say that the real projective plane is  $\mathbb{R}^2$  together with a line at infinity.

In a way, this distinction of finite and infinite objects is confusing and unnecessary. It is just meant as a dictionary to connect concepts of ordinary Euclidean space to concepts of projective space. As one may expect the projective space is much more homogeneous and symmetric than the usual Euclidean space. The projective space is governed by the following incidence properties (which are again analogues of the corresponding axioms of the projective plane):

- Any *two points* span a unique *line* as long as they do not coincide.
- Any *three points* span a unique *plane* as long as they are not collinear.
- Any *two planes* meet in a unique *line* as long as they do not coincide.
- Any *three planes* meet in a unique *point* as long as they do not meet in a line.
- Any pair of *point and line* span a unique *plane* as long as the point is not on the line.
- Any pair of *plane and line* meet in a unique *point* as long as the line is not contained in the plane.

It will be the task of the next few sections to express these operations of *join* and *meet* in a suitable way that also generalizes to higher dimensions.

## 12.2 Homogeneous Coordinates and Transformations

How do we represent these elements algebraically? Essentially the process is similar to the setup in the projective plane. Points in projective three-space are represented by nonzero *four-dimensional* vectors. They are the homogeneous coordinates of the points. Nonzero scalar multiples of the vectors are identified. In other words, we may represent the points of the projective three-space as

$$\mathbb{RP}^3 = \frac{\mathbb{R}^4 - \{(0, 0, 0, 0)^T\}}{\mathbb{R} - \{0\}}.$$

In the standard embedding we may “imagine” the Euclidean  $\mathbb{R}^3$  embedded in  $\mathbb{R}^4$  as an affine space parallel to the  $(x, y, z, 0)^T$  space of  $\mathbb{R}^4$ . If this is too hard to imagine, then one can also proceed purely formally. In the standard embedding a point  $(x, y, z)^T \in \mathbb{R}^3$  is represented by a four-dimensional vector  $(x, y, z, 1)^T \in \mathbb{R}^4$ . Nonzero scalar multiples are identified. The infinite points are exactly those nonzero vectors of the form  $(x, y, z, 0)^T$ . They do not correspond to Euclidean points. Vectors of the form  $(x, y, z, 0)^T$  may also be interpreted as homogeneous coordinates of the usual *projective plane* by ignoring the last entry.

Thus we can literally say that  $\mathbb{RP}^3$  consists of  $\mathbb{R}^3$  (the homogeneous vectors  $(x, y, z, 1)^T$ ) and a projective plane at infinity (the homogeneous vectors  $(x, y, z, 0)^T$ ). We may also interpret this process inductively and consider the projective plane itself as composed of the Euclidean plane  $\mathbb{R}^2$  (the vectors  $(x, y, 1, 0)^T$ ) and a line at infinity (the vectors  $(x, y, 0, 0)^T$ ). The line may be considered a Euclidean line (the vectors  $(x, 1, 0, 0)^T$ ) and finally a point at infinity (represented by  $(1, 0, 0, 0)^T$ ).

The objects dual to points in  $\mathbb{RP}^3$  will be *planes*. Similarly to points, planes will also be represented by four-dimensional vectors. In Euclidean terms we may consider the vector  $(a, b, c, d)^T$  as representing the parameters that describe the affine plane  $\{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz + d = 0\}$ . As usual, nonzero scalar multiples of the vector represent the same geometric object. If we interpret the equation in a setup of homogeneous coordinates, a point  $(x, y, z, w)^T$  is incident to a plane  $(a, b, c, d)^T$  if and only if

$$ax + by + cz + dw = 0.$$

As in the two-dimensional setup, incidence is simply expressed by a scalar product being zero. The plane at infinity has homogeneous coordinates  $(0, 0, 0, 1)^T$ .

Finding a plane that passes through three given points  $p_i = (x_i, y_i, z_i, w_i)$ ,  $i = 1, 2, 3$ , thus translates to the task of solving a linear equation:

$$\begin{pmatrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In the next section we will deal with methods of performing this calculation explicitly. However, before we do so we will consider transformations in this homogeneous projective setup. The situation here is almost completely analogous to the two-dimensional case. A transformation is represented by a simple matrix multiplication. This time we need a  $4 \times 4$  matrix. The cases of usual affine transformations in  $\mathbb{R}^3$  are again covered by special transformation

matrices in which certain entries are zero. The following matrices represent a linear transformation of  $\mathbb{R}^3$  with matrix  $A$ , a pure translation, a general affine transformation, and a general projective transformation, respectively:

$$\begin{pmatrix} \boxed{A} & 0 \\ & 0 \\ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}.$$

A “•” stands for an arbitrary number. In every case we must assume that the determinant of the matrix is nonzero.

Such a transformation  $T$  maps a point  $p$  to the associated image point. The corresponding transformation that applies to the homogeneous coordinates of a plane is (as in the two-dimensional case) given by  $(T^{-1})^T$ . By this choice incidence of points and planes is preserved by projective transformations:

$$\langle p, h \rangle = 0 \iff \langle Tp, (T^{-1})^T h \rangle = 0.$$

Instead of the transposed inverse  $(T^{-1})^T$  it is also possible to use the transposed adjoint  $(T^\Delta)^T$ , since they differ by only a scalar factor.

*Remark 12.1.* A word of caution: One should consider the setup of presenting points and planes by four-dimensional vectors and expressing coincidence by the standard scalar product as a kind of interim solution that will be replaced by something more powerful later on. The problem that we will have to face soon is that lines will be represented by *six-dimensional* vectors and we have to create a notational system that handles points, lines, and planes in a unified way. For this we will have to give up the concept of indexing a vector entry with the position where it is placed in a vector. We will return to this issue in Section 12.4.

## 12.3 Points and Planes in 3-Space

The task of this section is to give a closed formula for calculating the homogeneous coordinates of a plane spanned by three points. The corresponding two-dimensional situation is governed by the *cross product* operation. The line through two points  $p, q$  could be calculated by

$$l = p \times q = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} + \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \\ - \begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \\ + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{pmatrix}.$$

The entries of the coordinates of the line are the  $2 \times 2$  subdeterminants of the matrix

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \end{pmatrix}.$$

One way of explaining this effect is to observe that an arbitrary point  $\lambda p + \mu q$  on this line must have a zero scalar product with  $l$ . We obtain

$$\langle \lambda p + \mu q, l \rangle = \lambda \langle p, l \rangle + \mu \langle q, l \rangle = \lambda \det(p, p, q) + \mu \det(q, p, q).$$

The last equation holds since if we plug the expression for  $l$  into  $\langle a, l \rangle$ , we obtain the determinant  $\det(a, p, q)$ , as one can easily see if one develops this determinant by the first column.

In a similar way we can obtain the coordinates of a plane  $h$  through three points  $p, q, r$  in  $\mathbb{RP}^3$ . We get

$$h = \mathbf{join}(p, q, r) := \begin{pmatrix} + \begin{vmatrix} p_2 & q_2, & r_2 \\ p_3 & q_3, & r_3 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ - \begin{vmatrix} p_1 & q_1, & r_1 \\ p_3 & q_3, & r_3 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ + \begin{vmatrix} p_1 & q_1, & r_1 \\ p_2 & q_2, & r_2 \\ p_4 & q_4, & r_4 \end{vmatrix} \\ - \begin{vmatrix} p_1 & q_1, & r_1 \\ p_2 & q_2, & r_2 \\ p_3 & q_3, & r_3 \end{vmatrix} \end{pmatrix}.$$

So the coordinates are the  $3 \times 3$  subdeterminants of the matrix

$$\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{pmatrix}.$$

Equipped with alternating signs, they give the plane through these three points. The argument for the correctness of this calculation is the same as in the two-dimensional case. If we consider an arbitrary point  $a$ , then the development of the determinant

$$\det \begin{pmatrix} a_1 & p_1 & q_1 & r_1 \\ a_2 & p_2 & q_2 & r_2 \\ a_3 & p_3 & q_3 & r_3 \\ a_4 & p_4 & q_4 & r_4 \end{pmatrix}$$

by the first column is just the scalar product  $\langle a, h \rangle$  for the above definition of  $h$ . If  $a$  is spanned by  $p$ ,  $q$ , and  $r$ , then this determinant becomes zero (the point  $a$  is on  $h$ ). If the point is not spanned by  $p$ ,  $q$ , and  $r$ , then the columns are linearly independent and the determinant is nonzero. Thus  $a$  is not on  $h$ . Geometrically speaking, we compute a vector  $h$  that is simultaneously orthogonal to all three vectors  $p$ ,  $q$ , and  $r$ .

It may happen that the above operation (taking the four  $3 \times 3$  subdeterminants of the  $4 \times 3$  matrix) results in a zero vector. However, this could happen only if the three column vectors were linearly dependent. In this case the three points do not span a plane. Geometrically, there are several possibilities how this can happen. Either all points coincide (then the matrix has rank 1) or the points lie on a unique common line (then the matrix has rank 2). In both cases we have a degenerate situation in which the plane through the points is not uniquely determined.

As in the planar case, the procedure described here covers all possible cases of finite and infinite points. For instance, a plane through two finite points  $p$  and  $q$  and one infinite point  $r$  is the unique plane that contains the line  $\overline{pq}$  and parallels in direction of  $r$  (as long as  $r$  is not on  $\overline{pq}$ , which is a degenerate situation). The plane through three infinite points is the plane at infinity itself.

The same trick can also be used to calculate a point that is simultaneously contained in three planes. Let  $h, g, f$  be the homogeneous coordinates of three planes in  $\mathbb{RP}^3$ . Assume that the three planes do not have a line in common. In this case the planes contain a unique common point. The coordinates of this point can be calculated as the  $3 \times 3$  subdeterminants of the matrix

$$\begin{pmatrix} h_1 & g_1 & f_1 \\ h_2 & g_2 & f_2 \\ h_3 & g_3 & f_3 \\ h_4 & g_4 & f_4 \end{pmatrix}$$

equipped with alternating signs. As in the dual case, degenerate cases result in a zero vector. All special cases resulting, from finite or infinite points are automatically covered as well. For instance, if  $h$  and  $g$  are two finite planes and  $f$  is the plane at infinity, then the resulting point is the point at infinity on the intersection of  $h$  and  $g$ .

There is one more or less obvious but remarkable fact that we want to mention for further reference. If  $p, q$ , and  $r$  span a plane  $h$  and if  $a, b$ , and  $c$  span the same plane, then the join operation that calculates the plane from three points may have different results for  $p, q$ , and  $r$  and for  $a, b$ , and  $c$ . However, the two results may differ at most by a non-zero scalar multiple. One may view this result as a consequence of the fact that the plane  $h$  is uniquely determined and thus vectors representing it only may differ by a scalar multiple. However, one can obtain it also directly from the fact that the operation  $\mathbf{join}(p, q, r)$  is linear in each argument and anticommutative.

Each of the points  $a$ ,  $b$ ,  $c$  is a linear combination of  $p$ ,  $q$ , and  $r$ . So, if we, for instance, replace  $p$  by  $a$ , we obtain

$$\begin{aligned}\mathbf{join}(a, q, r) &= \mathbf{join}(\lambda p + \mu q + \tau r, q, r) \\ &= \lambda \cdot \mathbf{join}(p, q, r) + \mu \cdot \mathbf{join}(q, q, r) + \tau \cdot \mathbf{join}(r, q, r) \\ &= \lambda \cdot \mathbf{join}(p, q, r).\end{aligned}$$

As a consequence of antisymmetry, terms with repeated letters can be canceled.

## 12.4 Lines in 3-Space

Now comes the tricky part. What is the good way of representing lines of projective space? One could say that the answer to this question was, after introduction of homogeneous coordinates, one of the major breakthroughs of nineteenth-century geometry. More or less, the answer was independently discovered by at least two people. Perhaps the first was Hermann Günther Grassmann (1809–1877). In his work on *Lineare Ausdehnungslehre* [46] from 1844 he laid at the same time the basis for our modern linear algebra as well as for *multilinear algebra*. One of the essential parts was a formal method that made it possible to directly operate with points, lines, planes, etc. Unfortunately, Grassmann developed a kind of completely new mathematical terminology and notation to deal with these kinds of objects. This caused him to be more or less completely ignored by his contemporaries, and his ideas did not become common mathematical knowledge until he completely rewrote his book and published a second version [47] in 1862<sup>1</sup>.

The second person involved was Julius Plücker (whom we have already met frequently in this book). Not aware of Grassmann's work, in 1868 he published the first part of *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement* [101] (*New Geometry of space, based on the straight line as space element*) but died before the second part was complete. Felix Klein at this time was his assistant and essentially completed Plücker's thoughts in his doctoral dissertation [64].

From a modern perspective the basic ideas are a straightforward generalization of our considerations in the previous section. Nevertheless, these ideas opened whole new branches of mathematics starting from projective

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<sup>1</sup> The first edition of his book was of about 900 copies, from of about 600 were pulped since they simply could not be sold. The remaining 300 books were given away for free to anyone who showed interest. In his second edition, Grassmann expresses his regret that people do not take the time to follow another person's thoughts. It is little known among mathematicians that Grassmann was much more famous in his time for his works on Indo-Germanic linguistics. He published the first German translation of the Rig-Veda (an ancient indian document), and his Indo-German dictionaries are still in use today.

geometry in arbitrary dimensions, via multilinear or exterior algebra up to tensor calculus (which plays an omnipresent role in modern physics).

Let us return to our question: How do we represent a line in space? The correct generalization of our considerations so far is as follows. If we want to calculate the line spanned by two points  $p$  and  $q$ , then we consider the  $4 \times 2$  matrix

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \\ p_3 & q_3 \\ p_4 & q_4 \end{pmatrix}.$$

From this matrix we take all  $2 \times 2$  subdeterminants and collect them in a six-dimensional vector. For reasons we will investigate later, we will equip the entries of this vector only with positive signs. All in all, the coordinates for the line through the two points are

$$l = p \vee q := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \vee \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \\ + \begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \\ + \begin{vmatrix} p_1 & q_1 \\ p_4 & q_4 \end{vmatrix} \\ + \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \\ + \begin{vmatrix} p_2 & q_2 \\ p_4 & q_4 \end{vmatrix} \\ + \begin{vmatrix} p_3 & q_3 \\ p_4 & q_4 \end{vmatrix} \end{pmatrix}.$$

Let us start to collect a few properties of this new operation  $p \vee q$ .

**Theorem 12.1.** *The operation  $p \vee q$  is linear in each argument and anticommutative.*

*Proof.* The result is more or less obvious from the definition of the operation. Each entry of the six-dimensional vector is a  $2 \times 2$  matrix that is by itself linear in  $p$  and  $q$  and anticommutative. Thus the whole operation  $p \vee q$  must have these properties.  $\square$

This implies another immediate result, which is very similar to operations we have had so far.

**Theorem 12.2.** *Let  $p, q$  be homogeneous coordinates of two points that span a line  $l$  and let  $a, b$  be two other points that span the same line. Then  $p \vee q = \lambda \cdot (a \vee b)$  for a suitable factor  $\lambda$ .*

*Proof.* The result is an immediate consequence of multilinearity and anti-commutativity. If  $a$  and  $b$  are on  $l$ , then we may express them as linear combinations of  $p$  and  $q$ , say

$$a = \lambda_1 p + \mu_1 q \text{ and } b = \lambda_2 p + \mu_2 q.$$

Since  $a$  and  $b$  span the line, the determinant of the matrix  $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}$  is nonzero. Calculating the join, we obtain

$$\begin{aligned} a \vee b &= (\lambda_1 p + \mu_1 q) \vee (\lambda_2 p + \mu_2 q) \\ &= \lambda_1 \lambda_2 (p \vee p) + \lambda_1 \mu_2 (p \vee q) + \mu_1 \lambda_2 (q \vee p) + \mu_1 \mu_2 (q \vee q) \\ &= (\lambda_1 \mu_2 - \mu_1 \lambda_2) (p \vee q). \end{aligned}$$

The factor  $\lambda$  turns out to be the determinant  $(\lambda_1 \mu_2 - \mu_1 \lambda_2)$ .  $\square$

The last result is crucial. It claims that the six-dimensional vector  $p \vee q$  is (up to a scalar factor, as usual) a unique representation of the line spanned by  $p$  and  $q$ . It does not depend on the special choice of the two spanning points. This is similar to our observation in the last section, where we saw that the join of three points depends only on the plane spanned by these points, and not on their particular choice.

If we accept that a good representation of lines in space are six-dimensional vectors, then we have to face another problem. There are only *four* degrees of freedom necessary to parameterize lines in space. Here is a rough but instructive way to see this: Assume you have two parallel planes in  $\mathbb{R}^3$ . Then almost all lines in  $\mathbb{R}^3$  will intersect these planes (except those that are parallel to the planes). The two points of intersection (one on each plane) determine those lines uniquely. This makes four degrees of freedom—two for each plane. So, if we have just four degrees of freedom, how does this relate to the six-dimensional vector that represents the line? One of the degrees of freedom is “eaten up” by the irrelevant scalar factor that we have in any homogeneous approach. So there can be at least five relevant parameters in the six-dimensional vector. The reason that we have five parameters and not four is that the entries in our vector are not independent. They were defined to be the  $2 \times 2$  subdeterminants of a  $4 \times 2$  matrix. In Section 6.5 we learned about *Grassmann-Plücker relations* that form dependencies among such systems of sub-determinants. In particular, we have

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \begin{vmatrix} p_3 & q_3 \\ p_4 & q_4 \end{vmatrix} - \begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \begin{vmatrix} p_2 & q_2 \\ p_4 & q_4 \end{vmatrix} + \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \begin{vmatrix} p_1 & q_1 \\ p_4 & q_4 \end{vmatrix} = 0.$$

Thus in general, if we know five entries of a six-dimensional vector that describes a line, then we automatically know the last one (we have only to be a bit careful with degenerate cases).



Conversely, if we have a vector  $l = (a, b, c, d, e, f)^T$  that satisfies the equation  $af - be + cd = 0$ , Theorem 7.1 implies that there are two vectors  $p$  and  $q$  with  $l = p \vee q$ . Vectors that satisfy such a condition are called *decomposable*.

## 12.5 Joins and Meets: A Universal System ...

Before we play around with line coordinates in space we will first clear up our notation and present a universal system that is capable of dealing with points, lines and planes in a unified way. The situation is even better: we will present a system that is capable of dealing with linear objects in arbitrary projective spaces of any dimension. This will be a direct generalization of the three-dimensional case.

The main problem so far is that points and planes are represented by four-dimensional vectors, while lines are represented by six-dimensional vectors. So a priori it is not clear how to define operators that work reasonably on arbitrary collections of such objects. The key observation here is to give up the idea that the coordinate entries belong to certain positions in a vector. It will be much more useful to associate meaningful labels to the different entries of a vector. In the three-dimensional case these labels will be the subsets of the set  $\{1, 2, 3, 4\}$ . There are exactly *four* one-element subsets; they will label the coordinate entries of a point. There are *six* two-element subsets; they will label the entries of a line. And there are *four* three-element subsets and they will be used to label the coordinate entries of a plane. So points and planes are both represented by four-dimensional vectors, but they have explicitly different meanings.

It is also reasonable to include the empty set  $\{\}$  and the full set  $\{1, 2, 3, 4\}$  in our system. We will learn about their meaning soon. If we consider the numbers of subsets sorted by their cardinality, we obtain the sequence 1, 4, 6, 4, 1, which is simply a sequence of binomial coefficients—the fifth row of Pascal's triangle.

We will (at least partially) follow Grassmann's footprints in order to see how the whole system of *join* and *meet* operations arises. In modern terms, Grassmann states that if we want to work in projective  $(d - 1)$ -dimensional geometry, we have to fix first of all a system of  $d$  units

$$e_1, e_2, e_3, \dots, e_d$$

that are by definition independent. You may think of them as the  $d$ -dimensional unit vectors and associate the corresponding projective points to them. These units are in a sense the most fundamental geometric objects, and any other objects will be expressible in terms of units, real numbers, and admissible operations. Furthermore, it is allowed to form *products* of

units. These products are assumed to be anticommutative and linear in both factors. Thus we have

$$e_i e_j = -e_j e_i \text{ if } i \neq j \text{ and } e_i e_i = 0.$$

Products of  $k$  units are called rank- $k$  units. Thus  $e_1, \dots, e_d$  are rank-1 units,  $e_1 e_2, \dots, e_{d-1} e_d$  are rank-2 units, and so forth. Furthermore it is allowed to form products and sums of objects. A geometric object of rank- $k$  is a linear combination of several rank- $k$  units. One may think of rank- $k$  objects as representing a  $(k-1)$ -dimensional linear object in the corresponding projective space.

Let us see how these simple rules automatically create a system in which multiplication corresponds to well-known arithmetic operations. Let us start with the first nontrivial case  $d = 2$ . Rank-1 objects (points) are simply linear combinations

$$\lambda_1 e_1 + \lambda_2 e_2.$$

Up to sign change there is only one nonvanishing rank-2 unit, namely  $e_1 e_2 = -e_2 e_1$ . The product of two points turns out to be

$$\begin{aligned} & (\lambda_1 e_1 + \lambda_2 e_2)(\mu_1 e_1 + \mu_2 e_2) \\ &= \lambda_1 \mu_1 e_1 e_1 + \lambda_1 \mu_2 e_1 e_2 + \lambda_2 \mu_1 e_2 e_1 + \lambda_2 \mu_2 e_2 e_2 \\ &= (\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 e_2, \end{aligned}$$

which is up to the factor  $e_1 e_2$  just the determinant of the points. Products of more than three points will always vanish completely.

Next is  $d = 3$ . We have three units  $e_1, e_2, e_3$  and (up to sign) three rank-2 units  $e_1 e_2, e_1 e_3, e_2 e_3$ . Calculating the product of two points, we get

$$\begin{aligned} & (\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3) \\ &= \dots \text{expression with nine summands} \dots \\ &= (\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 e_2 + (\lambda_1 \mu_3 - \lambda_3 \mu_1) e_1 e_3 + (\lambda_2 \mu_3 - \lambda_3 \mu_2) e_2 e_3, \end{aligned}$$

which is essentially the cross product of the two points expressed as a linear combination of rank-2 units. Thus the join operation of two points pops out automatically (except for the sign change in the middle entry). If we proceed, we see that the product of three points turns out to be

$$\begin{aligned} & (\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3)(\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3) \\ &= \dots \text{expression with 27 summands} \dots \\ &= (\lambda_1 \mu_2 \tau_3 + \lambda_2 \mu_3 \tau_1 + \lambda_3 \mu_1 \tau_2 - \lambda_1 \mu_3 \tau_2 - \lambda_3 \mu_2 \tau_1 - \lambda_2 \mu_1 \tau_3) e_1 e_2 e_3, \end{aligned}$$

which is just the determinant of the point coordinates times  $e_1 e_2 e_3$ .

If we proceed in a similar manner, then we find out that for  $d = 4$  the points are represented by four-dimensional objects. The products of two points is a linear combination of the six rank-2 units whose coefficients are exactly the

entries of our join operation;

$$\begin{aligned}
 & (\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4)(\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4) \\
 &= \dots \text{expression with 16 summands} \dots \\
 &= +(\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 e_2 + (\lambda_1 \mu_3 - \lambda_3 \mu_1) e_1 e_3 + (\lambda_1 \mu_4 - \lambda_4 \mu_1) e_1 e_4 \\
 &\quad + (\lambda_2 \mu_3 - \lambda_3 \mu_2) e_2 e_3 + (\lambda_2 \mu_4 - \lambda_4 \mu_2) e_2 e_4 + (\lambda_3 \mu_4 - \lambda_4 \mu_3) e_3 e_4.
 \end{aligned}$$

(Now you see where the positive signs come from.) The product of three points gives the coordinates of the plane expressed in rank-3 units. Finally, the product of four points creates the determinant of the coordinate vectors of the points times the rank-4 unit  $e_1 e_2 e_3 e_4$ .

We see that all the  $e_i$  in our expressions carry essentially no information. The only thing that counts are the indices. Furthermore, the only combinations of indices that really contribute are those with nonrepeating letters, and it suffices to consider one unit for every subset of indices. Thus we can go ahead and say that a rank- $k$  object is a vector with  $\binom{d}{k}$  entries labeled by the  $k$ -element subsets of  $\{1, 2, \dots, d\}$ . We identify these subsets with the sequences of  $k$  ordered elements taken from  $\{1, 2, \dots, d\} =: E_d$ . Thus the index set of a  $k$ -flat (this is a  $(k-1)$ -dimensional linear object) in  $\mathbb{RP}^{d-1}$  is

$$\Lambda(d, k) := \{(i_1, \dots, i_k) \in E_d^k \mid i_1 < i_2 < \dots < i_k\}.$$

To further distinguish the Grassmann operation from ordinary multiplication we introduce the symbol “ $\vee$ ” and call this the *join* operation. In general the join operation can be determined by the following rules. If we are working in  $\mathbb{RP}^{d-1}$ , we can take the join of a  $k$ -flat  $P$  and an  $m$ -flat  $Q$  if  $k + m \leq d$ . If  $R = P \vee Q$ , then  $R$  is indexed by the elements of  $\Lambda(d, k + m)$ . For an index  $\lambda \in \Lambda(d, k + m)$  we can calculate the corresponding entry of  $R$  according to the formula

$$R_\lambda = \sum_{\substack{\lambda = \mu \cup \tau \\ \mu \in \Lambda(d, k) \\ \tau \in \Lambda(d, m)}} \text{sign}(\mu, \tau) P_\mu Q_\tau.$$

Here  $\text{sign}(\mu, \tau)$  is defined to be 1 or  $-1$  depending on the parity of transpositions needed to sort the sequence  $(\mu, \tau)$ . We will see in the next section how this formula is used in practice. It allows us to calculate the join of two arbitrary objects as long as the sum of their ranks does not exceed  $d$ . The above formula filters exactly those terms that do not cancel in the Grassmann product.

In a similar fashion we can define a *meet* operation. We will not develop here the theory behind the exact definition. It is essentially defined in a way that represents the dual of the join operation. If we work in  $\mathbb{RP}^{d-1}$ , we are allowed to take the meet of a  $k$ -flat  $P$  and an  $m$ -flat  $Q$  whenever  $k + m \geq d$ . We abbreviate the meet operator by “ $\wedge$ ”. If  $R = P \wedge Q$ , then  $R$  is indexed

by the elements of  $\Lambda(d, k + m - d)$ . For an index  $\lambda \in \Lambda(d, k + m - d)$  we can calculate the corresponding entry of  $R$  according to the formula

$$R_\lambda = \sum_{\substack{\lambda = \mu \cap \tau \\ \mu \in \Lambda(d, k) \\ \tau \in \Lambda(d, m)}} \text{sign}(\mu \setminus \lambda, \tau \setminus \lambda) P_\mu Q_\tau.$$

## 12.6 ... And How to Use It

In this section we will see what can be done with join and meet operations and how they are calculated in practice. We will restrict our considerations to  $d = 4$ , the spatial case. Everything carries over in a straightforward way to higher (and lower) dimensions. First of all, we will start with a kind of symbolic table that exemplifies the dimensions of the objects involved under join and meet operations. For the join we obtain

$$\begin{aligned} \text{point} \vee \text{point} &= \text{line}, \\ \text{point} \vee \text{line} &= \text{plane}, \\ \text{point} \vee \text{plane} &= \text{number}, \\ \text{line} \vee \text{line} &= \text{number}, \\ \text{point} \vee \text{point} \vee \text{point} &= \text{plane}, \\ \text{point} \vee \text{point} \vee \text{line} &= \text{number}, \\ \text{point} \vee \text{point} \vee \text{point} \vee \text{point} &= \text{number}. \end{aligned}$$

Every operation results in either a vector or a number. Whenever a zero vector or the number zero occurs as result, this indicates a degenerate situation in which the objects are dependent. For instance, usually the join of a point and a line results in the plane spanned by the point and the line. However, if the point is on the line, then the join results in the zero vector. The consecutive join of four points results in a number, and this number is just the determinant of the matrix formed by the vectors of the points as column vectors. If the points are coplanar, then this join returns the number zero. The join of two lines is zero if the two lines coincide. The join of a point and a plane is a number. In principle, this number is just the scalar product of the point and the plane. It is zero whenever the point is on the plane.

The meet operator performs the dual operations. Again the occurrence of a zero or a zero vector indicates a degenerate situation. In detail, the meets can be used to perform the following operations:

$$\begin{aligned}
&\mathbf{plane} \wedge \mathbf{plane} = \mathbf{line}, \\
&\mathbf{plane} \wedge \mathbf{line} = \mathbf{point}, \\
&\mathbf{plane} \wedge \mathbf{point} = \mathbf{number}, \\
&\mathbf{line} \wedge \mathbf{line} = \mathbf{number}, \\
&\mathbf{plane} \wedge \mathbf{plane} \wedge \mathbf{plane} = \mathbf{point}, \\
&\mathbf{plane} \wedge \mathbf{plane} \wedge \mathbf{line} = \mathbf{number}, \\
&\mathbf{plane} \wedge \mathbf{plane} \wedge \mathbf{plane} \wedge \mathbf{plane} = \mathbf{number}.
\end{aligned}$$

Again it should be mentioned that all operations based on join and meet fully support all elements of projective geometry. Thus in the standard embedding of  $\mathbb{R}^3$  elements at infinity are also processed correctly. For instance, the meet of two parallel planes results in a line at infinity. To achieve a standard embedding we have to make a choice of which the vectors  $e_1, \dots, e_4$  is chosen for homogenization purposes. If we choose  $e_4$  for this purpose, we may represent a point  $(x, y, z) \in \mathbb{R}^3$  as  $xe_1 + ye_2 + ze_3 + e_4$ . The plane at infinity then corresponds to  $e_1e_2e_3$ . The notation is much more familiar if we simply represent the objects with associated labels. A Euclidean point  $(x, y, z)$  and a Euclidean plane  $\{(x, y, z)^t \mid ax + by + cz + d = 0\}$  then correspond to the two vectors

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{pmatrix} -d \\ c \\ -b \\ a \end{pmatrix}.$$

The minus signs in the plane coordinates are used to compensate the sign changes caused by the join operator. The join operation of these two objects results in a single number (labeled by “1234”); this number is simply calculated as  $ax + by + cz + d$ , as desired. It is zero if the point and the plane coincide.

Let us perform a more elaborate example with concrete coordinates. Let us first calculate the join of two points:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix} \vee \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} -3 \\ 3 \\ 5 \\ 1 \end{pmatrix} = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} 3 \cdot 3 - (-2) \cdot (-3) \\ 3 \cdot 5 - 4 \cdot (-3) \\ 3 \cdot 1 - 1 \cdot (-3) \\ (-2) \cdot 5 - 4 \cdot 3 \\ (-2) \cdot 1 - 1 \cdot 3 \\ 4 \cdot 1 - 1 \cdot 5 \end{pmatrix} = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} 3 \\ 27 \\ 6 \\ -22 \\ -5 \\ -1 \end{pmatrix}.$$

The result is a six-dimensional vector representing a line. In particular, this vector must satisfy the Grassmann-Plücker relation. The Grassmann-Plücker relation for a line  $l$  can be simply expressed as  $l \vee l = \mathbf{0}$  (where  $\mathbf{0}$  is the zero vector). This has a direct geometric interpretation: a line is incident to itself. In our example, we get

$$3 \cdot (-1) - 27 \cdot (-5) + 6 \cdot (-22) = -3 + 135 - 132 = 0,$$

as expected. We proceed by forming the meet of this line with some plane. The result is the point where the plane intersects the line:

$$\begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} 3 \\ 27 \\ 6 \\ -22 \\ -5 \\ -1 \end{pmatrix} \wedge \begin{array}{c} 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{pmatrix} 4 \\ 2 \\ 1 \\ 5 \end{pmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 3 \cdot 1 - 27 \cdot 2 + 6 \cdot 4 \\ 3 \cdot 5 - (-22) \cdot 2 + (-5) \cdot 4 \\ 27 \cdot 5 - (-22) \cdot 1 + (-1) \cdot 4 \\ 6 \cdot 5 - (-5) \cdot 1 + (-1) \cdot 2 \end{pmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} -27 \\ 39 \\ 153 \\ 33 \end{pmatrix}.$$

If everything went right with our calculation, the resulting point must lie on the plane itself. Thus the join of the point and the plane must be simply zero. We obtain

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} -27 \\ 39 \\ 153 \\ 33 \end{pmatrix} \vee \begin{array}{c} 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{pmatrix} 4 \\ 2 \\ 1 \\ 5 \end{pmatrix} = {}_{1234} ((-27) \cdot 5 - 39 \cdot 1 + 153 \cdot 2 - 33 \cdot 4) = 0.$$

We finally will have a closer look at the representation of infinite lines (in the usual embedding of  $\mathbb{R}^3$  in  $\mathbb{RP}^3$ ). There are two ways to derive a line at infinity: either we join two infinite points or we meet two parallel planes. Either way leads to the same characterization of infinite lines, though the situations may be interpreted slightly differently from a geometric point of view. Joining two infinite points, we obtain

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 0 \end{pmatrix} \vee \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 0 \end{pmatrix} = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} x_1 y_2 - x_2 y_1 \\ x_1 y_3 - x_3 y_1 \\ 0 \\ x_2 y_3 - x_3 y_2 \\ 0 \\ 0 \end{pmatrix}.$$

A line is infinite if the only nonzero entries are those not involving the label 4. We also see that if we restrict the line to the labels 12, 13, and 23, then the join of infinite points behaves like a cross product.

Two planes are parallel if they differ only in the 123entry. The meet of two parallel planes is an infinite line:

$$\begin{array}{c} 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{pmatrix} d_1 \\ c \\ b \\ a \end{pmatrix} \wedge \begin{array}{c} 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{pmatrix} d_2 \\ c \\ b \\ a \end{pmatrix} = \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} d_1 c - d_2 c \\ d_1 b - d_2 b \\ cb - bc \\ d_1 a - d_2 a \\ ca - ac \\ ba - ab \end{pmatrix} = (d_1 - d_2) \cdot \begin{array}{c} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{pmatrix} c \\ b \\ 0 \\ a \\ 0 \\ 0 \end{pmatrix}.$$

Again, as expected, the 14, 24, and 34 entries are zero. The other three entries encode up to the usual sign changes the common normal vectors of the planes. The factor  $d_1 - d_2$  can be interpreted in the following way. If the planes coincide then  $d_1 = d_2$  and the meet operation results in a zero vector.





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## Diagram Techniques

*Notation, Notation, Notation.*

*Title of a book by Jim Blinn, 2002*

*We just reduced all of geometry to tensor multiplication (well, almost all). And there are no embarrassing transposes. Row-ness and column-ness is superseded by the more general concept of covariant and contravariant indices. Plus we can feel really cool by sharing notation with General Relativity.*

*Jim Blinn, 1992*

It sometimes happens that reading a mathematical article sheds a completely new light on subjects that one considered “personally well-under stood.” So it happened to me when I did some research related to solving cubic equations and stumbled across a series of papers written by the computer scientist Jim Blinn. The series was essentially about the expressive power of *tensor calculus* applied to geometry [8, 9, 10, 11]. I always hated working with tensors and avoided them wherever possible, because tensor notation tends either to be very abstract or to clutter all the essential information of a formula into indices and indices of indices and indices of indices of indices. Jim Blinn’s papers were different. There tensor formulas were encoded as diagrams, and suddenly all those terrible index battles became geometrically meaningful structures. Moreover, this approach gives a unifying setup for treating points, lines, planes, hyperplanes, transformations, quadratic forms, algebraic curves, and surfaces in all dimensions. Everything becomes a tensor, and every projective invariant becomes a diagram. The bookkeeping of how to combine the coordinate entries of the tensors (for instance to derive

the Plücker coordinates of a line) is completely driven by the structure of the diagrams. It is the aim of this chapter to give a brief introduction to these methods. As before, we want to remain as explicit as possible. We will first translate the objects of our previous investigations to tensors and then translate tensor formulas to tensor diagrams. The mathematically trained reader may excuse that we again focus on concrete objects and focus on tensors given by concrete coordinate values rather than taking the (more modern) abstract point of view: “A tensor is an abstract element of a tensor space, . . .”

## 13.1 From Points, Lines, and Matrices to Tensors

So far, we have dealt with several types of objects. In  $\mathbb{RP}^2$  we had points and lines. In  $\mathbb{RP}^3$  we had points, lines, and planes, and so on. Furthermore, there were transformations (represented by  $d \times d$  matrices) and quadratic forms (also represented by  $d \times d$  matrices). Our typical operations were scalar products (to determine coincidences), and cross products (for join and meet), exterior products, and matrix multiplications from the left and from the right, depending on the context. Our previous chapter introduced exterior products as a generalization of join and meet operations for the price that the bookkeeping of how to perform the operations became a little complicated (compare Section 12.5).

We will now (as a first step) consider all these objects as particular types of tensors, and we will consider all operations as multiplication of tensors with tensors. For this, let us look at a few of these operations from a very elementary perspective, in which we neglect the difference of rows and columns in matrices and vectors and represent all geometric quantities as *arrays of numbers* and where we rebuild all elementary operations from scratch. Let us consider the rank-three case  $\mathbb{RP}^2$  first. A point  $p$  has homogeneous coordinates  $(p_1, p_2, p_3)$ ; a line  $l$  has coordinates  $(l_1, l_2, l_3)$ ; the scalar product of the points and the line is

$$\sum_{i=1}^3 p_i l_i.$$

A transformation matrix  $T$  is a  $3 \times 3$  array of numbers with entries  $t_{ij}$ . The product of  $T$  and the point  $p$  is in elementary terms

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^3 p_i t_{1i} \\ \sum_{i=1}^3 p_i t_{2i} \\ \sum_{i=1}^3 p_i t_{3i} \end{pmatrix}.$$

The evaluation of the quadratic form  $p^T A p$  turns out to be

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ij} p_i p_j.$$

If we want to evaluate the quadratic form for the transformed point  $Tp$  we get

$$p^T T^T A T p = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 p_i t_{ij} a_{jk} t_{kl} p_l$$

(if you are patient you can check it). So, every single result or coordinate of a result turns out to be expressible as a summation of suitable monomials with carefully chosen indices. If we consider the vectors and matrices as one-dimensional, resp. two-dimensional arrays, the summation tells us explicitly how to perform the corresponding operation. Moreover, in the final summation formula, rowness, columnness, transposition, left- and right-multiplication, do not play a special role. Everything is encoded in the position of the indices.

So let us redefine these operations from scratch based entirely on the notion of multidimensional arrays and summation, thereby generalizing the concepts of numbers, vectors, and matrices (which are tensors of grade 0, grade 1, and grade 2, respectively). A *tensor* of grade<sup>1</sup>  $k$  is a multidimensional array of numbers  $a_{i_1, i_2, \dots, i_k}$ . Each of the numbers can be accessed by a suitable sequence of indices  $i_1, i_2, \dots, i_k$ . The range of the indices is for each  $j \in \{1, \dots, k\}$  restricted to a range  $1 \leq i_j \leq n_l$ . We will call such *hyperbox*-like structures that tell us the range of the indices the *shape* of a tensor. For most of our applications the limits  $n_l$  of the index ranges will all be equal to the rank of the projective geometry. So for the projective plane  $\mathbb{RP}^2$  we will deal with tensors of shape 3,  $3 \times 3$ ,  $3 \times 3 \times 3$ ,  $3 \times 3 \times 3 \times 3$ , etc. They have the grades 1, 2, 3, 4, ..., respectively. Numbers will be identified with tensors of grade 0. When we work in  $\mathbb{RP}^1$  we will have to deal with tensors of shape 2,  $2 \times 2$ ,  $2 \times 2 \times 2$ , etc. Grade-1 Tensors (which correspond to vectors) will not have a predefined way of writing them as rows or columns. The coordinate entries  $v_i$  will simply be accessed by the single index  $i$ . Similarly, 2-tensors (they correspond to matrices) will not have a row/column structure. The coordinates will be simply accessed by the two indices.

Let us consider the  $\mathbb{RP}^2$  case first. A point  $p$  has homogeneous coordinates  $(p_1, p_2, p_3)$ ; a line  $l$  has homogeneous coordinates  $(l_1, l_2, l_3)$ . The scalar product of  $p$  and  $l$  is

$$\sum_{i=1}^3 p_i l_i.$$

We have seen previously that expressions in projective geometry will tend to have many summation signs, which do not carry much information. They

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<sup>1</sup> Often in the literature the term *rank* is used instead of *grade*. However, in our context the term *rank* is already occupied in relation to the dimension of a projective geometry.

just tell us from where to where an index runs, but this is already specified by the shape of the tensors. So, let us drop the summation signs in our notation. We use a formal trick that is known as *Einstein's summation convention*. If we want to sum over an index, then the index should occur exactly twice in a formula, and it should be used once as a subscript and once as a superscript. Thus our scalar product may be simply written as

$$p_i l^i.$$

In a similar fashion we can represent transformation matrices as grade-2 tensors.

Using Einstein's summation convention we can write the multiplication of a matrix by a vector (the classical matrix multiplication) as

$$t_j^i p_i.$$

We should step back for a moment and check that we indeed get the right results. If we set  $q_j = t_j^i p_i$ , the three entries of  $q_j$  evaluate to

$$q_1 = \sum_{i=1}^3 t_1^i p_i, \quad q_2 = \sum_{i=1}^3 t_2^i p_i, \quad q_3 = \sum_{i=1}^3 t_3^i p_i,$$

which are exactly the components of the result of usual matrix multiplication.

The upper and lower indices have one additional semantic meaning. We have seen that lines were equipped with upper indices, whereas points were equipped with lower indices. In a sense, all lower indices will turn out to be “pointlike” and all upper indices will turn out to be “linelike.” In products of tensors indices with the same letter occur only between an upper and a lower entry. Thus in the product  $t_j^i p_i$ , the point  $p$  is connected via the index  $i$  to the transformation matrix. The only index not “eaten up” by the summation is the *lower* index  $j$ ; thus the result is again a pointlike object.

In our previous investigation into conics (Sections 9 to 11), quadratic forms were represented by a  $3 \times 3$  matrix  $A$ . Evaluation of a quadratic form at a point was done by the product  $p^T A p$ . In the context of tensors, the matrix  $A$  is different from a transformation matrix  $T$ . While the product of  $T$  with  $p$  is again a point, the product of  $A$  with  $p$  must be *linelike* so that it can be connected to another point. In other words, it must be possible to connect the quadratic form represented by  $A$  to two points. Thus the matrix  $A$  is represented as a grade-2 tensor with two upper indices,  $a^{ij}$ . The evaluation of the quadratic form can be written as

$$a^{ij} p_i p_j.$$

Observe that the order of the tensors becomes completely irrelevant, since the summation is completely driven by the distribution of the indices. We can also go one step further and combine points, transformations, and quadratic

forms in one formula. For instance, the application of the transformed point  $q = t_j^i p_i$  to the quadratic form  $a^{ij}$  becomes

$$p_i t_j^i a^{jk} t_k^l p_l.$$

Again the order of the different tensors in this formula is completely irrelevant. Only the position of the indices matters.

## 13.2 A Few Fine Points

Before we continue we will clarify a few points that may be confusing if one has no prior experience with tensors.

- *Tensors are neither vectors nor matrices:* Tensors are just multidimensional arrays of numbers. Semantically, grade-1 tensors are similar to vectors, and grade 2-tensors to matrices. However, they do not have a specific rowness or columnness. There is also nothing like a transpose of a tensor.
- *Tensor multiplication is commutative:* We are used to noncommutativity when we are dealing with matrices. In general, for matrices,  $AB$  is not the same as  $BA$ , however, for the corresponding tensors  $a_j^i$  and  $b_j^i$  the corresponding product is entirely expressed by the names and positions of the indices. Thus one of the above products (say  $AB$ ) corresponds to  $a_j^i b_i^k$  (which is the same as  $b_i^k a_j^i$ ). The other product is  $a_i^j b_k^i$  (which is the same as  $b_k^i a_i^j$ ).
- *Every summation needs a new index letter:* The indices are necessary formal elements of any tensor-product formula. An index may occur once or twice. If it occurs twice, then it has to occur as an upper and as a lower index. The result no longer contains this index. If it occurs once, then this index survives the multiplication. It becomes an index of the resulting tensor. The number of indices that occur exactly once corresponds to the grade of the resulting tensor. If every index occurs twice, then the result is a tensor of grade 0. This is just a number.
- *Position of indices matters:* In a tensor that has several upper or lower indices the sequence of the indices matters. The position of the indices replaces and generalizes the row/column structure of matrices.
- *Tensors do not have powers:* One might be afraid that the upper indices of tensors may be confused with powers of tensors. Fortunately, there is no such notation of the power of a tensor. If a tensor is repeatedly multiplied by itself, then this has to be written as  $t_a^b t_b^c t_c^d t_d^e \dots$
- *Tensors of the same shape may be added:* This is similar to vectors and matrices. Tensors are added componentwise. Tensors satisfy the left and right distributive laws.

We will see soon that tensors have a great unifying and clarifying power when one is dealing with objects of projective geometry. However, with the current notation (which is unfortunately the traditional one) there are a few obvious drawbacks. Much important information is encoded by the name and positions of the indices. As soon as we have more than 26 indices we will run out of letters, but most probably we will lose the overview earlier. This is the reason why tensor notation is not very common in books on projective geometry. Before we continue with our considerations on tensors, we will introduce a different notation that will make tensor operations more readable and give additional structural insight into the interplay of the different objects.<sup>2</sup>

### 13.3 Tensor Diagrams

The human brain is not very well suited for dealing with formulas in which everything depends on the names and positions of indices. This may be one of the reasons why tensor notation did not become really popular in contexts of incidence geometry (although it is extremely useful). This is where diagrams come to the rescue. What does a pair of upper and lower indices mean? The indices simply say that two tensors are connected in a certain way. For instances in the formula

$$p_i t_j^i a^{jk} t_k^l p_l,$$

the point  $p_i$  is connected to the transformation tensor  $t_j^i$ . It becomes transformed, and the result is connected to the quadratic form  $a^{jk}$  via the index  $j$ . Similarly, the point  $p_l$  is connected to the transformation  $t_k^l$ . The result is connected to the other “slot” of the quadratic form via the index  $k$ . In the diagram notation that will be introduced, the above product will be written as



So, here are the rules for converting tensor products to corresponding diagrams:

- (i) Every tensor in a product is represented by a node in a directed graph.
- (ii) Every lower index corresponds to an outgoing edge.
- (iii) Every upper index corresponds to an incoming edge.
- (iv) Tensors that share an index are connected by an arrow.

<sup>2</sup> Although we here will focus mainly on diagrammatic treatment of tensors, we want to mention at least one classical book by Gurevich [55] on tensors that gives a brilliant and deep introduction close to contexts of projective geometry with a special emphasis on invariant-theoretic aspects.

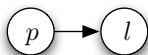
- (v) Indices that are used only once correspond to unconnected arrows.

Thus in the above diagram the arrows from left to right correspond to the index letters  $i, j, k$ , and  $l$  in this order. Furthermore, we will use lowercase letters for tensors of grade 1 and uppercase letters for other tensors. Each diagram will thereby represent a product of tensors. Since tensors of the same shape can also be added, tensor diagrams can also be added if they have the same number of incoming and outgoing edges of the same rank. However, it has to be specified exactly which of the free arrows of one summand correspond to which free arrows of the other summands. We will come to this issue later. Diagrams that have no incoming and outgoing edges will be called *closed*. Closed diagrams correspond to tensor products that will evaluate to numbers.

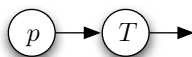
We will systematically translate geometric constructions and invariant formulas into the language of tensor diagrams. We start with the elementary objects first. Points, lines, transformations, and quadratic forms are represented by the following diagram nodes:



Semantically, it is meaningful to connect any two nodes along an incoming and an outgoing edge. The simplest closed diagram we can form is



This is nothing but the product  $p_i l^i$ , which is (in our old nomenclature) the scalar product of  $p$  and  $l$ . This closed diagram has an immediate geometric interpretation. If the diagram evaluates to zero, then  $p$  lies on  $l$ . Transformation of a point  $p$  by a transformation  $T$  can be expressed by the diagram



The outgoing arrow indicates that the result is again a point.

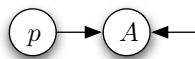
Let us now consider quadratic forms. Evaluation of the quadratic form  $p^T A p$  (classical notation) translates to



If this diagram evaluates to zero, then the point  $p$  lies on the conic described by  $A$ . We have to be a little careful here. Since  $A$  has two incoming edges,

it corresponds to a tensor  $A^{ij}$ . A priori the two arrows may have different meanings. This happens if there are index values  $i, j$  for which  $A^{ij} \neq A^{ji}$ . There is no problem if  $A$  is a *symmetric tensor*. This is a tensor for which the order of the indices does not matter. Then all arrows of this vector are equivalent. In our case  $A$  is symmetric if  $A^{ij} = A^{ji}$  for all values of  $i$  and  $j$ . Since quadratic forms always can be represented by symmetric matrices, we may also use a symmetric tensor.

What does the diagram



mean? The incoming free arrow indicates that the diagram represents a line-like object. In fact, it is nothing but the polar of the point  $p$  with respect to  $A$ . We get a closed diagram if we glue another point (say  $q$ ) to the free arrow of the above diagram. We obtain the diagram



This diagram evaluates to zero if  $q$  lies on the polar of  $p$ . Equivalently, in this case  $p$  lies on the polar of  $q$ . The little dotted line in the diagram indicates that one can consider the diagram as the polar of  $p$  connected to  $q$ .

## 13.4 How Transformations Work

There was something remarkable about the closed tensor diagrams we have considered so far. Whenever such a diagram evaluated to zero, this corresponded to a projectively invariant property. In our examples these properties were *coincidence of point and line*, *coincidence of point and conic*, and *q coincidences with the polar of q*. Thus the diagrams will turn out to be invariant under projective transformations. For this, however, we have to specify how an arbitrary tensor transforms under a projective transformation  $T$ . Let us recall the transformation behavior of our basic objects in the classical matrix/vector notation. If we transform a point  $p$  by multiplying it by a matrix  $T$ , we get the transformed point

$$p' = Tp.$$

In Section 3.6 we learned that under this transformation, lines have to be transformed by the matrix  $(T^{-1})^T$  according to

$$l' = (T^{-1})^T l.$$



In Section 10.4 we learned that if  $A$  is a matrix that represents a quadratic form, it has to be transformed according to

$$A' = (T^{-1})^T A T^{-1}.$$

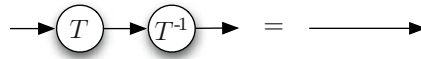
If  $S$  is a transformation itself and we consider how we have to represent  $S$  in a coordinate system after the transformation  $T$ , we obtain the similar matrix

$$S' = T S T^{-1}.$$

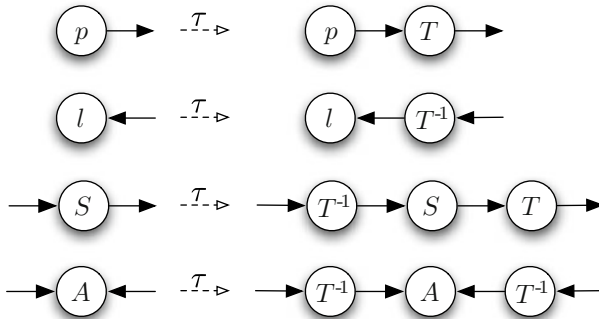
We can easily express all these transformations in terms of tensor diagrams. There all the problems of whether the matrices  $T$  and  $T^{-1}$  have to be multiplied from the left or from the right, transposed or nontransposed, will be resolved. For this let  $T^{-1}$  be the tensor representing the inverse matrix of  $T$ . We then have

$$T_i^j T^{-1}_j{}^k = E_i^k,$$

where  $E_i^k$  denotes the unit matrix. The unit matrix, however, may be simply represented by a plain arrow in a diagram. Thus in diagram notation the above equation reads



The tensor  $T^{-1}$  is uniquely determined by the tensor  $T$  in the same way as the inverse of a matrix is uniquely determined by the matrix. Now we can specify how a tensor behaves under a projective transformation  $\tau$ . This transformation determines a tensor  $T$  that transform the points. Lines are transformed by the tensor  $T^{-1}$ . The general rule for transforming tensors is as follows: *Each outgoing edge must be connected to the tensor  $T$ . Each incoming edge must be connected to the tensor  $T^{-1}$ .* Thus we obtain the following transformation rules for points, lines, quadratic forms, and transformations:



Retranslating these rules in terms of matrices gives exactly the old transformation rules we developed in previous chapters. These transformation rules imply that all diagrams composed of points, lines, and quadratic forms are projectively invariant in the following sense.

- (i) Assume that the diagram has no free arrows. After transforming each element according to the above rule the diagram will evaluate to exactly the same number as before.
- (ii) If the diagram has free arrows, the result of the evaluation will automatically be transformed according to the above rules after the diagram is transformed.

The reason for both effects is that along an arrow inside a diagram, transformation of the involved tensors generates a pair of consecutive tensors  $T$  and  $T^{-1}$  that cancel each other and leave the arrow unchanged. Consider, for instance, the diagram



If each tensor is transformed, we get



The dotted lines indicate the separation into the different transformed tensors. Each pair of tensors  $T$  and  $T^{-1}$  cancels, and we obtain the original diagram again. In particular, if the diagram evaluates to zero before the transformation, it will be zero after the transformation as well.

Let us spend a few words on common terminology to connect us to the terms of the standard literature. Lower indices are usually called *covariant indices*, while upper indices are usually called *contravariant indices*. Thus covariant indices correspond to outgoing (pointlike) arrows, while contravariant indices correspond to incoming (linelike) arrows. Under a projective transformation all covariant indices transform according to  $T$ , while all contravariant indices transform according to  $T^{-1}$ .

### 13.5 The $\delta$ -tensor

There is one ingredient in the classical theory of tensors that remains almost unnoticed in the context of tensor diagrams: The  $\delta$ -tensor  $\delta_i^j$ . This is a tensor that is defined as the Kronecker delta. Thus it is a two-dimensional grade-2 tensor  $\delta_i^j$  with one upper and one lower entry and defined by the property

$$\delta_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is nothing but a unit matrix in tensor form,  $\delta_i^j = E_i^j$ . The corresponding diagram for this transformation will simply be denoted by an arrow with no node at all. In the tensor world it “just” relabels the indices. For instance, we have  $p_j \delta_i^j = p_i$  and  $p_j \delta_i^j l^i = p_i l^i$ . So the  $\delta$ -tensor can be used to connect a covariant and a contravariant tensor with different indices. In the context of tensor diagrams a  $\delta$ -tensor plays the role of an intermediate extra arrow and remains essentially unnoticed. We will see that  $\delta$ -tensors play an essential role in this context when we start to transform diagrams into different-looking equivalent ones.

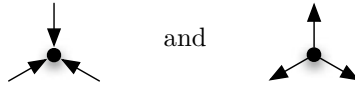
### 13.6 $\varepsilon$ -Tensors

The observant reader may have recognized that the diagrams we have considered so far were not very exciting. Each of them was essentially a linear chain of tensors with matrices in the middle and perhaps points or lines at the ends. The reason for this is that we only studied tensors that possess one or two arrows. There was no possibility to generate branched structures. This will now change dramatically, and this is the point where the game becomes really exciting. We will now introduce a special tensor, the so-called  $\varepsilon$ -tensor (or sometimes also called the Levi-Civita symbol). Unlike points, lines, quadratic forms, or transformations, the number of indices (the grade) of an  $\varepsilon$ -tensor depends on its rank. A rank-3  $\varepsilon$ -tensor will have three arrows, a rank-2  $\varepsilon$ -tensor will have two arrows, and a rank-4  $\varepsilon$ -tensor will have four arrows. For each rank there are two different variants of the  $\varepsilon$ -tensor: either purely covariant or purely contravariant. We will start with an investigation of the rank-3  $\varepsilon$ -tensor and discuss the rank-4 case later.

The rank-3  $\varepsilon$ -tensor occurs in two different forms,  $\varepsilon^{ijk}$  and  $\varepsilon_{ijk}$ . The tensor  $\varepsilon^{ijk}$  is a  $3 \times 3 \times 3$  tensor, since each index runs from 1 to 3. Thus it is defined by 27 coordinate entries. Its crucial defining properties are that it is *completely antisymmetric* and that  $\varepsilon^{123} = 1$ . Being antisymmetric means that interchanging two indices reverses the sign (for instance  $\varepsilon^{123} = -\varepsilon^{132}$ ). In particular, all entries where an index value occurs twice must be zero. Thus the only nonzero entries are those where none of the indices occurs twice. They are already determined by the alternating rule and  $\varepsilon^{123} = 1$ . We get

$$\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1, \quad \varepsilon^{132} = \varepsilon^{321} = \varepsilon^{213} = -1.$$

The diagrams for the contravariant and covariant  $\varepsilon$ -tensors are



The  $\varepsilon$ -tensor is the key tool for performing geometric operations in the context of tensors. As a first instance we investigate the tensor product

$$l^k = p_i q_j \varepsilon^{ijk}.$$

The index  $k$  runs from 1 to 3. For calculating  $l^1$  we have to sum over all nine possibilities for the indices  $i$  and  $j$ . There are only two such summands for which the  $\varepsilon$ -tensor does not vanish, and we get

$$l^1 = p_2 p_3 \varepsilon^{231} + p_3 p_2 \varepsilon^{321} = p_2 p_3 - p_3 p_2.$$

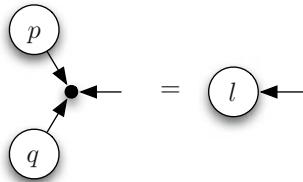
Similarly, we get

$$l^2 = p_1 p_3 \varepsilon^{132} + p_3 p_1 \varepsilon^{312} = -p_1 p_3 + p_3 p_1$$

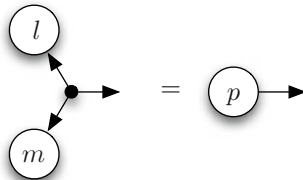
and

$$l^3 = p_1 p_2 \varepsilon^{123} + p_2 p_1 \varepsilon^{213} = p_1 p_2 - p_2 p_1.$$

Thus  $p_i q_j \varepsilon^{ijk}$  evaluates exactly to the tensor that represents the cross product of the points  $p$  and  $q$ . This tensor has one contravariant index and is therefore linelike. It is exactly the join of  $p$  and  $q$ . In diagram notation we can write



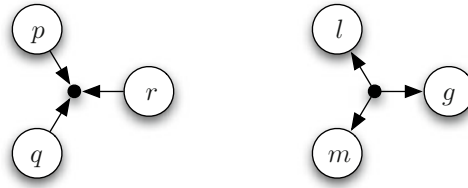
where  $l$  is the join of  $p$  and  $q$ . Similarly, we can calculate the meet of two lines using the covariant  $\varepsilon$ -tensor:



If we plug three different points  $p$ ,  $q$ , and  $r$  into a contravariant  $\varepsilon$ -tensor, we get exactly the determinant of a matrix whose columns are formed by the coordinates of the points:

$$p_i q_j r_k \varepsilon^{ijk} = \det \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}.$$

The reason for this is that the six nonzero entries of  $\varepsilon^{ijk}$  filter exactly the right summands with the correct signs for the determinant evaluation. Similarly, the expression  $l^i m^j g^k \varepsilon_{ijk}$  calculates the determinant of the coordinates of three lines. Thus vanishing of the diagrams



can be used to test whether three points are collinear or three lines meet in a point, respectively.

One has to be a bit careful to respect the anticommutativity of the  $\varepsilon$ -tensors when dealing with diagrams. Rotating the three slots of an  $\varepsilon$ -tensor does not change the sign, while interchanging two arrows changes the sign.

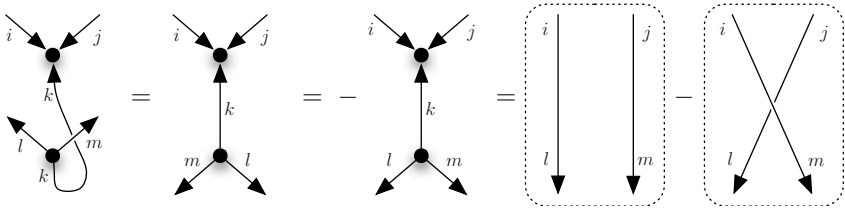
### 13.7 The $\varepsilon$ - $\delta$ Rule

We now want to investigate rules according to which diagrams can be transferred to equivalent diagrams. The key point here is an identity that allows us to express the direct connection of a covariant and contravariant  $\varepsilon$ -tensor by the sum of two simpler diagrams that do not contain any  $\varepsilon$ -tensor. This identity is called the  $\varepsilon$ - $\delta$ rule and reads

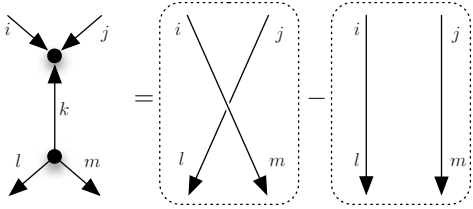
$$\varepsilon^{ijk} \varepsilon_{lmk} = \delta_l^i \delta_m^j - \delta_m^i \delta_l^j.$$

This identity can be easily checked by direct expansion. There is one problem in transferring this identity to a diagram. One has to be very careful about the signs of the involved summands. The following sequence of diagram identities leads to the diagrammatic form of the  $\varepsilon$ - $\delta$ rule that we will use later on. In order to make the diagram more readable, the different summands are marked by a dashed box. In the  $\varepsilon$ - $\delta$ rule the index  $k$  that is used by both  $\varepsilon$ -tensors is at the same position. This is represented by the first picture in the equation sequence. The second diagram is identical to the first one and is simply a topological transformation of the first. Observe that  $l$  and  $m$  become interchanged. We can reverse this interchange by simply multiplying the diagram by  $-1$ . The last term in the equation sequence is just the right

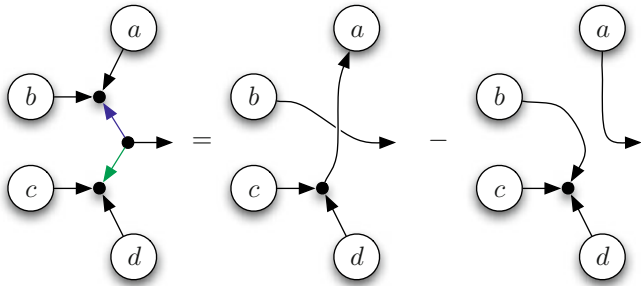
side of the  $\varepsilon$ - $\delta$ rule:  $\delta_l^i \delta_m^j - \delta_m^i \delta_l^j$  (the dotted boxes indicate the different terms of a sum where necessary):



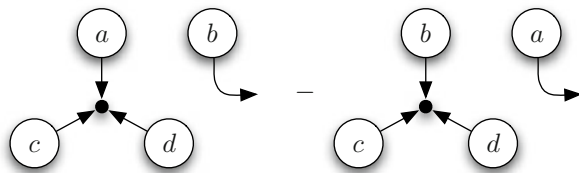
The last equation in the row (with both sides multiplied by  $-1$ ) is the version of the  $\varepsilon$ - $\delta$ rule that will be useful for our purposes. We summarize it in the following equation:



Let us investigate an application of the  $\varepsilon$ - $\delta$ rule. Consider two lines that are spanned by two pairs of points  $a, b$  and  $c, d$ , respectively. The meet of these two lines can be generated by the diagram on the left of the equal sign in the diagram below. Applying the  $\varepsilon$ - $\delta$ rule to the upper pair of  $\varepsilon$ -tensors (along the blue arrow), we get the diagram on the right of the equal sign (note that disconnected diagrams occurring in the same summand are simply multiplied):



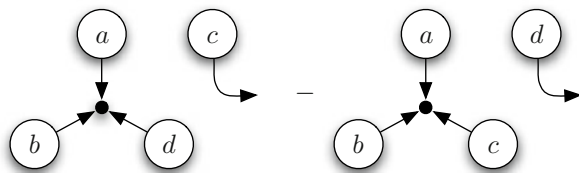
Cleaning up a little we get



In terms of bracket algebra this diagram means nothing but

$$[acd]b - [bcd]a.$$

This is exactly the bracket expression for this situation we derived in Section 6.3 using Plücker's  $\mu$ . Equivalently, we could have performed the same split along the green arrow. Then we would get the following equivalent expression for the meet:



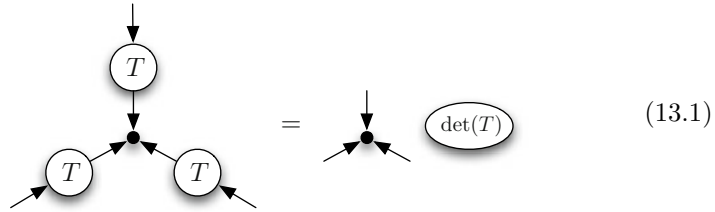
Since both expressions must be equivalent, we can also combine them into one equation and obtain

$$= 0$$

This is just Cramer's rule as we encountered it in Section 6.6. Plugging the join of two points  $e$  and  $f$  into all open arrows, we get the four-summand Grassmann-Plücker relation.

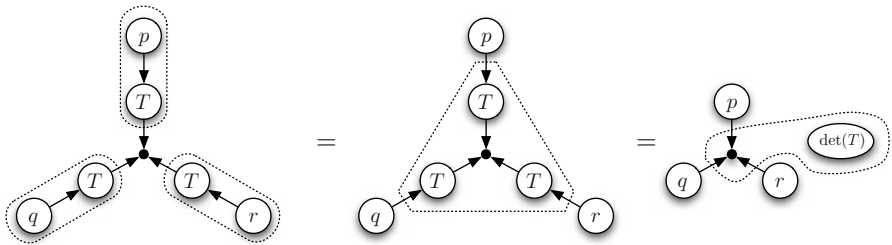
## 13.8 Transforming $\varepsilon$ -Tensors

We have seen that diagrams containing  $\varepsilon$ -tensors also create geometrically meaningful expressions. The deep reason for this is that  $\varepsilon$ -tensors behave well under projective transformations. Applying a projective transformation to an  $\varepsilon$ -tensor simply results in a tensor that is a multiple of the  $\varepsilon$ -tensor. The scaling factor is the determinant of the transformation matrix. Thus we have



$$(13.1)$$

Here the oval marked  $\det(T)$  on the right of the equation means that the  $\varepsilon$ -tensor is multiplied by this factor. There are several ways to convince oneself that this transformation rule is correct. First of all, one can simply expand the formula on the left (since it is nothing but an explicit tensor product  $T_i^a T_j^b T_k^c \varepsilon^{ijk}$ ) and observe that it expands to the  $\varepsilon$ -tensor times the determinant. One can also derive the formula by examining the behavior of points plugged into the open slots. We know that  $p_i q_j r_k \varepsilon^{ijk}$  gives just the determinant of the matrix with columns  $p, q, r$ . Thus  $p_a T_i^a q_b T_j^b r_c T_k^c \varepsilon^{ijk}$  must give  $\det(T)$  times this value. The diagram below illustrates this process:



We know that the diagram on the left where we connect three transformed points to an  $\varepsilon$ -tensor must be (in classical notation)  $[Tp, Tq, Tr]$ , which is nothing but  $\det(T)[p, q, r]$ . Regrouping the ingredients of this diagram, we see that it is nothing but  $p, q, r$  plugged into the transformed  $\varepsilon$ -tensor. The only possibility for this identity to hold for arbitrary  $p, q, r$  is that the transformed  $\varepsilon$ -tensor is the original  $\varepsilon$ -tensor scaled by a factor of  $\det(T)$ .

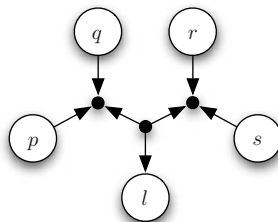
The transformation rule for the  $\varepsilon$ -tensor has a deep consequence for the transformation behavior of closed diagrams consisting of tensors and  $\varepsilon$ -tensors. We know that any such diagram evaluates to a single number if concrete geometric objects are plugged in. If the objects are transformed, this number is scaled by a number that depends only on the determinant of the transformation matrix and the numbers of covariant and contravariant  $\varepsilon$ -tensors. The exact relation is as follows:

**Theorem 13.1.** *Let  $\mathcal{D}$  be a closed tensor diagram consisting of tensors that represent geometric objects and  $\varepsilon$ -tensors. If the geometric objects of  $\mathcal{D}$  (all tensors except for the  $\varepsilon$ -tensors) are transformed by a projective transformation given by a  $3 \times 3$  matrix  $T$ , then the evaluation of the diagram is scaled*

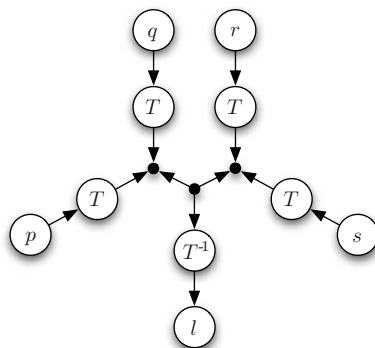


by a factor  $\det(T)^k$ , where  $k$  is the difference of covariant and contravariant  $\varepsilon$ -tensors.

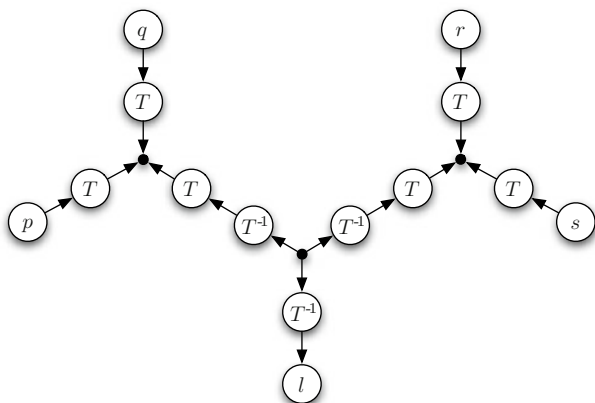
*Proof.* Rather than giving a strictly formal proof, we confine ourselves here to a “proof by example” that exemplifies the main principles in diagram notation. We consider the following diagram, which consists of four points, one line, and three  $\varepsilon$ -tensors (two of them are contravariant and two of them are covariant):



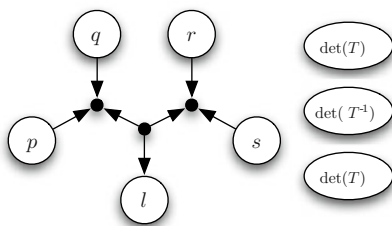
How does the evaluation change if we transform the geometric objects by a projective transformation given by the pair  $(T, T^{-1})$ ? Covariant arrows (our points) of geometric objects have to be transformed by  $T$ , contravariant arrows (our lines) have to be transformed by  $T^{-1}$ , and we obtain



For evaluating this diagram we “blow up” the interior arrows by replacing them by a pair of transformations  $\rightarrow \textcircled{T^{-1}} \rightarrow \textcircled{T} \rightarrow$ . After this procedure our diagram looks as follows:



Now each contravariant  $\varepsilon$ -tensor is surrounded by a halo of  $T$ -tensors and each covariant  $\varepsilon$ -tensor is surrounded by  $T^{-1}$ -tensors. We can replace each such  $\varepsilon$ -tensor together with its surrounding transforming matrices by an  $\varepsilon$ -tensor multiplied by the corresponding determinant. Thus we get



This is just our original diagram multiplied by  $\det(T)^2 \cdot \det(T^{-1})$ . Since the determinant of  $T^{-1}$  is the inverse of the determinant of  $T$ , the final factor is  $\det(T)$ . In general, we obtain a factor that is  $\det(T)^k$ , where  $k$  is the number of contravariant  $\varepsilon$ -tensors minus the number of covariant  $\varepsilon$ -tensors.  $\square$

The situation is closely related to the situation of Theorem 6.1, where we considered multihomogeneous bracket polynomials as projective invariants. Each monomial summand of such a polynomial can be considered a tensor diagram. For each bracket we get a contravariant  $\varepsilon$ -tensor connected to three points. Thus the overall factor of such a bracket polynomial under a transformation  $T$  is  $(\det(T))^k$ , where  $k$  is the number of brackets in this summand.

## 13.9 Invariants of Line and Point Configurations

The previous section showed that diagrams formed by geometric objects and  $\varepsilon$ -tensors form invariants under projective transformations in the same sense as we encountered invariants in Section 7.2. There we discussed invariants of configurations of points only and stated (Theorem 7.3) that each such invariant can be expressed by a multihomogeneous bracket polynomial. This is one version of the first fundamental theorem of invariant theory. Depending on the class of objects under consideration and on the type of transformations allowed, there are different versions of this theorem. One very general version can be stated in terms of tensor algebra:

*Every relatively invariant function of a collection of tensors can be expressed as a (multihomogeneous) linear combination of closed diagrams that involve only these tensors and  $\varepsilon$ -tensors.*

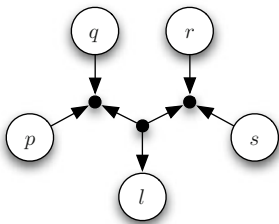
Here *relatively invariant* means that the evaluation of the linear combination of diagrams is invariant up to a factor of  $(\det(T))^k$ . This theorem is very strong and requires some technical lemmas to prove it. We will not do this here (as we did not prove Theorem 7.3), since this is beyond the scope of this text. A proof may be found in [55]. However, we will see how this general form can be used to derive the corresponding version of the first fundamental theorem if we consider configurations that contain points as well as lines.

**Theorem 13.2.** *Every relatively invariant function of a collection of points and lines can be expressed by a (multihomogeneous) polynomial that involves determinants of points, determinants of lines, and scalar products between points and lines.*

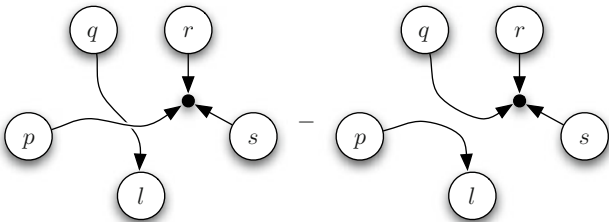
*Proof.* The proof is a simple consequence of the  $\varepsilon$ - $\delta$ -rule. By the fundamental theorem in its tensor version one can express each diagram as a multihomogeneous linear combination of closed diagrams. We focus on one summand of this linear combination. The corresponding diagram may involve many  $\varepsilon$ -tensors. If there are no internal arrows connecting two  $\varepsilon$ -tensors then the diagram contains only factors of the following form: three points plugged into a contravariant  $\varepsilon$ -tensor, three lines plugged into a covariant  $\varepsilon$ -tensor, or points that are directly connected to lines. These three factors correspond to determinants of points, determinants of lines, and scalar products between points and lines, respectively.

Thus we have to consider the case that the diagram contains internal arrows that connect two  $\varepsilon$ -tensors. In this case we may apply the  $\varepsilon$ - $\delta$ -rule. By this the diagram is replaced by the difference of two other diagrams that contain at least one such internal arrow fewer than in the original diagram. By proceeding inductively we arrive at a linear combination of diagrams that do not have arrows between  $\varepsilon$ -tensors.  $\square$

We will exemplify this theorem by the example of our last section. This diagram contains three  $\varepsilon$ -tensors and connects four points and one line:



Applying the  $\varepsilon$ - $\delta$ rule once, we obtain



This diagram expresses the multihomogeneous expression

$$[rqs]\langle q,l\rangle - [rqs]\langle p,l\rangle.$$

*When the proofs, the figure,  
                                were ranged in columns before me,  
When I was shown the charts and diagrams,  
                                to add, divide, and measure them,...  
Two lines of a poem by Walt Whitman (1819–1892)*

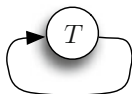
The value of diagram techniques even at this rudimentary level should be clear by now: it is easier to visualize where simplifications may be found in a complicated network by searching for a reducible linkage than by examining a complicated algebraic expression.

Geoffrey E. Stedman,  
*Diagram Techniques in Group Theory* (1990)

So much for the preliminaries. Now let us come to the real stuff about diagrams. In this chapter we will investigate more advanced applications of tensors and diagrams. In particular, we will link tensor diagrams more closely to geometric theorems and concepts we encountered in previous chapters. The invariant character of closed diagrams and the  $\varepsilon$ - $\delta$  rule will play a prominent role in this context. Moreover, we will revisit the theorems of Pappos and Pascal once again.

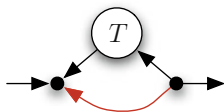
## 14.1 The Simplest Property: A Trace Condition

We will now consider closed diagrams in their simplest forms and examine the geometric relevance of these diagrams. In particular, if a closed diagram evaluates to zero, we have a projectively invariant property. The simplest closed diagram we may think of consists of just one tensor and one arrow. It looks as follows:

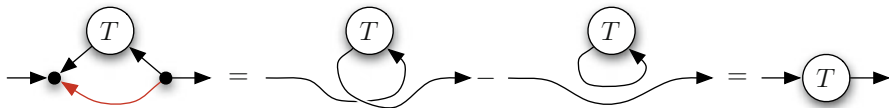


It is a transformation tensor with the same index co- and contravariant. Algebraically this is  $T_i^i$ , which is the sum over all diagonal entries of the corresponding matrix: the *trace* of the transformation. Hence we may conclude that a transformation  $T$  with  $\text{trace}(T) = 0$  is different from other projective transformations.

We can observe a first special property of a trace-zero transformation  $T$  if we apply the  $\varepsilon$ - $\delta$  rule to the pair of  $\varepsilon$ -tensors in following diagram:



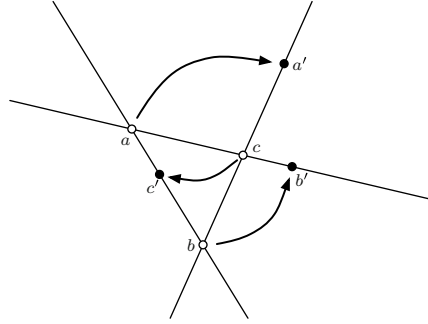
Assuming  $\text{trace}(T) = 0$ , we obtain the following chain of equivalent diagrams:



The first equal sign comes from the  $\varepsilon$ - $\delta$  rule. In the next expression the second summand vanishes according to the trace-zero condition. Unwinding the diagram, we see that the right side is just equivalent to the original transformation. Thus for a trace-zero transformation we can exchange any occurrence of  $T$  by the more complicated expression on the left. (One might wonder what this is good for. We will see an application of this in the next theorem.)

Transformations with vanishing trace are also characterized by the following geometric property. This property characterizes the trace-zero condition via incidence relations of images and preimages of three points (compare Figure 14.1).

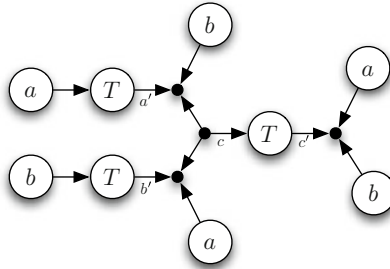
**Theorem 14.1.** *Let  $\tau$  be a projective transformation in  $\mathbb{RP}^2$  whose corresponding matrix  $T$  satisfies  $\text{trace}(T) = 0$ . Let  $a$  and  $b$  be two arbitrary*



**Fig. 14.1** Geometric characterization of trace-zero transformation.

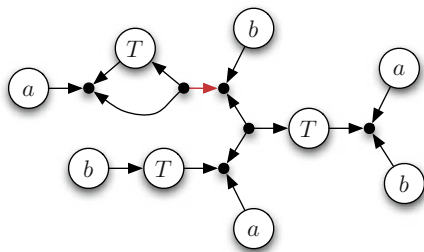
points and  $a'$ ,  $b'$  their images under  $\tau$ . Define  $c$  as the meet of  $\text{join}(a, b')$  and  $\text{join}(a', b)$ . Then the image  $c'$  of  $c$  lies on the join of  $a$  and  $b$ .

*Proof.* It is not too hard to prove this fact entirely by methods of linear algebra. However, we will investigate how the result can be obtained by shuffling around diagrams. The incidence relation stated in the theorem can be expressed as follows: For every trace-zero transformation  $T$  and for arbitrary points  $a$  and  $b$ , the following diagram vanishes:

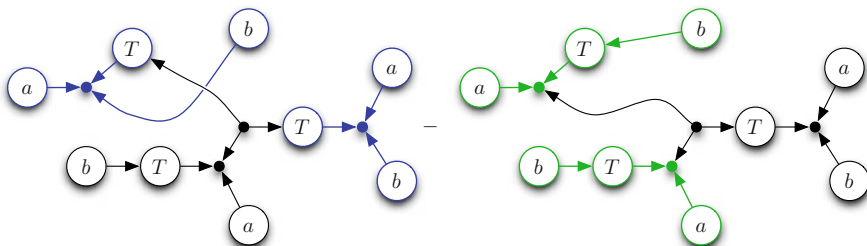


The small letters at the arrows refer to the corresponding points in the theorem ( $a'$  is the image of  $a$ ,  $b'$  is the image of  $b$ ,  $c$  is the join of the two lines, and  $c'$  is the image of  $c$ ). The diagram encodes the construction of the theorem from left to right. The rightmost  $\varepsilon$ -tensor expresses that  $c'$  is incident to the join of  $a$  and  $b$ . So, how do we prove that this diagram always vanishes? Applying the  $\varepsilon$ - $\delta$  rule to any of its internal arrows decomposes the diagram into smaller units and we have no chance to apply the trace-zero condition. However, we can proceed as follows:

First we exchange one occurrence of  $T$  by the more complicated diagram we mentioned above and obtain



This diagram is equivalent to the above one if  $T$  has vanishing trace. Applying the  $\varepsilon$ - $\delta$ rule to the red arrow in this diagram gives



Now we see that obviously both summands vanish, since in the first summands the two blue parts are identical (up to sign) and plugged into an  $\varepsilon$ -tensor, and in the second summand the two green parts are identical and plugged into an  $\varepsilon$ -tensor. Thus the entire diagram must be identically equal to zero.  $\square$

## 14.2 Pascal's Theorem

We will now demonstrate how the bracket condition that characterizes that six points are on a common conic (compare Sections 10.1 and 10.2) can be derived by straightforward diagram manipulations starting from Pascal's theorem. According to Pascal's theorem (compare Section 1) six points are on a conic if and only if the three points

$$X = \text{meet}(\text{join}(a, b), \text{join}(e, d)),$$

$$Y = \text{meet}(\text{join}(e, f), \text{join}(c, b)),$$

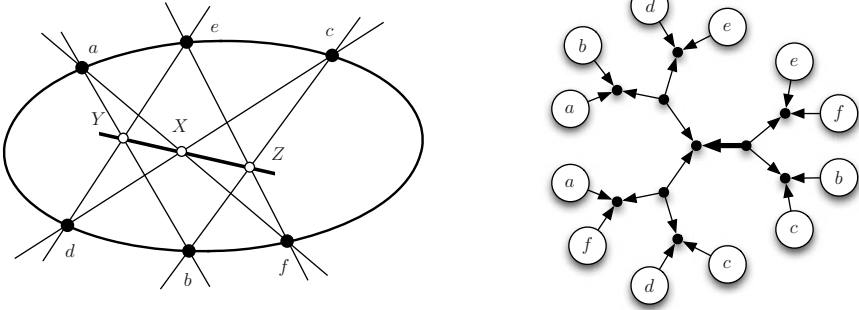
$$Z = \text{meet}(\text{join}(c, d), \text{join}(a, f)),$$

are collinear. On the other hand, we learned in Section 10.2 that the algebraic characterization of coconicality can be expressed by the bracket expression

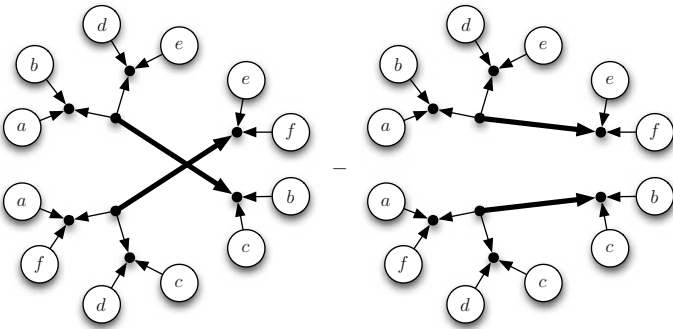


$$[abc][deb][cdf][fae] - [abe][def][cdb][fac] = 0.$$

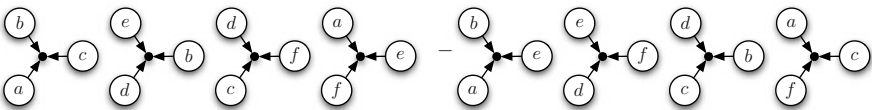
We will now demonstrate how the algebraic expression follows from the geometry in a straightforward manner. Since the construction of the points  $X, Y, Z$  in Pascal's theorem can be expressed as a sequence of join and meet operations, the collinearity of  $X, Y, Z$  corresponds to the vanishing of a certain diagram, which turns out to be a tree of  $\varepsilon$ -tensors:



Applying the  $\varepsilon$ - $\delta$ -rule to the darkened edge, we arrive at the following diagram:



Applying the  $\varepsilon$ - $\delta$ -rule again to all connected subdiagrams at the darkened edges, we have an intermediate expression, in which every connected subdiagram is just an  $\varepsilon$ -tensor with three soldered points. In half of these subdiagrams one letter will appear twice (check it!). Collecting the remaining terms, we end up with



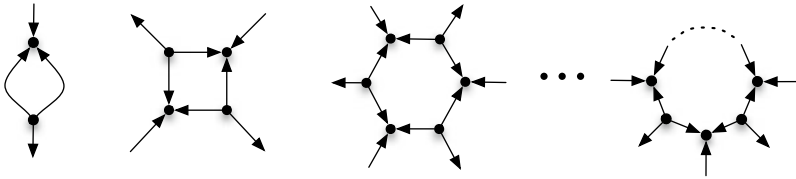
This is exactly the characterization of coconicality we were heading after:

$$[abc][deb][cdf][fae] - [abe][def][cdb][fac] = 0.$$

Conversely (if one assumes this expression as given), one can read the diagrammatic calculations in the opposite directions and interpret them as a proof of Pascal's theorem.

### 14.3 Closed $\varepsilon$ -Cycles

As a further example let us consider closed rings of  $\varepsilon$ -tensors (alternately co- and contravariant) and investigate their geometric meaning under various aspects. Since all our arrows are directed, the number of  $\varepsilon$ -tensors has to be even, in order to get a closed cycle:



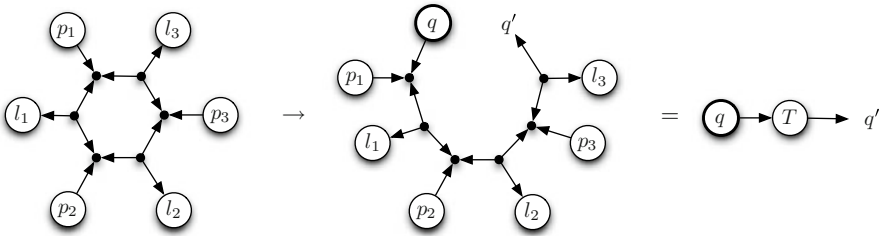
As a first instance, we will consider the meaning of a 2-cycle. This little gadget will turn out to be useful as an auxiliary calculation that will be needed in order to simplify the diagrams later on. The calculation is presented in the following diagram and explains how double arrows between  $\varepsilon$ -tensors can be replaced by a single arrow and a factor:

The first equality is just a topological deformation; the second is an application of the  $\varepsilon$ - $\delta$  rule. The third equality is due to the fact that the closed loop  $\bigcirc = \delta_i^i = 3$  is nothing but the trace of the unit matrix (here the dimension enters the calculation as a number).

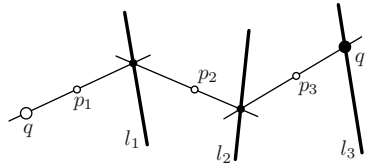
Let us now consider larger cycles. Each of these cycles has an alternating sequence of incoming and outgoing arrows along the boundary. We will analyze the cases in which we attach points and lines in an alternating manner. The vanishing of such a cycle encodes a projective condition that must be fulfilled by the points and lines that are involved. For each cycle we will

interpret this condition in two different ways. First we will give a geometric interpretation of such a cycle being closed for general cycles. After this we will analyze alternative geometric and algebraic characterizations by applying the  $\varepsilon$ - $\delta$ -rule.

Let us first consider a general  $2n$ -cycle with points and lines attached in alternation (as a running example we consider a six-cycle; our considerations, however, will apply to the general case). Assume that the points and lines are attached in the interleaved order  $p_1, l_1, p_2, l_2, \dots, p_n, l_n$ . By breaking the cycle at a specific arrow between two  $\varepsilon$ -tensors (say the  $\varepsilon$ -tensor attached to  $p_1$  and the  $\varepsilon$ -tensor attached to  $l_n$ ) we derive a transformation  $T$  that takes a point  $q$  as input and “calculates” a point  $q'$ :

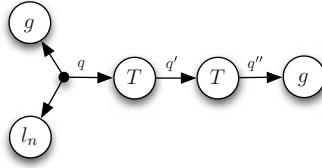


The vanishing of the closed cycle corresponds to the transformation  $T$  having trace zero, since  $\text{tr}(T) = 0$  by our cycle condition. Furthermore,  $T$ , considered as a matrix, must have at most rank 2, since the transformed point  $q'$  will always lie on  $l_n$ . Geometrically, this transformation corresponds to the chain of *join* and *meet* operations as indicated by the following picture:



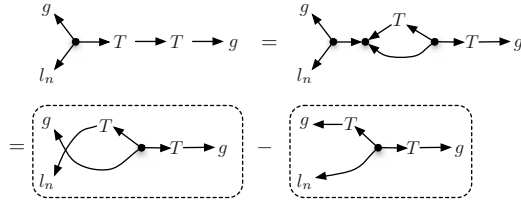
The final point  $q'$  will always lie on line  $l_n$ . Thus if we assume that point  $q$  was chosen also to lie on  $l_n$ , then we obtain a (projective) transformation from  $l_n$  to  $l_n$ . The cycle condition now translates to the following geometric criterion:

**Theorem 14.2.** *With all settings as above we assume that the points and lines  $p_1, l_1, p_2, l_2, \dots, p_n, l_n$  are chosen such that the  $\varepsilon$ -cycle evaluates to zero. Then for an arbitrarily chosen line  $g$  the following diagram will be zero as well:*



Before we prove this theorem we will analyze its geometric significance. If no degeneracy occurs along the construction, then by intersecting  $g$  with  $l_n$  (the first  $\varepsilon$ -tensor) we get an arbitrary point  $q$  on  $l_n$ . Applying the chain of joins and meets *twice*, we arrive at the point  $q''$ , which will again lie on  $g$  and thus be identical to  $q$  (since it also lies on  $l_n$ ). Thus either the sequence of operations degenerates at some point (which results in a zero tensor) or the mapping  $T$  restricted to  $l_n$  is an involution.

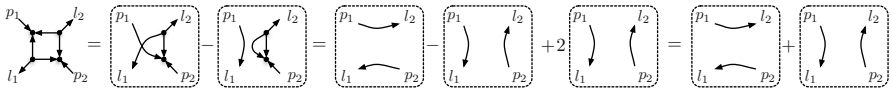
*Proof.* We make simplifications according to the rule for trace-zero transformations derived at the beginning of Section 14.1 and to the  $\varepsilon$ - $\delta$ -rule (circles around tensors are omitted):



Both summands of the final expression vanish, the first one since  $T$  produces a point that lies on  $l_n$ , the second one because of the fact that one  $\varepsilon$ -tensor is connected to two identical chains of tensors.  $\square$

Now we will calculate algebraic expressions for the 4-cycle and for the 6-cycle. The 4-cycles turn out to be intimately related to harmonic relations of points; 6-cycles turn out to be related to Pappos's theorem.

**4-cycles:** A 4-cycle involves two points and two lines. If we apply the  $\varepsilon$ - $\delta$ -rule to a four-cycle, we arrive at the following algebraic expression:

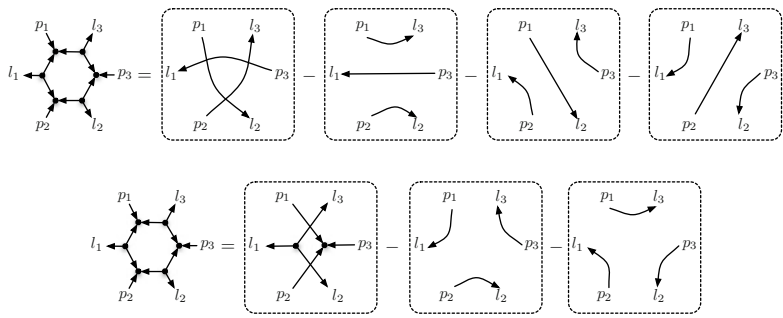


Vanishing of the final expression in vector language is  $\frac{\langle p_1, l_1 \rangle \langle p_2, l_2 \rangle}{\langle p_1, l_2 \rangle \langle p_2, l_1 \rangle} = -1$ , which is the algebraic condition for the situation that the line joining  $p_1$  and  $p_2$  intersects  $l_1$  and  $l_2$  in two points that are harmonic to the pair  $(p_1, p_2)$ .

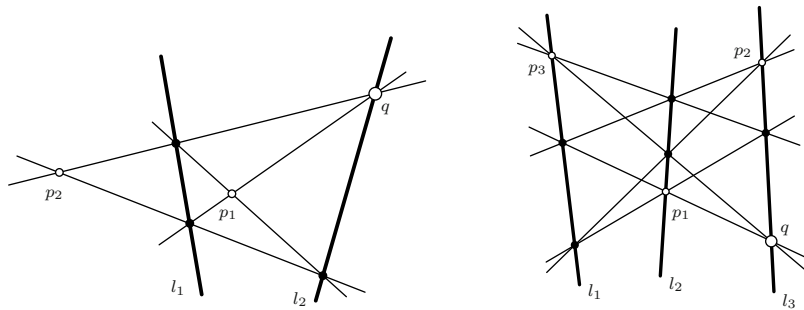
In Figure 14.2 on the left a corresponding situation is shown in which  $p_1, p_2, l_1, l_2$  satisfy the 4-cycle condition. The geometric statement derived

by Theorem 14.2 reflects the algebraic condition for the harmonic situation. The algebraic reduction for the 4-cycle that we performed by the  $\varepsilon$ - $\delta$ -rule is unique in the sense that no matter in which order we perform  $\varepsilon$ - $\delta$ -rules to the arrows, we will arrive at the same expression. Thus if we greedily perform  $\varepsilon$ - $\delta$ -rules whenever two  $\varepsilon$ -tensors are connected, we always end up with the same result.

**6-cycles:** A similar calculation to the one above can be performed for 6-cycles. However, here the result of a greedy reduction process is not unique. We obtain (intermediate reduction steps omitted) the following two irreducible forms:



We now study *particular* geometric situations under which the 6-cycle expression vanishes. First of all, since the 6-cycle is linear in the points and lines, it follows that for every position of five of the elements there is *at least* a one-dimensional space for the final object to obtain a vanishing cycle. We will now consider special cases in which the cycle vanishes. These come from the vanishing of the four summands in the first of the above expansions or from vanishing of the three summands in the second expansion. The sum-



**Fig. 14.2** Geometry of 4-cycles and 6-cycles.

mands consisting of three arrows will vanish if at least one of the point/line pairs connected by an arrow are incident. The summand with the two  $\varepsilon$ -tensors will vanish if either the three points are collinear or the three lines are concurrent. Whenever the cycle expression vanishes, Theorem 14.2 tells us that starting with a point  $q$  on  $l_3$  and performing a cycle of six consecutive join/meet operations (in the order specified by the cycle), we will either run into a degenerate situation or the resulting point will coincide with  $q$ . A careful case-by-case analysis shows that most of these cases lead to degenerate situations. There is only one possibility of coincidences that make the cycle expression vanish and does not lead to degeneracy. This is achieved, if the pairs  $(p_1, l_2), (p_2, l_3), (p_3, l_1)$  define coincidences. In this case, the first expansion formula for the 6-cycle vanishes. The geometric situation is drawn in Figure 14.2 on the right. If we start with a point  $q$  on  $l_3$  and cycle around twice (as indicated by Theorem 14.2), we must end up at the starting point  $q$ . This is nothing but our well-known Pappos's theorem.

*Remark 14.1.* Another interesting fact concerning the expansions of a 6-cycle arises if one considers the fact that both expansions must be algebraically the same. By this we obtain an algebraic expansion of the product of two determinants in terms of a linear combination of scalar products. Again removing the points and the lines from this expression, we get the following well-known identity on  $\varepsilon$ - and  $\delta$ -tensors:

$$\varepsilon^{abc}\varepsilon_{ijk} = \det \begin{pmatrix} \delta_i^a & \delta_i^b & \delta_i^c \\ \delta_j^a & \delta_j^b & \delta_j^c \\ \delta_k^a & \delta_k^b & \delta_k^c \end{pmatrix}.$$

This is essentially equivalent to the expansion formula for determinants. To see this, one must simply connect three-dimensional vectors  $p_a, p_b, p_c, p^i, p^j, p^k$  to both sides of the equation. The left side calculates the product of two  $3 \times 3$  determinants formed by these vectors. The right side is the determinant of the product of two matrices formed by rows  $p_a, p_b, p_c$  and columns  $p^i, p^j, p^k$ , respectively. If the vectors  $p_a, p_b, p_c$  are chosen to be the unit vectors, we get the usual expansion formula.

## 14.4 Conics, Quadratic Forms, and Tangents

Let us now study some diagram invariants involving quadratic forms. We start with diagrams that contain a contravariant symmetric quadratic form  $A$ . In rank 3, such a quadratic form has two incoming arrows  $\rightarrow \textcircled{A} \leftarrow$ . Being symmetric means that in any diagram we can interchange both arrows without changing the value of the diagram. Connecting twice the same point to such a diagram produces (as already mentioned in Section 13.3) the equation for a conic. What is the simplest interesting closed diagram that we can produce

exclusively by  $\rightarrow A \leftarrow$  and  $\varepsilon$ -tensors? Connecting both arrows of  $\rightarrow A \leftarrow$  to the same  $\varepsilon$ -tensor forces the entire diagram to collapse, since  $A$  is symmetric and the  $\varepsilon$ -tensor is antisymmetric. Hence we have

$$\begin{array}{c} \text{A} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \rightarrow = \begin{array}{c} \text{A} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \rightarrow = - \begin{array}{c} \text{A} \end{array} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \rightarrow = \begin{array}{c} 0 \end{array} \rightarrow$$

The first equality holds since we can interchange the arrows of the  $A$ -tensor without changing the value of the diagram. The second equality holds since interchanging the arrows of the  $\varepsilon$ -tensor reverses the sign. So in consequence the diagram has to be the zero tensor. Connecting this tensor to any diagram makes the diagram vanish.

So we need at least two different  $\varepsilon$ -tensors. We will first explore the meaning of a simple non trivial (open) diagram involving two  $\varepsilon$ -tensors and two  $A$ -tensors.

$$\begin{array}{c} \text{A} \\ \text{A} \end{array} \quad (14.2)$$

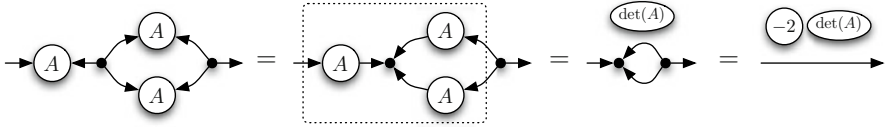
We can derive the geometric meaning of this diagram if we attach another copy of  $A$  and study the diagram

$$\begin{array}{c} \text{A} \end{array} \quad \begin{array}{c} \text{A} \\ \text{A} \end{array}$$

The easiest way to see what the latter diagram represents is via a little formal trick. Without changing the evaluation of a diagram we may reverse the orientation of any interior arrow, if we consistently transfer covariant to contravariant tensor indices. In the above diagram  $\rightarrow A \leftarrow$  corresponds to an entirely contravariant tensor  $A^{ij}$ . We now define a tensor  $A_i^j$  according to the rule  $A_i^j = A^{ij}$  for all  $i, j \in \{1, 2, 3\}$ . Both tensors have exactly the same entries. We simply interpret one index differently (so formally they represent different geometric objects). Since altering the direction of the arrow does not change the summation in the diagram, we get

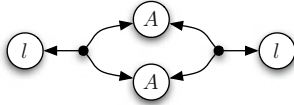
$$\begin{array}{c} \text{A} \end{array} \begin{array}{c} \text{A} \\ \text{A} \end{array} = \begin{array}{c} \text{A} \end{array} \begin{array}{c} \text{A} \\ \text{A} \end{array}$$

In the second diagram we may now, however, interpret  $A$  as a transformation. Thus we can apply equations (13.1) and (14.1) and obtain

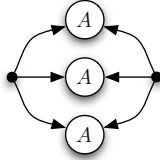


In other words, applying first  $\rightarrow \textcircled{A} \leftarrow$  and then the diagram (14.2) we get the identity transformation times  $-2 \det(A)$ . Hence the diagram (14.2) is  $-2 \det(A)$  times the inverse of  $A$ , or equivalently  $-2$  times the adjoint of  $A$ .

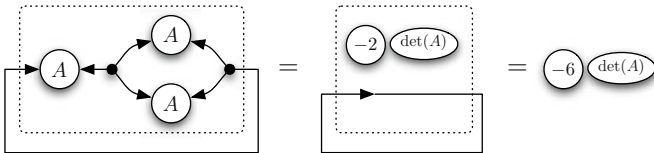
In Section 9.3 we learned that the adjoint of  $A$  is the quadratic form for the dual of the conic represented by  $A$ . Hence we get the following nice closed-diagram encoding the condition for a line  $l$  being tangent to the conic represented by  $A$ :



Let us now switch to closed diagrams containing only  $A$ - and  $\varepsilon$ -tensors. The simplest closed diagram we can form under the premise that no  $\varepsilon$ -tensor is connected to the same  $A$  twice needs two  $\varepsilon$ -tensors and three  $A$ -tensors. It is



Since it is a closed diagram, it must either vanish or represent a proper projective invariant. Using our last result, we get



The factor 3 in the last equality is induced by the closed ring, which calculates the trace of the unit matrix. Thus the diagram evaluates to the determinant of  $A$  times  $-6$ . Vanishing of this determinant is a projectively invariant property.



## 14.5 Diagrams in $\mathbb{RP}^3$

How does the machinery of tensor diagrams generalize to other dimensions? As one might expect, most concepts can be carried over in a most natural way. However, already in  $\mathbb{RP}^3$  there are some nice twists that enter the game according to the fact that the most natural representation of a line in  $\mathbb{RP}^3$  is by its six-dimensional Plücker vector. In what follows we will briefly sketch some of the most crucial effects. The reader should always be aware that we only scratch the surface and will leave aside many fascinating aspects of this subject. For a more elaborate treatment we recommend [8].

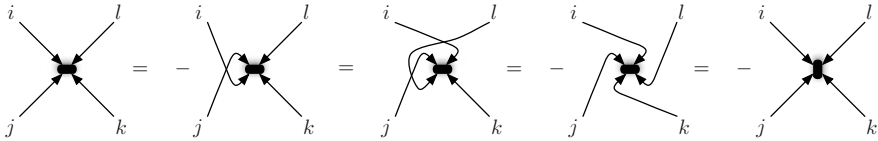
We start with the obvious parts. As we saw in Section 12, the points in  $\mathbb{RP}^3$  have to be represented by four-dimensional vectors. They in turn are represented by a covariant tensor of grade 1 with four entries. In a diagram such a point is simply represented by a node  $\textcircled{p} \rightarrow$  with one outgoing arrow. Similarly, a plane is represented by a contravariant tensor of grade 1 with four entries, diagrammatically expressed by a node  $\leftarrow \textcircled{h}$  with one incoming arrow. Incidence of a point  $p$  and a plane  $h$  corresponds to vanishing of the diagram  $\textcircled{p} \rightarrow \textcircled{h}$ . Transformations are grade-two tensors  $T_i^j$  of shape  $4 \times 4$  with the entries of the corresponding transformation matrix and the diagrammatic notation  $\rightarrow \textcircled{T} \rightarrow$ .

Matters start to become interesting as soon as  $\varepsilon$ -tensors are involved. For applications in  $\mathbb{RP}^3$  (i.e., rank 4) these are completely antisymmetric  $4 \times 4 \times 4 \times 4$  tensors. There is a completely covariant version  $\varepsilon_{ijkl}$  and a completely contravariant version  $\varepsilon^{ijkl}$ . As in rank 3, they are defined by the property that they are completely antisymmetric and  $\varepsilon_{1234} = \varepsilon^{1234} = 1$ . The only coordinate entries are 0, 1, and  $-1$ . There are exactly 24 nonzero entries that appear at those places for which all indices  $i, j, k, l$  differ. If  $e_1, \dots, e_4$  are the four unit vectors in  $\mathbb{R}^4$ , then we get a nice and easily memorizable description of the coordinate entries of the  $\varepsilon$ -tensors by

$$\varepsilon_{ijkl} = \varepsilon^{ijkl} = \det(e_i, e_j, e_k, e_l).$$

In this equation the indices are not merely placeholders but concrete entries  $(i, j, k, l) \in \{i, j, k, j\}^4$ . Now, what is a good diagrammatic representations of  $\varepsilon_{ijkl}$  and  $\varepsilon^{ijkl}$ ? Each such tensor must be represented by a node with four outgoing, resp. four incoming, arrows. Since the  $\varepsilon$ -tensors are completely antisymmetric, interchanging of two indices (i.e., arrows) induces a sign-switch. In fact, a perfect representation for these tensors would be to embed the diagrams in three-dimensional space and let the four arrows point from the center to the four vertices of a tetrahedron. Rotations of this three-dimensional diagram node would not affect the sign, while a mirror reflection of the node would result in a sign inversion. This is elegant, but we do not want to cook up three-dimensional diagrams here, in order not to overstress the geometry behind the diagrams. We want to keep them as close to usual graphs with

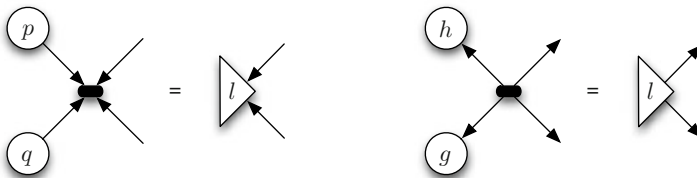
nodes and edges as possible. We will represent a rank-4  $\varepsilon$ -tensor as a two-dimensional graph node with four outgoing (or incoming) arrows. However, we must keep in mind that interchanging two arrows causes a sign-switch. In particular, this means that a cyclic shift of the arrows produces a sign-switch, since this corresponds to an odd permutation. To take care of this effect we draw the  $\varepsilon$ -tensor with a  $180^\circ$  symmetry, having in mind that a  $90^\circ$  rotation produces a sign switch. The following chain of diagrams shows a covariant rank-4  $\varepsilon$ -tensor and illustrates how a chain of three index alternations can be interpreted as a  $90^\circ$  turn that reverses the sign.



Here one has to be very careful what “rotation about  $90^\circ$ ” exactly means. Whenever you have some diagram that involves a rank-4  $\varepsilon$ -tensor, then cutting the  $\varepsilon$ -tensor out (i.e., disconnecting its arrows), rotating it about  $90^\circ$ , and soldering the shifted arrows again into the diagram causes a sign-reversal. Thus the  $i, j, k, l$  in the above diagrams indicate the connections of the  $\varepsilon$ -tensor to the rest of the world. Thus the  $180^\circ$  rotational symmetry of the rank-4  $\varepsilon$ -tensornode represents the identity

$$\varepsilon_{ijkl} = -\varepsilon_{jkli} = \varepsilon_{klij} = -\varepsilon_{lijk}.$$

In many respects the rank-4  $\varepsilon$ -tensor behaves like its rank-3 counterpart. Soldering it to four points calculates the determinant of the corresponding  $4 \times 4$  determinant. Soldering it to three points creates a contravariant tensor that represents the plane through these four points. The antisymmetry of the  $\varepsilon$ -tensor implies that as soon as two of the arrows are connected to the same object, the whole diagram evaluates to the zero tensor. An interesting situation arises when we connect the rank-4  $\varepsilon$ -tensor to just two points. We get a contravariant tensor with exactly two incoming arcs. This tensor resembles a line through the two points. Dually, we can connect the tensors of two planes with a covariant  $\varepsilon$ -tensor. We get a completely covariant tensor that represents the intersection of the two planes. We will now study the most important aspects of these two representations of a line:

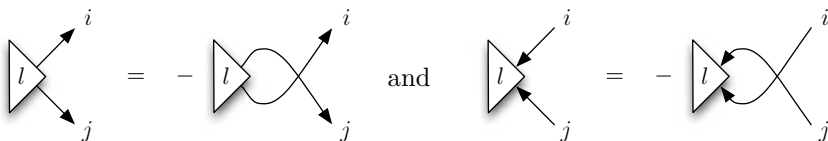


The diagram above shows the two situations whereby a line can be generated. If we want to refer to the line as a single entity, we will represent it by one triangular node that has two outgoing or incoming arrows. We will call them respectively the contravariant, and covariant representations of a line.

Let us first focus on the contravariant representation that arises by connecting two points  $p$  and  $q$  to an  $\varepsilon$ -tensor. The resulting contravariant rank-4 order-2 tensor is a representation of the join of these two points. It is easy to check that up to a scalar factor this tensor does not depend on the specific choice of the points on a fixed line (just use linearity and antisymmetry to prove this property). So as in the case of Plücker coordinates we get a (up to nonzero scalar multiples) unique contravariant representation of a line. This representation is perfect for example, for calculating a join operation. Connecting a point  $r$  to the line node  $l$  generates a contravariant grade-1 rank-4 tensor that represents the plane  $h$  that is the join of the line and the point. Dually, we can use the covariant representation of a line  $l$  to calculate the meet  $p$  of  $l$  with some plane  $f$ . Both operations are represented in the diagrams below:



One important point has to be mentioned here. Since line tensors are obtained by connecting two elements to an  $\varepsilon$ -tensor, they are as well antisymmetric. Interchanging the arrows (i.e., indices) results in a sign-reversal. In diagram notation we have



So far, there is one really unfortunate issue in our system. We have *two* different representations of a line, a covariant one that is good for performing meet operations and a contravariant one that is good for performing join operations. In one of the next sections we will overcome this difficulty by explaining how these two representations can be mutually translated into one another. However, before we do so we have to elaborate on the  $\varepsilon$ - $\delta$ -rule in rank 4.

## 14.6 The $\varepsilon$ - $\delta$ -rule in Rank 4

In Section 13.6 we learned the  $\varepsilon$ - $\delta$ -rule for rank-3 tensors. In usual tensor language it reads as

$$\varepsilon^{ij\alpha} \varepsilon_{lm\alpha} = \delta_l^i \delta_m^j - \delta_m^i \delta_l^j = \det \begin{pmatrix} \delta_l^i & \delta_l^j \\ \delta_m^i & \delta_m^j \end{pmatrix}.$$

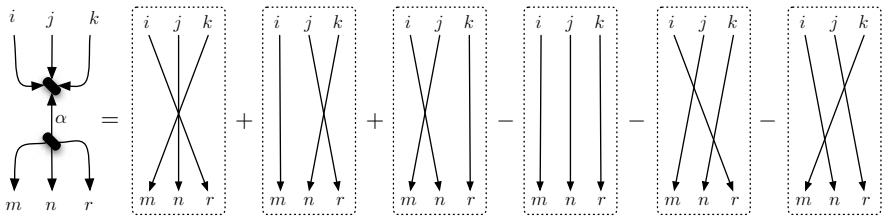
This expression straightforwardly generalizes to higher ranks. In particular, for the rank-4 case we get

$$\varepsilon^{ijk\alpha} \varepsilon_{mnr\alpha} = \det \begin{pmatrix} \delta_m^i & \delta_m^j & \delta_m^k \\ \delta_n^i & \delta_n^j & \delta_n^k \\ \delta_r^i & \delta_r^j & \delta_r^k \end{pmatrix}.$$

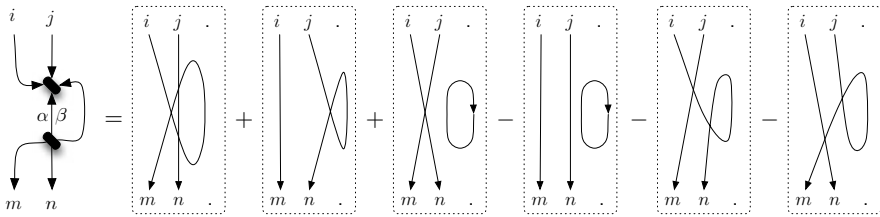
We will not prove this here, since the proof is mainly an index and coefficient battle that can be obtained by brute-force expansion of the implicit summation.

*Remark 14.2.* Although we will not prove this  $\varepsilon$ - $\delta$ -rule here, we will point out the resemblance to the formula we observed in Remark 14.1. While the formula there was essentially the factorization formula for determinants of the product of two matrices, the general  $\varepsilon$ - $\delta$ -rule may be considered an expression of the famous Cauchy-Binet formula. This formula expresses the determinant of an  $n \times m$  and an  $m \times n$  matrix as a summation over products of two determinants of corresponding minors of the two matrices. The  $\varepsilon$ - $\delta$ -rule generates exactly this formula, where the summation is implicitly generated by the summation over the indices shared by both  $\varepsilon$ -tensors.

Diagrammatically the rank-4  $\varepsilon$ - $\delta$ -rule translates to the identity presented below. (As always, one should be aware of the fact that the signs and the orientations of the  $\varepsilon$ -tensors require a careful translation process.)



We will also need another version of the  $\varepsilon$ - $\delta$ -rule that expands the term  $\varepsilon^{ij\beta\alpha} \varepsilon_{mn\beta\alpha}$ . Also, here we could obtain the formula by brute force expansion (we get  $(-2)(\delta_l^i \delta_m^j - \delta_m^i \delta_l^j)$ ). Instead, this time we will show how this formula is derived directly by diagram calculations from the previous one. We simply have to solder the two exits  $k$  and  $r$  in all subdiagrams. We get

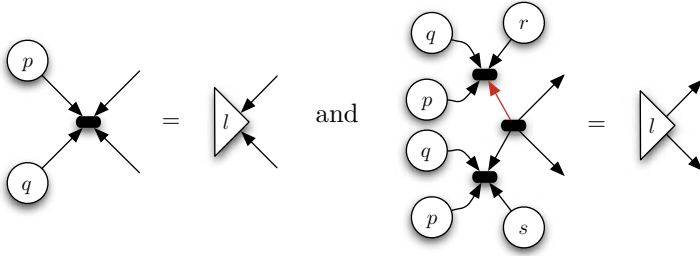


After straightening the arrows, collecting crosswise and parallel connected diagrams (thereby taking into account that a loop is the trace of a unit matrix and evaluates to 4), and rotating one of the  $\varepsilon$ -tensors by  $90^\circ$  (and taking care of the induced sign change), we get the following nice diagram formula (which is up to the factor  $-2$  analogous to the rank-3  $\varepsilon$ - $\delta$ -rule):

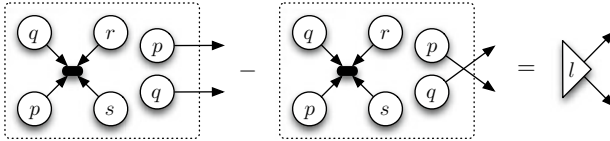
$$\begin{array}{c} i \\ \swarrow \\ \bullet \\ \searrow \\ j \end{array} \begin{array}{c} \bullet \\ \swarrow \\ m \\ \searrow \\ n \end{array} = \begin{array}{c} i \\ \swarrow \\ \bullet \\ \searrow \\ j \end{array} \begin{array}{c} \bullet \\ \swarrow \\ m \\ \searrow \\ n \end{array} = -2 \begin{array}{|c|} \hline \begin{array}{c} i \quad j \\ \swarrow \quad \searrow \\ m \quad n \end{array} \\ \hline \end{array} + 2 \begin{array}{|c|} \hline \begin{array}{c} i \quad j \\ \downarrow \quad \downarrow \\ m \quad n \end{array} \\ \hline \end{array} \quad (14.3)$$

## 14.7 Co- and Contravariant Lines in Rank 4

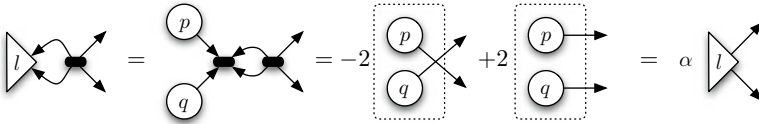
So, why did we introduce  $\varepsilon$ - $\delta$ -rules for rank-4 diagrams? We wanted to understand the relation between covariant and contravariant representations of lines in rank 4. It turns out that the  $\varepsilon$ - $\delta$ -rule is exactly the desired translation tool needed. For this, consider the following situation. Let  $l$  be a line generated as the join of two distinct points  $p$  and  $q$ . Consider the same line as generated as meet of two planes  $h$  and  $g$ . We may consider  $h$  and  $g$  as being spanned by point triples  $(p, q, r)$  and  $(p, q, s)$ , respectively. Here  $r$  and  $s$  are additional points on  $h$  and  $g$  not incident with the line  $l$ . Thus we get the following contra- and covariant representations of the same line:



Applying the rank-4  $\varepsilon$ - $\delta$ -rule to the red arrow in the second representation splits the left side of the equation into six summands. Each of them consists of an  $\varepsilon$ -tensor connected to four points multiplied by the product of two other points. Four of these summands will vanish due to the fact that either  $p$  or  $q$  is connected twice to the  $\varepsilon$ -tensor. The only remaining terms can be rearranged as



The determinantal factor in both summands is identical, so it can be factored out. Hence the covariant form of the line has up to a scalar factor exactly the same shape as the left side of equation (14.3) (connected to  $p$  and  $q$ ). Thus we obtain for a suitably chosen factor  $\alpha$  the following simple conversion formula:



We obtain (up to a scalar factor) the covariant line representation by simply connecting an  $\varepsilon$ -tensor to the contravariant line representation—and dually vice versa. Inspecting this chain of equations also reveals another interesting property. The covariant representation of a line can (up to an unimportant factor) be calculated by summation of two different products of the vectors  $p$  and  $q$  (this is what the last equality says). We can also think of the covariant representation of  $l$  as a  $4 \times 4$  matrix  $L$ . In matrix language the last equation reads  $\alpha L = 2pq^T - 2qp^T$ .

It is an interesting exercise to calculate what happens if we append  $\varepsilon$ -tensors twice to a line tensor. After all, we said that we must, up to a scalar

factor, get a copy of the line tensor we started with. By applying the  $\varepsilon$ - $\delta$ -rule to the double link of the  $\varepsilon$ -tensors we obtain

$$\text{Diagram 1} = -2 \text{Diagram 2} + 2 \text{Diagram 3} = +4 \text{Diagram 4}$$

The last equality is a consequence of the antisymmetry of the  $l$  tensor. Thus transforming a contravariant tensor by appending a chain of two  $\varepsilon$ -tensors results in multiplication by a factor of 4.

## 14.8 Tensors versus Plücker Coordinates

If we compare the representation of lines by tensors with the representation of lines by Plücker coordinates, at first sight there is an amazing structural difference. While the Plücker coordinates of a line  $l$  were six-dimensional vectors  $(g_{12}, g_{13}, g_{14}, g_{23}, g_{24}, g_{34})^T$  that satisfy the Grassmann-Plücker relations

$$g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23} = 0,$$

the tensor representation looks quite different. A  $4 \times 4$  tensor represents essentially a matrix  $M$  with 16 different entries. However, both the covariant and contravariant representations of the line tensor are antisymmetric. If  $M$  is such a matrix we have  $M = -M^T$ . This implies that each of the four diagonal entries vanishes.

Furthermore, the lower left triangle determines the complete shape of the matrix. This makes exactly *six* free entries. As in the case of Plücker coordinates! The matrix  $M$  has the structure

$$M = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix}.$$

We will briefly analyze how these entries relate to the Plücker coordinates of a line. For this we have to go down to the coordinate level once again. Let  $p = (p_1, p_2, p_3, p_4)^T$  and  $q = (q_1, q_2, q_3, q_4)^T$  be two points that span the line  $l$ . The Plücker coordinates of  $l$  then are  $g_{ij} = \begin{vmatrix} p_i & q_i \\ p_j & q_j \end{vmatrix}$ . Let  $l^{ij} = \varepsilon^{\alpha\beta ij} p_\alpha q_\beta$  be the contravariant tensor that represents  $l$  and let  $\tilde{l}_{ij} = p_i q_j - p_j q_i$  be the covariant representation of the line. We then get for specific values of  $i$  and  $j$

$$l^{ij} = \text{sign}(i, j, \alpha, \beta) \begin{vmatrix} p_\alpha & q_\alpha \\ p_\beta & q_\beta \end{vmatrix} \quad \text{and} \quad \tilde{l}_{ij} = \begin{vmatrix} p_i & q_i \\ p_j & q_j \end{vmatrix},$$

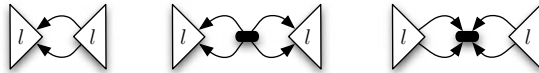
or more comprehensively in matrix notation,

$$l = \begin{pmatrix} 0 & -g_{34} & g_{24} & -g_{23} \\ g_{34} & 0 & -g_{14} & g_{13} \\ -g_{24} & g_{14} & 0 & -g_{12} \\ g_{23} & -g_{13} & g_{12} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{l} = \begin{pmatrix} 0 & -g_{12} & -g_{13} & -g_{14} \\ g_{12} & 0 & -g_{23} & -g_{24} \\ g_{13} & g_{23} & 0 & -g_{34} \\ g_{14} & g_{24} & g_{34} & 0 \end{pmatrix}.$$

The entries of the different line representations are noting but the entries of the Plücker coordinates, suitably arranged. It comes as no surprise that also on the tensor level we have immediate translations for the Grassmann-Plücker relation of the coordinates. We get

$$l^{ij}\tilde{l}_{ij} = 4(g_{12}g_{34} - g_{13}g_{24} + g_{14}g_{23}).$$

In tensor diagram notation we get various equivalent ways to express the Grassmann-Plücker relation on the entries of the line tensor and have several almost trivial ways to prove that this relation vanishes. Here are a few of these representations:



All these diagrams are essentially equivalent, and each of them vanishes identically.

The leftmost diagram is a literal translation of the tensor expression  $l^{ij}\tilde{l}_{ij}$  that we just discovered to be equivalent to the Grassmann-Plücker relation. We simply connect a co- and a contravariant representation of the same line. The middle diagram connects two contravariant expressions via a covariant  $\varepsilon$ -tensor. Up to a factor it is essentially identical to the leftmost diagram, since we can group the  $\varepsilon$ -tensor together with its right neighbor to form a covariant representation of the line. The dual statement holds for the rightmost diagram. To prove the Grassmann-Plücker relation diagrammatically we simply have to verify that one of these diagrams vanishes. We invite the reader to find his/her personal simplest diagrammatic proof for one of these conditions.

We will end our little excursion on diagrams here with a few reading suggestions on diagrammatic approaches to algebra. First traces of a diagrammatic treatment of geometric invariants can be traced back as far as to Clifford and Sylvester [24, 128] in 1878. There are also amazing connections of tensor diagrams to knot theory [62] and to mathematical approaches to quantum (information) theory [23, 123]. In Part III of this book we will make an in-depth study of representing metric properties in a projective language. Also there the diagrammatic approach can be very helpful. We invite



the reader who is (after digesting at least Chapters 16–19) interested in this direction to have a look at the articles [83, 84].



## Configurations, Theorems, and Bracket Expressions

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*Beauty depends on size as well as symmetry.*

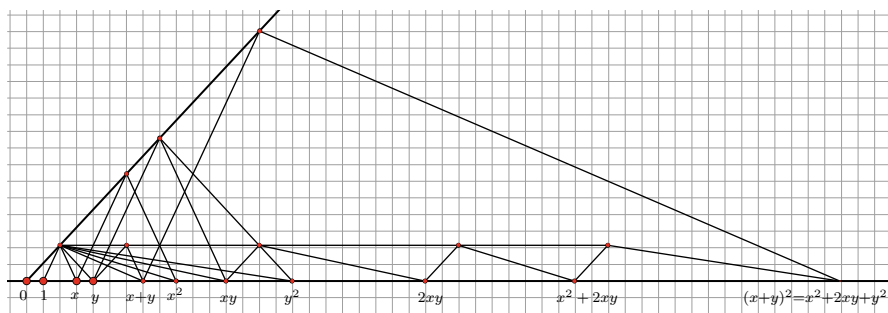
*Aristotle (384–322 BCE), Poetics*

*Characteristic of Weyl was an aesthetic sense which dominated his thinking on all subjects. He once said to me, half-joking, “My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful.” (Hermann Weyl (1885–1955))*

*F. Dyson, in Nature, March 10, 1956*

This section is devoted to several specific examples of theorems and configurations in projective geometry. Clearly, our considerations in Chapter 5 demonstrated that there is an infinite variety of incidence theorems in real projective geometry. Any polynomial identity like  $(x+y)^2 = x^2 + 2 \cdot x \cdot y + y^2$  can be translated into an incidence theorem via von Staudt constructions. For this, one models every elementary addition or multiplication by a suitable subconfiguration. The equality in the equation translates to a final coincidence of two lines that forms the conclusion of the theorem. Figure 15.1 shows a suitable geometric construction for the equation above (lines that appear to be parallel are assumed to be parallel).

Clearly, most of the incidences obtained *in this way* will be very boring, since they represent only more or less trivial facts about algebraic expressions. In fact, the simplest algebraic expressions will surprisingly lead to nicer geometric theorems than the complicated ones. For instance, we saw in Section 5.7 that Pappos’s theorem expresses the equation  $x \cdot y = y \cdot x$ .



**Fig. 15.1** An incidence theorem from  $(x + y)^2 = x^2 + 2 \cdot x \cdot y + y^2$ .

It is unreasonable to try to find a formal notion of *mathematical beauty*, but nevertheless many geometric theorems are considered to be beautiful mathematical objects. Typically there are several (interrelated) ingredients that make a geometric theorem a nice result:

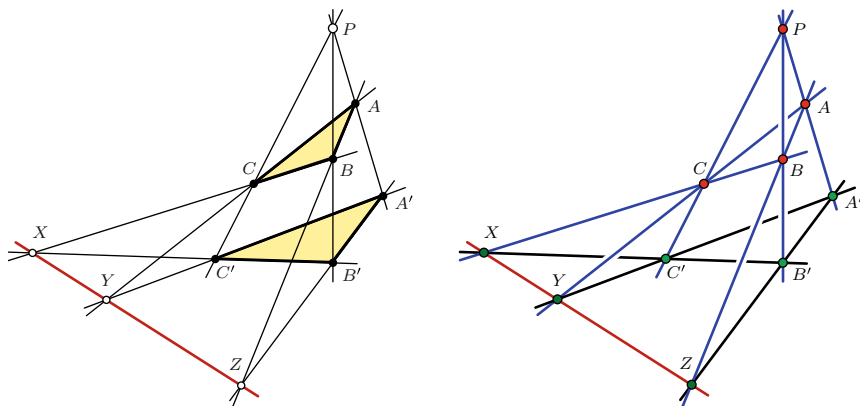
- *simplicity* (it is easy to state the theorem),
- *symmetry* (are there repeated patterns),
- *surprising conclusions* (is the conclusion somehow unexpected),
- *size* (the fewer hypotheses the better),
- *generalizability* (does the theorem extend to a whole class of objects).

In this section we want to explore several theorems and configurations that are expressible in terms of projective geometry. In particular, we will explore relations to bracket expressions and explore their combinatorial symmetries and the symmetries of the corresponding bracket expressions. We will also use this section to demonstrate several techniques that are useful in relating geometric scenarios to algebraic expressions. Several times we will provide different proofs for the same facts in order to demonstrate different approaches. I hope the reader will share the appreciation of the mathematical beauty of many of these structures.<sup>1</sup>

## 15.1 Desargues's Theorem

The first theorem we will meet is an incidence theorem similar to Pappos's theorem. However, in contrast, it requires 10 points and 10 lines. Each point

<sup>1</sup> Some of the proofs presented here already occurred in the introductory Chapter 1, where we explored different approaches to Pappos's theorem. For matters of completeness they are also included in this chapter.



**Fig. 15.2** Desargues's theorem (left). A spatial interpretation (right)

lies on three lines and each line contains three points. The theorem is based on a remarkable symmetric configuration in which the role of the conclusion can be played by any line. Again we will restrict our considerations to real projective planes. In fact, Desargues's theorem holds in any projective plane over a field, but not in arbitrary projective planes.

**Theorem 15.1 (Desargues's theorem).** *In the real projective plane let  $A, B, C$  and  $A', B', C'$  be two triples of points such that  $A \wedge A'$ ,  $B \wedge B'$ ,  $C \wedge C'$  are distinct and meet in a point  $P$ . Then the three points  $(A \wedge B) \vee (A' \wedge B')$ ,  $(B \wedge C) \vee (B' \wedge C')$ , and  $(C \wedge A) \vee (C' \wedge A')$  are collinear.*

The theorem could as well be restated in a more compact manner.

**Theorem 15.2 (Desargues's theorem).** *If two triangles are perspective with respect to a point  $P$ , then the intersections of corresponding triangle sides are collinear.*

There are several approaches to proving Desargues's theorem. One of them makes use of the surprising fact that the underlying configuration can be interpreted as a projection of a spatial configuration. The following proof is based on this observation.

*Proof.* Take the situation of Figure 15.2 on the left as starting point. Since the formulation of the theorem requires only concepts from projective geometry, we may without loss of generality assume that (perhaps after a suitable projective transformation) all points are in a finite position in the drawing. We assume that the position of the points of the configuration is given by  $(x, y)$  coordinates in the drawing plane. We want to interpret this situation as a projection of a spatial configuration. For this we assign altitudes to the points  $P$ ,  $A$ ,  $B$ , and  $C$ . All other points remain in the drawing plane  $H'$

(compare Figure 15.2 (right)). We may interpret the situation as follows: We consider three planes in space which meet in a point  $P$ . Any pair of these planes meet in a line that passes through  $P$ . There are three of these lines. On each line we take a pair of points. We label these pairs  $(A, A')$ ,  $(B, B')$ , and  $(C, C')$ . The three points  $A$ ,  $B$ , and  $C$  span a plane  $H$ , and the three points  $A'$ ,  $B'$ , and  $C'$  span a plane  $H'$  (this is the drawing plane). These two planes meet in a line. The four points  $A, A', B$ , and  $B'$  are on one of the three initial planes. The intersection of this plane,  $H$ , and  $H'$  is the point  $Z$  that also lies on  $\ell$ . Similarly, the points  $X$  and  $Y$  lie on  $\ell$ , which proves the theorem.  $\square$

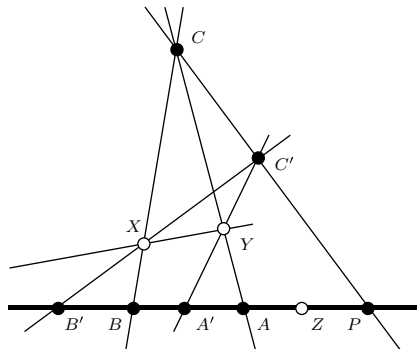
The method of proof used here can be nicely generalized to prove other theorems. Furthermore, it gives a nice insight into the combinatorial structure of the underlying configuration of Desargues's theorem. The spatial situation consists altogether of five planes (the initial three planes together with  $H$  and  $H'$ ), all ten intersections of pairs of these planes (the lines of the configuration), and all ten intersections of triples of these planes (the points of the configuration). The projection of the lines and planes forms the planar Desargues's configuration. This demonstrates that the configuration possesses a very high degree of symmetry. In fact the combinatorial automorphism group has 120 elements and is isomorphic to  $S_5$  the group of all permutations of five elements.

In the next sections we will see also different algebraic proofs of Desargues's theorem that essentially make use of bracket calculations.

## 15.2 Binomial Proofs

The following proof technique is very far-reaching. It translates the hypotheses and the conclusion of the theorem into multihomogeneous bracket polynomials. The conclusion is then algebraically composed from the polynomials of the hypotheses. *Binomial proofs* are now special in the respect that the cancellation patterns between the algebraic terms are particularly simple. Many proofs that can be found in classical literature on projective geometry follow exactly the pattern we present here. The way we will approach binomial proofs here has the advantage that one can even use them to implement automatic provers for geometric theorems. Binomial proofs were first studied in the context of oriented matroids [15] and were later used in the context of automated theorem-proving [30, 109].

The theorems we will consider are (on the first level) those whose hypotheses and conclusions can be expressed as collinearity conditions of points. Furthermore, we will also have to deal with *nondegeneracy conditions* that prevent our configurations from being too special. Algebraically, the nondegeneracy conditions prevent us from dividing by zero. The nondegeneracy conditions will always be such that certain points are not allowed to be



**Fig. 15.3** A degenerate case of Desargues's theorem.

collinear. Most often it will be sufficient to require that no two configuration lines that meet in a configuration point coincide. We will call this the *generic nondegeneracy assumptions*. A version of Desargues's theorem in this setup could be stated as follows.

**Theorem 15.3.** *Let  $P, A, B, C, A', B', C', X, Y, Z$  be ten points in the projective plane. If the triples  $(P, A, A')$ ,  $(P, B, B')$ ,  $(P, C, C')$ ,  $(A, B, Z)$ ,  $(A', B', Z)$ ,  $(B, C, X)$ ,  $(B', C', X)$ ,  $(A, C, Y)$ ,  $(A', C', Y)$  are collinear and the generic non-degeneracy assumptions hold then also  $(X, Y, Z)$  is collinear.*

Figure 15.3 exemplifies the necessity of the nondegeneracy conditions. In this picture, all collinearity conditions of the hypotheses are satisfied, yet the conclusion of the theorem does not hold. The reason for this is that the lines  $(A, A')$  and  $(B, B')$  coincide, and hence the position of point  $Z$  (the intersection of these lines) can be anywhere on this line.

We will now provide a binomial proof based on bracket calculations for Desargues's theorem. First consider one of the collinearities of the theorem, say  $(A, B, Z)$ . Let  $R$  and  $Q$  be two arbitrary points of the projective plane. Via the Grassmann-Plücker relation

$$[Z, A, B][Z, Q, R] - [Z, A, Q][Z, B, R] + [Z, A, R][Z, B, Q] = 0,$$

this collinearity implies the equation

$$[Z, A, Q][Z, B, R] = [Z, A, R][Z, B, Q].$$

We will call such an expression with a bracket monomial on the right and a bracket monomial on the left a *binomial bracket expression*. Now a *binomial proof* consists of a collection of binomial bracket expressions (each one coming from a hypothesis of the theorem) such that multiplying all left-hand sides, multiplying all right-hand sides, and canceling all brackets that occur on

both sides gives a bracket binomial that corresponds to the conclusion of the theorem.

*Proof. Binomial proof for Desargues's theorem.* The following collection of binomials does the job for Desargues's theorem:

$$\begin{array}{ll}
 [CYA] = 0 & \implies [CYB][CAX] = \underline{[CXY]}[CAB] \\
 [ABX] = 0 & \implies [ABC][AXP] = \underline{[ABP]}[AXC] \\
 [PA'A] = 0 & \implies [PA'X][PAB] = \underline{[PA'B]}[PAX] \\
 [PCC'] = 0 & \implies [PCZ][PC'B'] = \underline{[PCB']}[PC'Z] \\
 [C'ZB'] = 0 & \implies [C'ZP][C'B'A'] = \underline{[C'ZA']}[C'B'P] \\
 [A'YC'] = 0 & \implies [A'YB']\underline{[A'C'Z]} = \underline{[A'YZ]}[A'C'B'] \\
 [PBB'] = 0 & \implies [PBA']\underline{[PB'C]} = \underline{[PBC]}[PB'A'] \\
 [CBZ] = 0 & \implies \underline{[CBP]}[CZY] = \underline{[CBY]}[CZP] \\
 [A'B'X] = 0 & \implies [A'B'P]\underline{[A'XY]} = \underline{[A'B'Y]}[A'XP]
 \end{array}$$

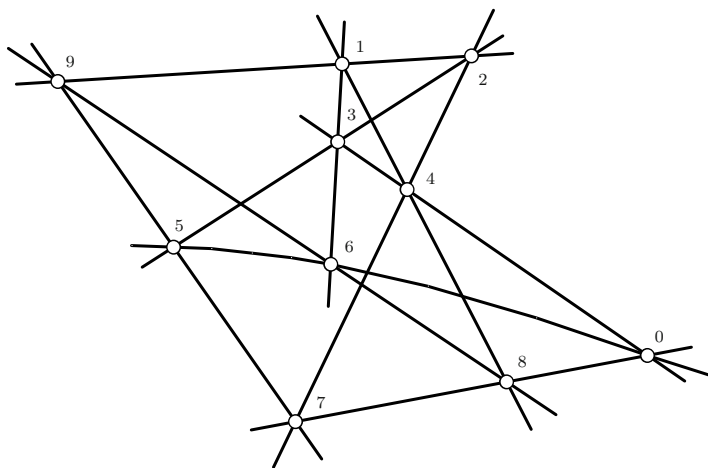
Each row corresponds to one of the hypotheses. The binomial equation is (via a suitable Grassmann-Plücker relation) a consequence of one of the hypotheses. Multiplying all left sides and all right sides gives a binomial equation that has 18 brackets on the left and 18 brackets on the right. Taking into account the alternating determinant rules, one observes that all but four brackets (the underlined ones) occur on both sides. Canceling these brackets, we are left with the expression  $[YXA']\underline{[YZC]} = \underline{[YXC]}[YZA']$ . The cancellation process requires that none of the brackets that is canceled be zero. This, in turn, is established by our nondegeneracy assumptions. Using again a Grassmann-Plücker relation, one can conclude that

$$[YXA']\underline{[YZC]} = \underline{[YXC]}[YZA'] \implies [YXZ] = 0 \text{ or } [YCA'] = 0.$$

The collinearity of  $(Y, C, A')$  would violate our nondegeneracy assumptions. Thus finally the points  $(X, Y, Z)$  must be collinear.  $\square$

In Chapter 1 we presented a similar proof of Pappos's theorem. The crucial point in finding such a proof is the translation of collinearities to binomial expressions. There is a lot of freedom in this translation. First observe that for each triple of collinear points one of the points plays a special role in the Grassmann-Plücker relation. Furthermore, there are two other configuration points that are needed for the Grassmann-Plücker relation. They must be chosen from the remaining points. Thus if the theorem involves  $n$  points, there are altogether  $3 \cdot \binom{n-2}{2}$  possibilities to translate a collinearity into a binomial expression. For each collinearity in the hypotheses one has to find one (or more) suitable representations such that a combination of all binomial expressions can be translated into the conclusion. If a "brute-force" approach were the only way to find such a proof, one would end up with a computationally rather hard problem. Fortunately, the problem can be translated into a linear problem by considering the exponent vectors. In this setup one has to test whether a representation of the conclusion lies in the (linear) span



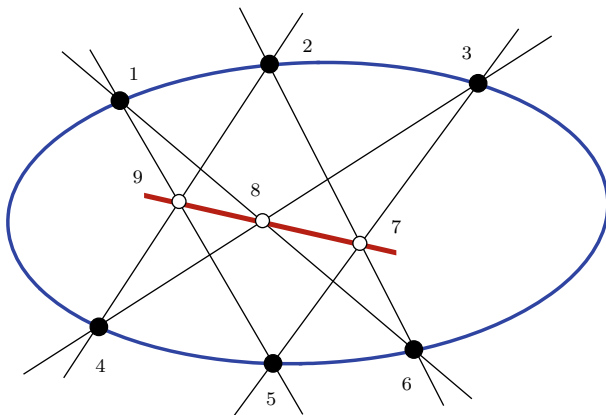


**Fig. 15.4** A nonrealizable  $10_3$  configuration.

of all representations of the hypotheses, which leads to algorithmically very well behaved algorithms [109]. In this approach the determinants themselves are treated as formal symbols (variables), and one has never to go down to the concrete level of coordinates.

As another example of this kind of proof let us consider the configuration shown in Figure 15.4. It shows a certain  $10_3$ -configuration [81] (10 points, 10 lines, and three points on each line) that has the property that it is geometrically not realizable without additional degeneracies (observe that the line (5, 6, 0) is slightly bent). The equations below demonstrate that if all 10 collinearities are satisfied as indicated in the picture, then (by the usual cancellation argument) we can conclude that also  $[157][250] = [150][257]$  holds:

$$\begin{array}{l}
 (129) \Rightarrow [128][179] = +[127][189] \\
 (136) \Rightarrow [146][130] = -[134][160] \\
 (148) \Rightarrow [124][168] = -[128][146] \\
 (235) \Rightarrow [234][250] = -[245][230] \\
 (247) \Rightarrow [127][245] = -[124][257] \\
 (304) \Rightarrow [134][230] = +[130][234] \\
 (056) \Rightarrow [160][570] = +[150][670] \\
 (759) \Rightarrow [157][789] = -[179][578] \\
 (869) \Rightarrow [189][678] = -[168][789] \\
 (780) \Rightarrow [578][670] = +[570][678] \\
 \hline
 [157][250] = +[150][257]
 \end{array}$$



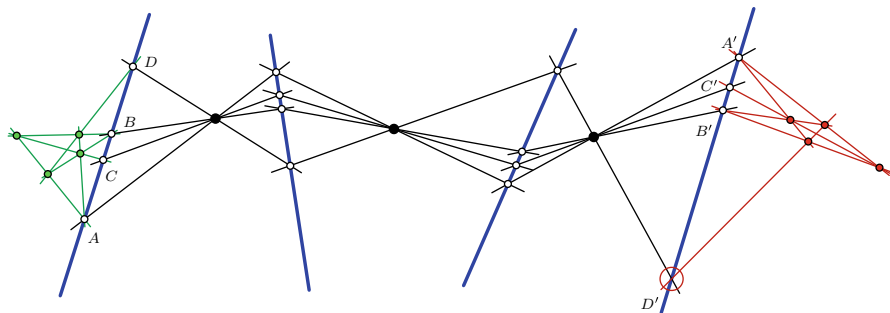
**Fig. 15.5** Pascal's theorem.

This, however, implies that either  $(5, 1, 2)$  or  $(5, 7, 0)$  is collinear. Both cases force a massive degeneration of the configuration.

The above proof technique was successfully applied in [30, 109] to create algorithms that prove many theorems in projective geometry automatically. The algorithm has the particularly nice feature that if it finds a proof, the proof admits a clearly readable structure, such that its correctness can be checked by hand easily. This is an interesting contrast to automatic proof techniques that are based on methods of commutative algebra (such as Gröbner bases or Ritt's algebraic decomposition method (see [22, 76, 134, 135]) that produce proofs consisting of large polynomials of generally high degree.

One might wonder whether the method of binomial proofs applies only to theorems that have collinearity conditions as hypotheses and conclusions. Fortunately, this is not the case. The method has a chance to be applied whenever one succeeds in expressing the geometric statements that are involved as a bracket binomial. As an example we show a proof of Pascal's theorem that takes advantage of the coconicality for the six-points condition we proved in Section 10.2:

$$\begin{array}{ll}
 \text{conic:} & \Rightarrow [125][136][246][345] = [126][135][245][346] \\
 (1, 5, 9) & \Rightarrow [517][592] = [512][597] \\
 (1, 6, 8) & \Rightarrow [612][683] = [613][628] \\
 (2, 4, 9) & \Rightarrow [245][297] = [247][295] \\
 (2, 6, 7) & \Rightarrow [247][286] = [246][287] \\
 (3, 4, 8) & \Rightarrow [346][385] = [345][386] \\
 (3, 5, 7) & \Rightarrow [513][587] = [517][583] \\
 \hline
 (9, 8, 7) & \Leftarrow [728][759] = [729][758]
 \end{array}$$



**Fig. 15.6** Transforming cross-ratio information.

Later, in Chapter 18, we will see how similar proof techniques can be applied also to Euclidean theorems. For this it will be essential also to study certain points with complex coordinates that help to encode Euclidean geometric properties.

## 15.3 Chains and Cycles of Cross-Ratios

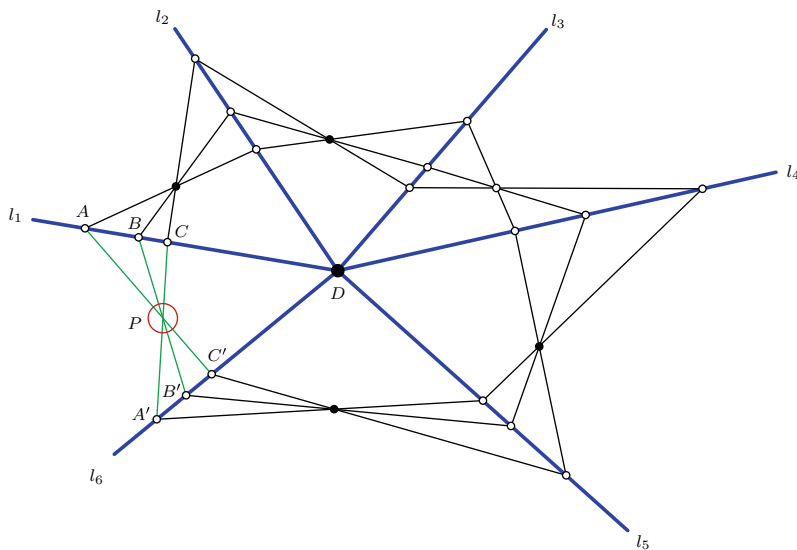
The famous theorems of Pappos, Pascal, and Desargues are more or less a bit solitary or sporadic in the sense that a very special small configuration of points and lines finally closes up to become a theorem. In this section we will deal with structures that generate infinite classes of theorems. Perspectivities—the central projection of points of one line to another line—and cross-ratios will play a crucial role here. Each of these theorems will essentially be based on an iterative application of the fact that the cross-ratio is preserved under a perspectivity (compare Section 4.4.3).

So here is a prototypical application of this fact. Consider the image in Figure 15.6. There a green configuration forces four points  $A, B, C, D$  on a line to be in harmonic position, i.e.,  $(A, B; C, D) = -1$ . A chain of perspectivities “transports” the cross-ratio  $-1$  of these points to the cross-ratio of corresponding points on other lines. Finally, the figure is closed by a red configuration that “tests” whether the final four points are also in harmonic position. The last incidence has to be satisfied automatically by construction.

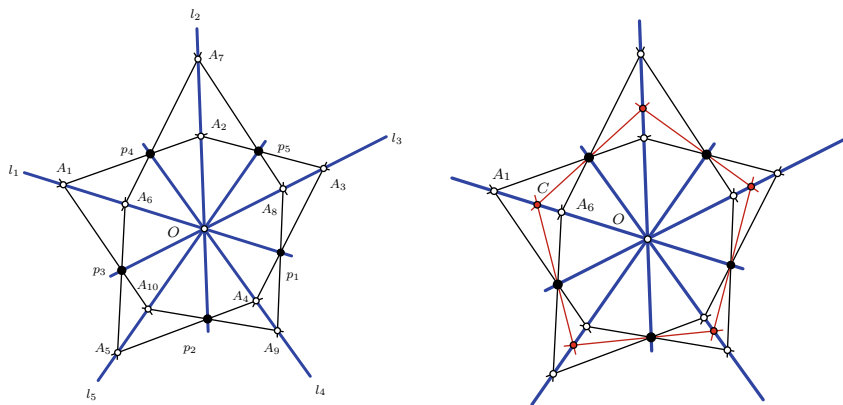
The idea to transport cross-ratios can also be used to build up cyclic structures [119]. A first instance is given in Figure 15.7. There  $n = 6$  blue lines  $l_1, \dots, l_n$  meet in one point  $D$ . On the first line, three points  $A, B, C$  are chosen. They are transferred by a chain of perspectivities to the last line  $l_n$ . Since all lines  $l_1, \dots, l_n$  meet in a point, each of these perspectivities leaves the point  $D$  invariant. Hence the corresponding points  $A', B', C'$  on the

last line must be such that  $(A, B; C, D) = (A', B'; C', D)$ . Now consider the meet  $P$  of the two lines  $\mathbf{join}(A, A')$  and  $\mathbf{join}(C, C')$ . Forming a perspective from  $l_n$  to  $l_1$  through  $P$  maps  $A'$  to  $A$ ,  $B'$  to  $B$ , and  $D$  to  $D$ . Thus (since the cross-ratio of the four corresponding points is identical) it must also map  $C'$  to  $C$ . In other words,  $\mathbf{join}(C', C)$  also passes through  $P$ . Clearly, this construction produces a theorem for any integer  $n$ .

There is another nice variant of this construction. For this consider the left drawing in Figure 15.8. Start with an odd number  $n$  of (blue) lines  $l_1, \dots, l_n$  that meet in one common point  $O$ . On each of these lines  $l_i$  choose a point  $p_i$  (black) that will serve as a projection center of a perspectivity. Now start with a point  $A_1$  on  $l_1$  and  $i = 1$  and form a perspectivity through  $p_{i+(n-1)/2}$  on  $l_{i+(n-1)/2}$ . Call the new point  $A_2$  and proceed with this projection process iteratively by an index shift of 1. After wrapping around *twice*, one ends up at a point  $A_{2n+1}$  that coincides with  $A_1$ . This theorem was originally discovered by Armin Saam [118]. In principle, it is possible also to prove this theorem by a “transfer of cross-ratio” argument. This argument is subdivided into two steps. One of them is technically easy; the other is not. Assume that one moves the point  $A_1$  in the drawing along  $l_1$  toward the center  $O$  of the configuration. Alternatingly even and odd  $A_i$  will move inward and outward with respect to the center  $O$ . In particular, the other point on  $l_1$  (the point  $A_{(n+1)/2}$ ) will move in the opposite direction of  $A_1$ . Eventually, there is a position  $C$  where the moved points  $A_1$  and  $A_{(n+1)/2}$  coincide (this was the



**Fig. 15.7** Cycles of cross-ratios.



**Fig. 15.8** Saam's theorem.

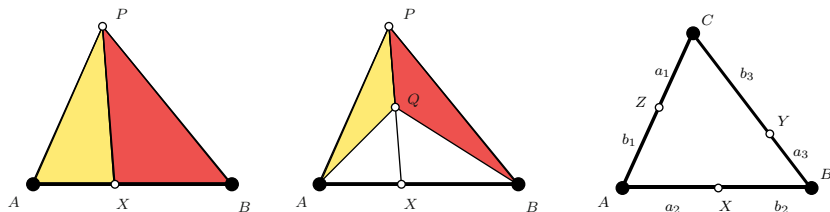
easy part). Now one can observe that the original points  $A_1$  and  $A_{(n+1)/2}$  are in harmonic position to  $C$  and  $O$  (hard part!). The rest of the proof is easily done by “transfer of cross-ratio.” We will not work out this proof in detail. Instead, we will postpone to Section 15.5 a more elegant proof that avoids the hard part and even generates a generalization of the theorem.

The theorems of this section were essentially based on chains or one-dimensional cycles of perspectivities. In what follows we will elaborate on the idea of cyclic structures. However, we will generalize the concept to cycles on two-dimensional manifolds. For this we will first, in the next section, study small configurations that serve as basic building blocks for the manifolds. In Section 15.5 we will show how the building blocks can be glued together to generate interesting theorems.

## 15.4 Ceva and Menelaus

In this section we deal with two structurally very simple theorems: the theorems of Ceva and Menelaus (see [28]). At first sight, they have the flavor of Euclidean, resp. affine, theorems, since they involve ratios (and not cross-ratios) of lengths along the sides of a triangle. However, a slightly closer inspection of these theorems reveals their projective nature. Moreover, on the level of bracket algebra the proofs of these theorems are almost trivial. Nonetheless, these theorems turn out to be of key importance for understanding the structures of other projective and Euclidean incidence theorems [110].<sup>2</sup>

<sup>2</sup> Interesting generalizations of the theorems of Ceva and Menelaus have been intensively studied by Grünbaum and Shephard [51, 52, 53, 54, 120].



**Fig. 15.9** Oriented length ratios and segment cuts in a triangle.

We begin our considerations with the Euclidean version of these theorems. For both theorems the notion of *oriented distance ratio* is crucial. Let  $A, X, B \in \mathbb{R}$  be three points on the real number line. Then the oriented distance ratio  $\frac{|A, X|}{|X, B|}$  is defined as

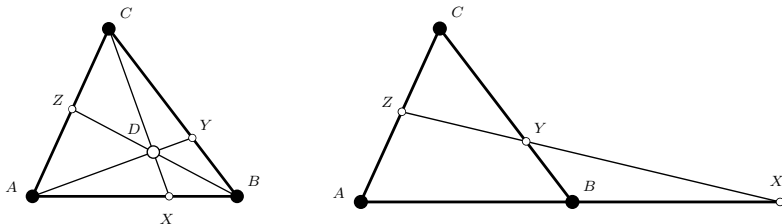
$$\frac{|A, X|}{|X, B|} = \frac{A - X}{X - B}.$$

Similarly, we can define the oriented length ratio for an arbitrary line by considering this line as the real number line  $\mathbb{R}$ . If  $X$  is between  $A$  and  $B$  then this ratio is positive; if  $X$  is outside the line segment  $A, B$  then the ratio is negative. For calculating the oriented distance ratio we can equivalently calculate the ratio of ordinary distances and equip it with the appropriate sign. It is important to notice that the sign of the distance ratio can be assigned in a reasonable way only if the three points that are involved lie on a common line. All cases that we will consider will be of this type.

There is also a way to express the oriented length ratio by quotients of *oriented triangle areas*. For this let  $\Delta(A, B, P)$  be the oriented area of a triangle  $(A, B, P)$ . The sign is positive if the points are in counterclockwise order and negative if they are in clockwise order. Let us assume for a moment that  $A, B, P$  are in the standard embedding with last homogeneous coordinate being 1. Then we have  $\Delta(A, B, P) = \frac{1}{2}[A, B, P]$ . Since the triangle area can be calculated by  $\frac{|A, B| \cdot h_P}{2}$ , where  $h_P$  is the altitude of  $P$  over the side  $(A, B)$ , we get for the oriented distance ratios

$$\frac{|A, X|}{|X, B|} = \frac{\Delta(A, X, P)}{\Delta(X, B, P)} = \frac{[A, X, P]}{[X, B, P]}. \quad (15.1)$$

Here  $P$  is an arbitrary point not on the line  $(A, B)$  (compare Figure 15.9 (left)). Notice that the sign turns out to be automatically correct. We will also need another expression for the oriented distance ratios. For this assume that  $Q$  is another point on the line  $(X, P)$  distinct from  $P$ , again in the standard embedding. We then have  $X = (1 - \lambda)P + \lambda Q$ , and obtain



$$\text{Ceva's Thm: } \frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1 \quad \text{Menelaus's Thm: } \frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = -1$$

**Fig. 15.10** Theorems of Ceva and Menelaus.

$$\frac{|A, X|}{|X, B|} = \frac{|A, X, P|}{|X, B, P|} = \frac{|A, (1 - \lambda)P + \lambda Q, P|}{|(1 - \lambda)P + \lambda Q, B, P|} = \frac{|A, Q, P|}{|Q, B, P|}. \quad (15.2)$$

We will need (15.1) and (15.2) later.

The theorems of Ceva and Menelaus now deal with triangles  $(A, B, C)$  for which each of the sides is dissected by one of the points  $X, Y, Z$ . In what follows we will always assume that  $X$  is the cutting point for the segment  $(A, B)$ , that  $Y$  is the cutting point for  $(B, C)$ , and that  $Z$  is the cutting point for  $(C, A)$ . We also assume that none of the points involved coincides with another (compare Figure 15.9 (right)). The product

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} \quad (15.3)$$

is now the first-class citizen in the theorems of Ceva and Menelaus; they are illustrated in Figure 15.10.

**Theorem 15.4 (Ceva's theorem).** *If in a triangle  $(A, B, C)$ , with cutting points  $X, Y, Z$  as above, the lines  $(A, Y)$ ,  $(B, Z)$ , and  $(C, X)$  are concurrent, then we have*

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1.$$

**Theorem 15.5 (Menelaus's theorem).** *If in a triangle  $(A, B, C)$ , with cutting points  $X, Y, Z$  as above, the points  $X, Y, Z$  are collinear, then we have*

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = -1.$$

Before presenting proofs of these two theorems we will first explore their *projective* nature. For this consider the following bracket formula on the points  $A, B, C, X, Y, Z$  and one additional point  $P$  in generic position.

$$\frac{[PAX]}{[PXB]} \cdot \frac{[PBY]}{[PYC]} \cdot \frac{[PCZ]}{[PZA]}. \quad (15.4)$$

By our considerations in Chapter 6 this is clearly a projective invariant function, since it is the quotient of two multihomogeneous bracket polynomials with same multidegree. On the other hand, we may assume that each of the points is represented in standard homogeneous coordinates with the last coordinate entry being 1. Then the brackets correspond to oriented triangle areas and we get by applying 15.1,

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = \frac{[PAX]}{[PXB]} \cdot \frac{[PBY]}{[PYC]} \cdot \frac{[PCZ]}{[PZA]}.$$

In other words, the above expression (15.3) looks as if it were a Euclidean expression (since it involves quotients of segment lengths) but is indeed a projective invariant. Applying a projective transformation does not alter the value of (15.3). If we deal only with the bracket expression, we may even drop the assumption that the points are in the standard embedding. The theorems of Ceva and Menelaus deal with special cases of the values of the expression (15.3): the cases  $+1$  and  $-1$ .

We will give two different proofs of each of the two theorems. The first proof for the two theorems will take direct advantage of the fact that (15.4) is a *projective* invariant. The second proof will present an explicit bracket cancellation pattern. We refer to Figure 15.10 for the labeling.

*First proof of Ceva's theorem:* The observation that the product of the ratios is a projective invariant gives an almost trivial proof of Ceva's theorem. Since by a projective transformation four points in general position can be mapped to four other arbitrary points in general position, we can freely adjust the positions of  $A, B, C, D$  in the drawing. Thus it is sufficient to verify Ceva's theorem just for one special case. By far the simplest situation is an equilateral triangle with sides cut by the symmetry axes. There the three oriented distance ratios are all equal to 1 and their product is 1 as well.  $\square$

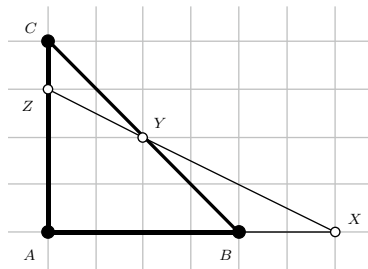
*First proof of Menelaus's theorem:* A proof in the same spirit can be given for Menelaus's theorem. A projective transformation carries exactly enough freedom to adjust the three corners of the triangles and the secant line. Thus verifying one example suffices to prove the theorem. A suitable example is given in Figure 15.11. There the calculation is

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = \frac{6}{-2} \cdot \frac{2}{2} \cdot \frac{1}{3} = -1.$$

$\square$

*Second proof of Ceva's theorem:* Our second proof of Ceva's theorem relies on the equality (15.2). In order to prove Ceva's theorem we consider the obvious





**Fig. 15.11** Proof by example.

identity

$$\frac{[ACD]}{[CBD]} \cdot \frac{[BAD]}{[ACD]} \cdot \frac{[CBD]}{[BAD]} = 1,$$

(note that each determinant in the denominator occurs as well in the numerator). This expression again is a projective invariant. Applying (15.2) we get.

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = \left( \frac{[ACD]}{[CBD]} \right) \cdot \left( \frac{[BAD]}{[ACD]} \right) \cdot \left( \frac{[CBD]}{[BAD]} \right) = 1$$

which proves Ceva's theorem.  $\square$

*Second proof of Menelaus's theorem:* Similarly, a proof of Menelaus's theorem is derived. For this consider the special line as being generated by two points  $D$  and  $E$ . We have

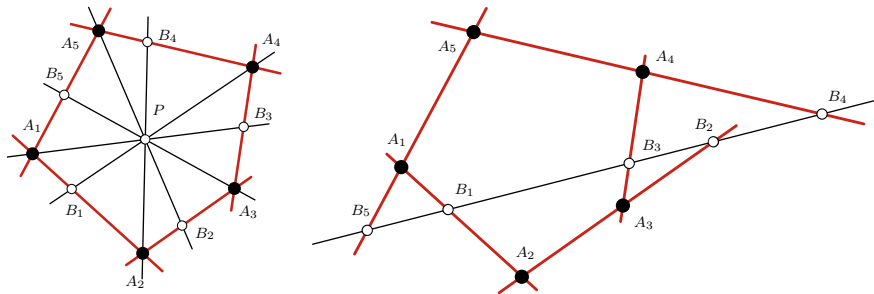
$$\frac{[ADE]}{[BDE]} \cdot \frac{[BDE]}{[CDE]} \cdot \frac{[CDE]}{[ADE]} = 1.$$

Applying the identity (15.2) and taking into account the alternating determinant rules, we get

$$\begin{aligned} \frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} &= \left( \frac{[ADE]}{[DBE]} \right) \cdot \left( \frac{[BDE]}{[DCE]} \right) \cdot \left( \frac{[CDE]}{[DAE]} \right) \\ &= \left( -\frac{[ADE]}{[BDE]} \right) \cdot \left( -\frac{[BDE]}{[CDE]} \right) \cdot \left( -\frac{[CDE]}{[ADE]} \right) \\ &= -1, \end{aligned}$$

which yields Menelaus's theorem.  $\square$

There are also interesting generalizations of the theorems of Ceva and Menelaus to larger cyclic products of oriented distance ratios. The



**Fig. 15.12** Hoehn's theorem and the generalized Menelaus theorem for  $n = 5$ .

generalization of Ceva's theorem is known as Hoehn's theorem (see Figure 15.12, left):

**Theorem 15.6 (Hoehn's theorem).** *Let  $n$  be an odd number and let  $A_1, \dots, A_n$  and  $P$  be  $n + 1$  points in the plane such that  $P$  is distinct from the lines  $(A_i, A_{i+1})$ , indices modulo  $n$ . Then let the points  $B_i$  be the intersection of the line  $(A_i, A_{i+1})$  with the line  $(P, A_{i+(n+3)/2})$ , for  $i = 1, \dots, n$  with indices modulo  $n$ . Then we get*

$$\prod_{i=1}^n \frac{|A_i, B_i|}{|B_i, A_{i+1}|} = 1.$$

*Proof.* The proof is analogous to the bracket proof of Ceva's theorem. We count indices modulo  $n$  and apply (15.2) in the standard embedding. We get with  $k = (n + 3)/2$ ,

$$\frac{|A_i, B_i|}{|B_i, A_{i+1}|} = \frac{[A_i, A_{i+k}, P]}{[A_{i+k}, A_{i+1}P]},$$

and hence

$$\prod_{i=1}^n \frac{|A_i, B_i|}{|B_i, A_{i+1}|} = \prod_{i=1}^n \frac{[A_i, A_{i+k}, P]}{[A_{i+k}, A_{i+1}P]}.$$

In the last expression (since  $k$  was chosen appropriately) each term in the denominator occurs also in the numerator. Thus the whole expression must be equal to 1.  $\square$

The reader is invited to translate our investigations of Section 8.3 to derive a corresponding generalization of Menelaus's theorem. An appropriate drawing is given in Figure 15.12 on the right.

## 15.5 Gluing Ceva and Menelaus Configurations

*The World Serpent was an enormous snake which wrapped around the world and bit his own tail.*

*The legend of Thor*

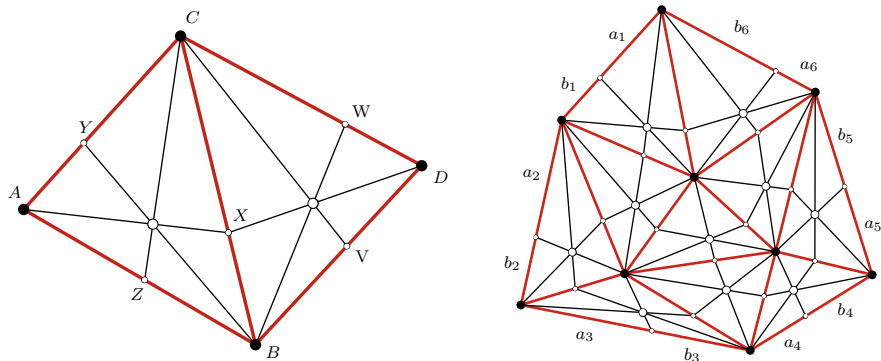
The reason why we spend so much time on the theorems of Ceva and Menelaus is that according to their simple structure they are ideal basic building blocks for more complicated statements. What is missing so far is a kind of *glue* to link several copies of Ceva's or Menelaus's configurations. Indeed, gluing them turns out to be an extremely natural (and powerful) process [110]. For this consider two different Ceva configurations that are identified along one edge such that the special point on the edge is shared by both configurations. Before we do so, we fix several naming conventions. When in the future we talk about the "Cevaexpression" (or "Menelausexpression") for the triangle  $A, B, C$ , the letters  $A, B$ , and  $C$  are assumed to be ordered as in (15.3). The ordering  $A, C, B$  would generate the reciprocal expression. Furthermore, we will call the points  $X, Y$ , and  $Z$  the *edge points* of the configuration. The points  $A, B$ , and  $C$  will be called the *vertices* of the configuration. Point  $D$  in a Ceva configuration will be called the *Ceva point*, and the secant line in a Menelaus configuration is the *Menelaus line*.

Now consider the situation in which two triangles that are equipped with a Ceva configuration share an edge and the corresponding edge point on this edge (see the Figure 15.13). The triangle  $A, B, C$  yields a relation  $\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1$ , while the triangle  $C, B, D$  yields  $\frac{|CX|}{|XB|} \cdot \frac{|BV|}{|VD|} \cdot \frac{|DW|}{|YW|} = 1$ . The quotient  $\frac{|BX|}{|XC|}$  occurs in the first expression, and its reciprocal occurs in the second expression. If we multiply both expressions, this quotient cancels and we are left only with terms that live on the boundary of the figure. We obtain

$$\frac{|AZ|}{|ZB|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|BV|}{|VD|} \cdot \frac{|DW|}{|YW|} = 1.$$

We now consider a triangulated topological disk. All triangles of the triangulation should be equipped with Ceva configurations that have the additional property that points on interior edges are the shared edge points of the two adjacent triangles. We consider the product of all corresponding Cevaexpressions.

If the triangles are oriented consistently (adjacent triangles use the common edge in opposite directions), all quotients related to inner edges will cancel. We are left with an expression that depends only on the position of



**Fig. 15.13** Pasting copies of Ceva's theorem.

the boundary points (including the edge points along the boundary edges). If in Figure 15.13 on the right the letters  $a_1, b_1, \dots, a_6, b_6$  correspond to the oriented lengths around the boundary, we can conclude immediately that we must have

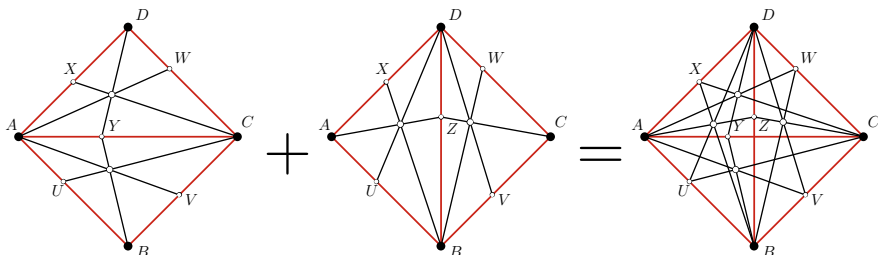
$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} = 1.$$

Now consider any triangulated manifold that forms an oriented 2-cycle. This cycle serves as a kind of *frame* for the construction of an incidence theorem. It is important to mention in what category we understand the term “triangulated manifold.” We consider compact, orientable 2-manifolds without boundary and subdivisions by CW-complexes whose faces are triangles. For practical purposes this means that we have a collection of oriented triangles glued along edges such that on a combinatorial level every edge is shared exactly by two triangles in opposite directions. We do not care about coincidence or overlapping of the triangles. So in principle, a subdivision of a 2-sphere by two topological triangles, which are identified along the edges, would be a feasible object for our considerations.

Consider such a cycle as being realized by flat triangles (it does not matter whether these triangles intersect, coincide, or are coplanar as long as they represent the combinatorial structure of the cycle). By the above argument the presence of Ceva configurations on all but one of the faces will imply automatically the existence of a Ceva configuration on the final face. Thus at the final face the three lines connecting the edge points and the vertices will meet automatically, and we have an incidence theorem. In what follows we will study several concrete examples of this amazingly rich construction technique.

As a first example take the projection of a tetrahedron ( $ABCD$ ) to  $\mathbb{R}^2$ . Now choose points  $U, V, W, X, Y, Z$ , one on each of the edges of the

tetrahedron. Assume that for three of the faces these points form a Ceva configuration. Then they automatically form a Ceva configuration on the last face: an incidence theorem. Here is a graphical representation, in which the tetrahedron has been decomposed into a front and a back part—each carrying two Ceva triangles:



Although the proof of this incidence theorem is already evident by our above homotopy argument, we still want to present the algebraic cancellation pattern in detail. Consider the following formula:

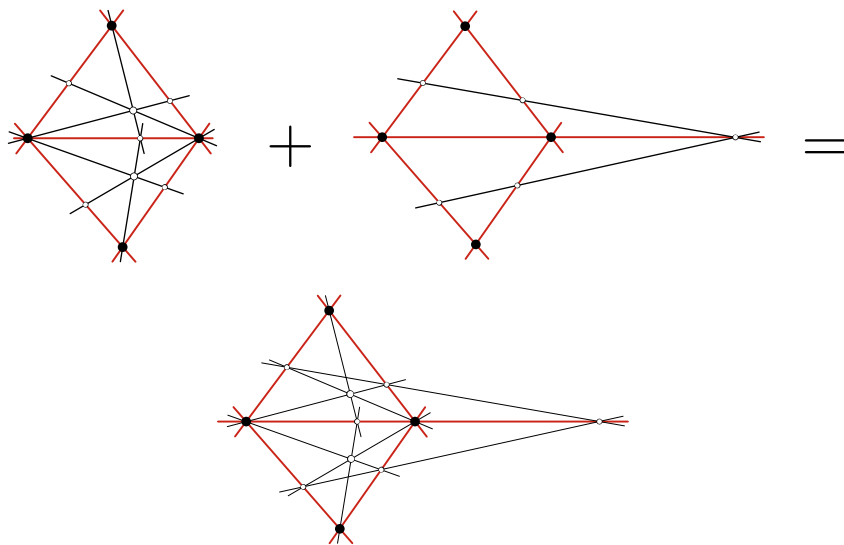
$$\left( \frac{|AU|}{|UB|} \cdot \frac{|BV|}{|VC|} \cdot \frac{|CY|}{|YA|} \right) \cdot \left( \frac{|CW|}{|WD|} \cdot \frac{|DX|}{|XA|} \cdot \frac{|AY|}{|YC|} \right) \cdot \left( \frac{|AX|}{|XD|} \cdot \frac{|DZ|}{|ZB|} \cdot \frac{|BU|}{|UA|} \right) \cdot \left( \frac{|BZ|}{|ZD|} \cdot \frac{|DW|}{|WC|} \cdot \frac{|CV|}{|VB|} \right) = 1.$$

This formula is obviously true, since all lengths of the numerator occur in the denominator as well and vice versa (this property is inherited from the cyclic structure). On the other hand, each of the factors in brackets being 1 states the Ceva condition for one of the faces. Thus three of these conditions imply the last one. The essential fact that makes this proof work is that whenever two faces meet in an edge, the two corresponding ratios cancel. In general we obtain (see [110]) the following

*For any triangulated oriented 2-CW-cycle choose a point on each edge such that for every face either a Ceva or a Menelaus condition is generated. If altogether an even number of Menelaus configurations is involved, then the conditions on all but one of the triangles automatically imply the condition on the last triangle.*

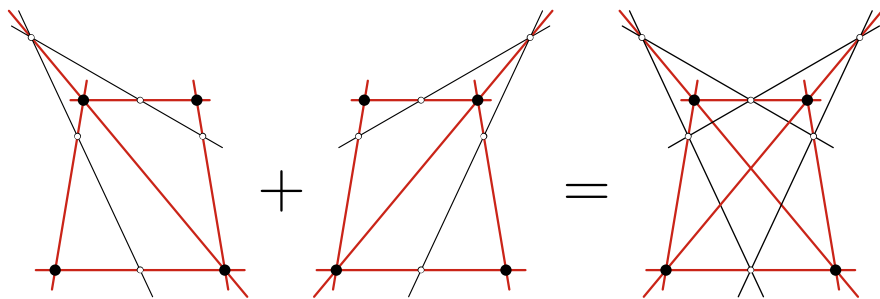
We need an even number of Menelaus configurations, since each Menelaus configuration accounts for a factor of  $-1$  in the product. We will not exploit the full power of this method. Rather than that, we will restrict ourselves to a few illuminating examples.

As a second example we consider again four triangles. Two adjacent triangles should carry Ceva configurations, and two other copies of the same (!) triangles should carry Menelaus configurations. We identify the two sub-configurations along the four boundary edges and make sure that the edge points coincide. The following figure illustrates the situation:

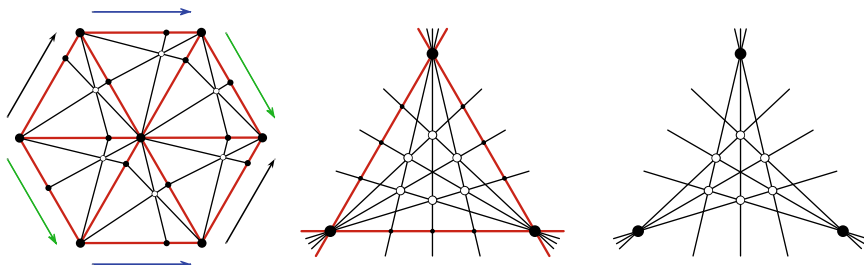


By our considerations above it is clear that the final coincidence in this configuration has to be satisfied automatically. We again have an incidence theorem. This incidence theorem has a nice projective interpretation. The upper and the lower parts of the picture correspond to the construction of a harmonic point that we met in Figure 5.1. The incidence theorem shows that the construction is independent of the concrete choice of the auxiliary construction points.

As a next example we consider again a tetrahedral structure, but now we equip each of its triangular faces with a Menelaus configuration. The following picture shows the resulting configuration:



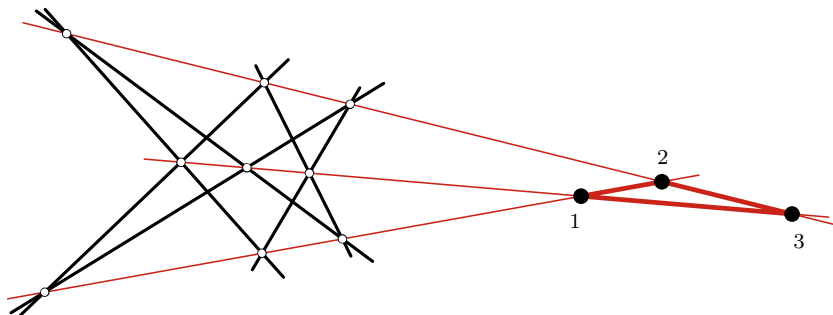
This is again an incidence theorem. It consists of ten points and ten lines and is nothing but Desargues's theorem, which we met in Section 15.1.



**Fig. 15.14** A Cevaproof of Pappos's theorem.

How about Pappos's theorem? The amazing fact about Ceva/Menelaus proofs of Pappos's Theorem is that in that case the topological genus of the underlying cycle must be more complicated. In fact, the proof of Pappos's theorem "lives on a torus." For this consider six triangles that are glued such that they form a hexagon where the six triangles meet at the center (see Figure 15.14, left). In this figure we now identify opposite boundary edges of the hexagon. The resulting manifold has the topological type of a torus. In fact, the only way to "realize" this framework is to place the triangles such that they all share the same three vertices, and these coincide. Now on each of the triangles we place a Ceva configuration and make sure that the edge points of identified edges coincide. A corresponding drawing is shown in Figure 15.14 in the middle. The drawing consists of 18 points (three vertices, six Ceva points, and nine edge points) and 12 lines (the three triangle lines and the nine interior lines). None of the edge points contribute to the incidence theorem, since they turn out to be the intersection of just two lines. Removing them, we see that also the triangle sides do not contribute to the incidence theorem. Removing them as well, we are left with nine points and nine lines and an incidence theorem on them: Pappos's theorem!

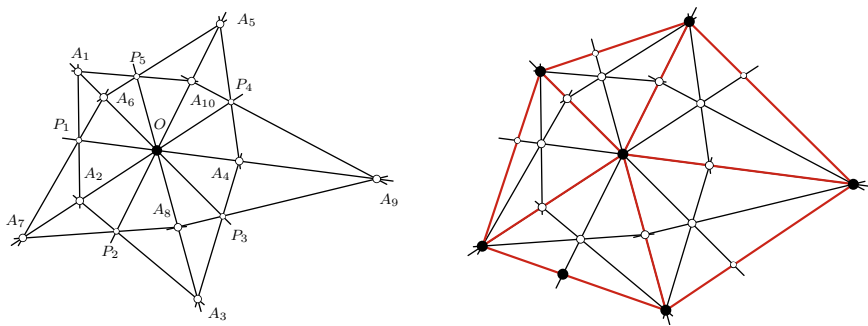
It is a really surprising fact that also in the case in which we assign to all six triangles a Menelaus configuration we get a drawing of Pappos's theorem. Figure 15.15 shows a drawing of this configuration. The original triangle of the manifold is the dark triangle in the right of the drawing. The additional six lines in the drawing are the six Menelaus lines that cut all three edges of the triangles. The nine points on the left of the configuration correspond to the nine edges of the triangles in our torus. Similar to the fact that in the last proof the edges of the triangles could be neglected, this time, the three vertices are superfluous. A proof along these lines was first given in [28]. It is amazing that this kind of proof technique seems in the case of Pappos's theorem to be intimately related to a torus. The combinatorial symmetry of Pappos's theorem is also intimately related to a torus. Its incidence graph (a graph in which the nodes are the points and lines and the edges are the



**Fig. 15.15** A Menelausproof of Pappos's theorem.

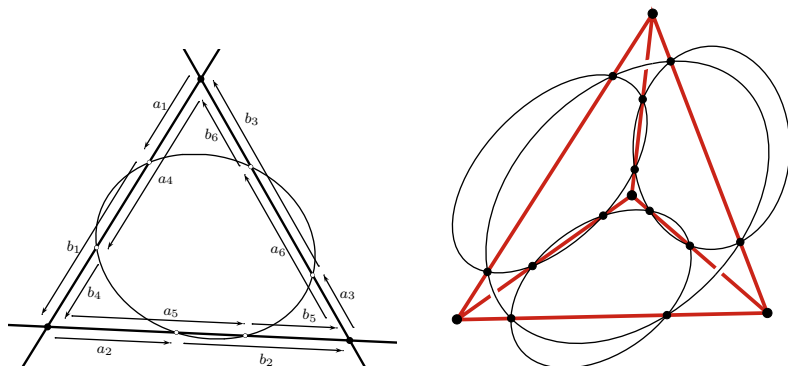
incidences of a configuration) can be embedded without intersections in a torus, but not in a sphere (see [27] and [111]).

As a final example we return to Saam's theorem, which we encountered in Section 15.3 and whose proof was still left open. We get the proof as an application of Hœhn's Theorem as a basic building block in our gluing technique. Figure 15.16 once more gives the configuration that underlies Saam's theorem. We recall that for this theorem one starts with a central point and an odd number of lines passing through it. On each of these lines one chooses a projection point  $P_i$ . Then one starts with a point  $A_1$  (compare to the picture) and projects this point through  $P_1$  onto the next line. If one continues projecting, then after cycling around twice one again reaches the initial point  $A_1$ . The picture on the right shows a way to prove this theorem by a cycle construction. One forms an  $n$ -gon by the points  $A_i$  with odd  $i$  and forms a pyramid over this  $n$ -gon. On all triangular faces one imposes



**Fig. 15.16** Saam's theorem and its underlying cycle.





**Fig. 15.17** Carnot's theorem and a cycle theorem involving conics.

a Ceva configuration, and on the  $n$ -gon one imposes a Hohn configuration. The usual cancellation procedure proves the theorem. It also becomes obvious that Saam's theorem is only a special case of a more general theorem that has the same underlying manifold proof, but in which the apex of the pyramid does not coincide with the centerpoint of Hohn's configuration.

## 15.6 Furthermore ...

We have only just touched the surface. There are still many interesting issues that one could explore in the context of brackets, cycles, theorems, and proofs. For instance it can be proved that binomial proofs and Ceva/Menelaus cycle proofs can be converted into each other (the translation is not trivial) [1, 110]. There are generalizations to higher dimensions; symmetry considerations; relations to liftings; cycle proofs can also be applied to nonlinear situations; finding cycle or binomial proofs can be automated to implement automatic proving machines, etc. It is a striking fact that the rather abstract series of papers by Dress and Wenzel (see [34, 35, 36, 87, 130]) centered on the topic of *Tutte groups of a matroid* has a very close relation to the Ceva/Menelaus setup presented in this chapter.

As a final example we will give a glimpse of the application of cycle proofs for configurations that involve conics. Figure 15.17 (left) shows a theorem of Carnot, which can be considered a generalization of Ceva's theorem. In Carnot's theorem we consider a triangle with exactly two (distinct) edge points per edge. We assume that the edge points are labeled  $1, \dots, 6$  and that the corresponding length ratios are  $a_i/b_i$ . Carnot's theorem states that we have

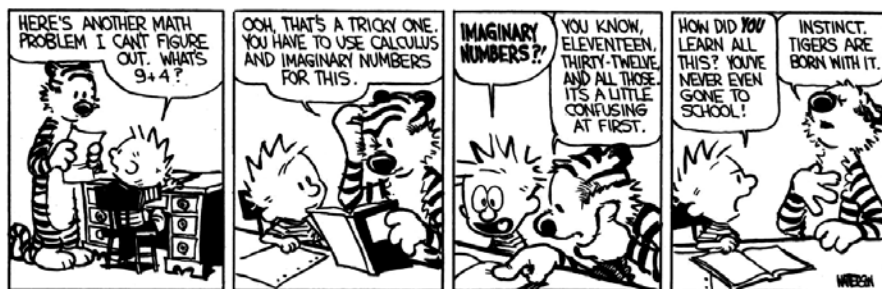
$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} = 1$$

if and only if the six edge points lie on a common conic. We can immediately use Carnot's configuration as a "primitive" to build theorems that also involve conics. In Figure 15.17 on the right we present a small theorem that involves only Carnotfaces: If one has a tetrahedron with two distinct points on each edge and if the six edge points of three faces are coconical, then they will be coconical for the last face automatically. In fact, there is nothing special about the tetrahedron. Any oriented triangulated 2-manifold would serve as a frame as well.

# Part III

## Measurements





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*Bill Watterson, Calvin and Hobbes, 1988*

Perhaps the most annoying thing about our current setup for projective geometry is the fact that concepts that one usually immediately associates with geometry (such as circles, distances, and angles) do not have an immediate correspondence. We obtained a very beautiful algebraic system for dealing with geometry, but we paid a high price. Many interesting and nice results from geometry cannot even be expressed by our language developed so far. Essentially most concepts that belong to Euclidean geometry were abandoned from our setup. We could talk about conic sections, but not about circles. We could measure cross-ratios but not distances; we could intersect arbitrary pairs of lines, but we lost the notion of angles.

Thus it is, for instance, impossible to formulate a simple statement like “*The altitudes in a triangle meet in a point*”, since we do not have a concept of perpendicularity. We also do not have any equivalents to *Thales’ theorem* or the *Pythagorean theorem*, since we cannot speak about circles and right triangles. It would seem then that we have developed a beautiful algebraic system at the price of giving up Euclidean geometry!

Here comes the good news: This is not true at all—there is a beautiful system that allows us to express all Euclidean concepts and magnitudes in a purely projective framework. Thus we can perform all nice operations of projective geometry and still retain the expressive power and richness of Euclidean geometry. Things are even better: if we properly understand how to incorporate Euclidean geometry into projective geometry we will see that Euclidean geometry is just a special case of several other ways to embed other geometries. If we then play with the parameters, we will see that there is a whole variety of other geometries expressible in a similar (and algebraically smooth) manner. Euclidean geometry will just be a special case of a much richer system. The larger system will contain prominent inhabitants such as *hyperbolic geometry* and *relativistic space-time geometry*.

It is the purpose of this part of the book to develop this general theory. As before, we will pay special attention to the question of how geometric primitive operations can be carried out in the most general and at the same time the simplest way. The key to everything will be the proper use of *complex numbers*.

---

## Complex Numbers: A Primer

*The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity.*

*Gottfried Wilhelm Leibniz, 1702*

So far, almost all our considerations have dealt with *real* projective geometry. The main reason for this is that we wanted to stay with all our considerations as concrete and close to imagination as possible. Nevertheless, almost all considerations we have made so far carry over in a straightforward way to other underlying fields. Only in very rare cases are small twists necessary, and if so, they result from one of the following two facts:

- Depending on the field, certain equations may be solvable or not.
- Other fields may have other field automorphisms.

We now will be interested in particular in projective geometry over the *complex* numbers. Since the complex numbers are algebraically closed (every polynomial equation is solvable over the complex numbers), objects will have intersections in general. For instance, over the complex numbers there is always an intersection of a conic and a line, in contrast to the real case in which we may have situations in which the two objects do not intersect. This is the first benefit we will get from the use of complex numbers. They will help in our efforts to exclude special cases.

Compared to real numbers, the complex numbers have more field automorphisms. Besides the identity (the only automorphism of the real numbers), there is also the automorphism that sends  $x + iy$  to  $x - iy$ .

automorphism of a field. This automorphism corresponds to complex conjugation.<sup>1</sup> Considering this fact under the light of the fundamental theorem of projective geometry (compare Section 5.4), we see that over the complex numbers there will be harmonic maps and collineations that do not correspond to a matrix multiplication but come from complex conjugation. We will have to deal with this difference later.

This section is meant as a very brief introduction to complex numbers. Here we will highlight all ingredients we will need later. Readers who already feel very familiar with complex numbers can skip this chapter with no harm.

## 16.1 Historical Background

The historical roots of complex numbers are closely related to the task of finding solutions to polynomial equations [137]. It is an interesting fact that complex numbers were first discovered in the context of cubic rather than quadratic equations. Roughly, the story goes as follows:

quadratic equation

equation!quadratic In antiquity it was known how to solve quadratic equations  $x^2 + px + q = 0$ . The solution is given by the well-known formula

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

It was clear that there are instances in which no (real) solution would be possible due to the fact that the square of a (real) number is always positive. So for instance, the equation  $x^2 = -1$  was considered unsolvable.

Since then it remained a major open question how to generalize the solution of the quadratic equation to the next more complicated case, the cubic equation.

cubic equation

equation!cubic In fact, it took several hundred years until a solution was found (in the sixteenth century), and one can say that the developments triggered by the discovery of this formula form the initial point of what one would consider “modern mathematics.” The history around the discovery of the solution (we already used a variant of it in Section 11.4 when we intersected two conics) is exciting, full of personal tragedy, amusing—a prototype of a mathematical “crime story” on priority disputes. Again, since this is not a book on the history of mathematics we will only briefly outline the basic plot and refer to history books for details [48, 137]. We also recommend the novel [61]. Briefly, Scipione del Ferro was essentially the first to solve (a special case of) the general cubic equation  $x^3 + ax^2 + bx + c = 0$  around 1515.

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<sup>1</sup> These are the only continuous automorphisms of  $\mathbb{C}$ . Indeed, in the presence of the axiom of choice there are also uncountably many *wild automorphisms* of  $\mathbb{C}$ .



Unfortunately, he did not publish his result during his lifetime (he died in 1526). However, he told the solution to his scholar and relative Anton Maria Fior (a more or less mediocre mathematician). After Scipione's death, Anton Maria Fior challenged the mathematician Nicolo Tartaglia (1499–1557) to a mathematical tournament (which were quite common at this time). Tartaglia was the best-known mathematician in Italy at this time and practiced in Venice. Such a tournament consisted of 30 mathematical challenges, which were given from either opponent to the other. Somehow, Tartaglia found out that Fior knew the secret of how to solve cubic equations. Tartaglia suspected correctly that all challenges he had to face would involve cubic equations. He figured out how to solve (a special case of) the cubic equation by himself and won the tournament handily—he solved all 30 problems within 2 hours. At that time another Italian mathematician, Girolamo Cardano (1501–1576) was working on a book that covered the mathematical knowledge known at the time (Tartaglia had a similar project). Cardano importuned Tartaglia to reveal his formula so that he could include it in his book. Tartaglia first resisted, but after several attempts, he agreed to tell him the formula under the condition that Cardano would not publish the formula in his own book. Cardano even vowed not to tell Tartaglia's formula in speech or in writing. However, a few years later Cardano found out that Tartaglia was not the first to have solved the cubic equation. Therefore, he felt released from the vow and included Tartaglia's formula in his book *Ars magna* (1545). Cardano even generalized the solution and was able to solve all special cases of the equation. For Tartaglia this was a great shock, and he began a long and public priority fight with Cardano. The solution formula was until recently always called “Cardano's formula.” In recent years (and after some historical research) it is more common to call it the “del Ferro/Tartaglia/Cardano formula.”

As we will see, despite all the priority disputes, Cardano made a contribution in this context that is perhaps more important than the solution of cubic equations itself. Tartaglia's general formula for solving the cubic equation  $x^3 + px + q = 0$  (every cubic equation can be easily transformed into this form) can in modern terms be formulated as:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

For instance, if one wants to find a solution of  $x^3 - 24x - 72 = 0$ , one gets

$$\begin{aligned}
x &= \sqrt[3]{-\frac{-72}{2} + \sqrt{\frac{72^2}{4} + \frac{(-24^3)}{27}}} + \sqrt[3]{-\frac{-72}{2} - \sqrt{\frac{72^2}{4} + \frac{(-24^3)}{27}}} \\
&= \sqrt[3]{36 + \sqrt{1296 - 512}} + \sqrt[3]{36 - \sqrt{1296 - 512}} \\
&= \sqrt[3]{36 + 28} + \sqrt[3]{36 - 28} \\
&= \sqrt[3]{64} + \sqrt[3]{8} \\
&= 4 + 2 \\
&= 6.
\end{aligned}$$

And indeed,  $x = 6$  is a solution of the cubic equation:  $6^3 - 24 \cdot 6 - 72 = 216 - 144 - 72 = 0$ .

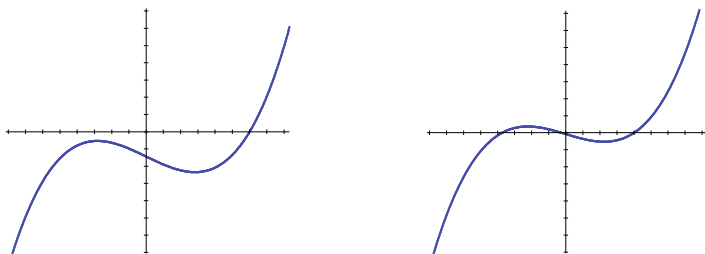
Now, it is clear that *every* cubic equation of the above form must have at least one solution, since as  $x$  runs from  $-\infty$  to  $+\infty$ , the cubic itself runs continuously from  $-\infty$  to  $+\infty$  and therefore has to pass through zero. Unfortunately, Tartaglia's formula applied in a naive way does not always lead to a solution. Consider, for instance, the problem  $x^3 - 15x - 4 = 0$ . If we proceed in the same way as before, we get

$$\begin{aligned}
x &= \sqrt[3]{-\frac{-4}{2} + \sqrt{\frac{4^2}{4} + \frac{(-15^3)}{27}}} + \sqrt[3]{-\frac{-4}{2} - \sqrt{\frac{4^2}{4} + \frac{(-15^3)}{27}}} \\
&= \sqrt[3]{2 + \sqrt{4 - 125}} + \sqrt[3]{2 - \sqrt{4 - 125}} \\
&= \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\
&= \dots
\end{aligned}$$

... and we are stuck! What should one do with the term  $\sqrt{-121}$ ? There is no number (at least for del Ferro and Tartaglia) whose square is  $-121$ . Nevertheless, there must be a solution of the cubic (in contrast to the quadratic case, where nobody saw a reason for searching for a solution of  $x^2 = -1$ ). This was the place where Cardano made his brilliant innovation. He simply went on calculating, assuming that one could take the term  $\sqrt{-1}$  as a purely formal expression with which one could do arithmetic. It behaves almost like a usual number but must be used subject to the rule  $\sqrt{-1} \cdot \sqrt{-1} = -1$ . If we do so, we can continue the above calculation

$$\begin{aligned}
&\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\
&= \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} \\
&= \dots
\end{aligned}$$

... and now we need two numbers  $a$  and  $b$  with the property that  $a^3 = 2 + 11\sqrt{-1}$  and  $b^3 = 2 - 11\sqrt{-1}$ . In fact,  $a = 2 + \sqrt{-1}$  and  $b = 2 - \sqrt{-1}$  do the job (check it!) and we can proceed:



**Fig. 16.1** Function plots of the two cubic polynomials  $(x^3 - 24x - 72)/50$  and  $(x^3 - 15x - 4)/50$ .

$$\begin{aligned}
 & \sqrt[3]{2 + \sqrt{-1}} + \sqrt[3]{2 - \sqrt{-1}} \\
 &= (2 + 11\sqrt{-1}) + (2 - 11\sqrt{-1}) \\
 &= 4.
 \end{aligned}$$

Et voilà, right before the last equation the mysterious  $\sqrt{-1}$  disappears again, and we end up with a nice, real (and correct) solution  $x = 4$ . Testing the result we obtain  $4^3 - 15 \cdot 4 - 4 = 64 - 60 - 4 = 0$ .

In “modern times” things have been smoothed out. A new symbol “ $i$ ” was introduced that plays the role of the mysterious  $\sqrt{-1}$  and behaves subject to  $i^2 = -1$ . This number  $i$  is usually called the *imaginary unit*. We now usually consider  $\mathbb{C} = \mathbb{R}[i]$  as a field extension of the real numbers, so that we can consider complex numbers as numbers of the form  $x + i \cdot y$ . Complex numbers play a great unifying role in modern mathematics. With their help, seemingly unrelated effects and topics may be interpreted as different sides of the same coin. We will have a brief look at some of them.

## 16.2 The Fundamental Theorem

The fundamental theorem of algebra is a great example for the unifying power of complex numbers. Complex numbers were originally introduced to perform the calculations to solve cubic equations. However, they also generalize the structure of the solution set. A view to the solution set of cubic equation from the “real perspective” tells us that they will have one, two, or three solutions, depending on the values of their parameters. From a “complex perspective” one can prove that a cubic  $x^3 + ax^2 + bx + c$  can always be written in the form  $(x - x_1)(x - x_2)(x - x_3)$ , where  $x_1, x_2, x_3$  are three (possibly complex) numbers. Since this expression is zero if and only if  $x$  itself equals one of these numbers,  $x_1, x_2, x_3$  must be solutions of the cubic equation  $x^3 + ax^2 + bx + c = 0$ . Thus in a certain sense we could say that a cubic equation always has three solutions.

They may occur with a *multiplicity* if the same linear expression is used more than once in the expression  $(x - x_1)(x - x_2)(x - x_3)$ .

This is a special case of a much more general theorem: the fundamental theorem of algebra. This theorem states that every polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

of degree  $n$  may be written as a product of  $n$  linear factors

$$(x - x_1) \cdot (x - x_2) \cdot \cdots \cdot (x - x_n).$$

The numbers  $x_1, \dots, x_n$  are all solutions of the equation  $f(x) = 0$ . These numbers may be real or complex, and they may occur with a certain multiplicity in the product expression.

Thus not only do complex numbers help to solve quadratic and cubic equations, they also allow one to find all solutions of arbitrary polynomials. The fundamental theorem is by far not easy to prove, and both proofs of this theorem first known (one by Gauss and one by d'Alembert) turned out to have some minor flaws.<sup>2</sup> It is important to mention that the fundamental theorem of algebra does not tell how to find the solutions of a polynomial equation. It only states their existence.

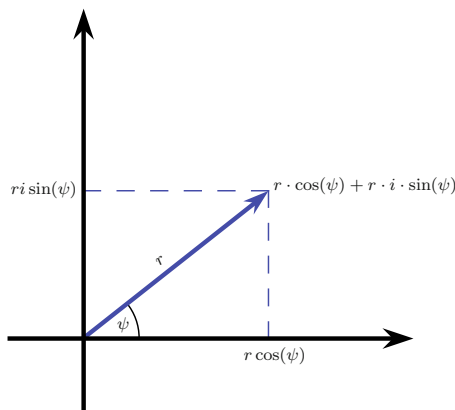
Applied to geometric problems, the fundamental theorem has many important consequences concerning the intersection multiplicity of geometric objects. It gives us the right to speak of intersections of objects even if we do not see them. Thus a line and a nondegenerate conic will always have two intersections, either real or complex. These two intersections coincide if the line is tangent to the conic. Similarly, two conics will in general have four points of intersection. Again these intersections may coincide.

## 16.3 Geometry of Complex Numbers

One of the most important aspects in relation to our investigations will be a geometric interpretation of complex numbers. A complex number  $a + i \cdot b$  may be associated to the point  $(a, b)$  in the real plane  $\mathbb{R}^2$ . Thus we may identify the field of complex numbers  $\mathbb{C}$  with the real Euclidean plane  $\mathbb{R}^2$ . Every statement about complex numbers immediately possesses a geometric counterpart. It is amazing that this (from a modern perspective almost obvious) interpretation was made quite a while after the invention of complex numbers. The geometric interpretation was first published by Caspar Wessel in 1799 (this is about 250 years after complex numbers were introduced!). Later,

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<sup>2</sup> In fact, in 1799 Gauss published a proof in response to d'Alembert's proof, since he thought that this proof was not rigorous. However Gauss's proof (based on a topological argument) also contained some flaws. Perhaps the first complete and correct proof was given in 1816 by Gauss.



**Fig. 16.2** Expressing complex numbers by trigonometric functions.

the geometric interpretation was rediscovered independently by Argand and by Gauss. The geometric interpretation of complex numbers as points in the plane will be crucial for all our further investigations.

Let us see how elementary arithmetic operations translate into geometric terms. If we add two complex numbers  $z_1 = a_1 + i \cdot b_1$  and  $z_2 = a_2 + i \cdot b_2$  we simply have to add the real and imaginary parts, and we obtain

$$z_1 + z_2 = a_1 + i \cdot b_1 + a_2 + i \cdot b_2 = (a_1 + a_2) + i \cdot (b_1 + b_2).$$

This is nothing but usual addition of vectors. Thus we can say that adding a complex number  $z = a + i \cdot b$  causes a translation by the vector  $(a, b)$ .

Multiplication is a bit more intricate. Using our rules for calculations with  $i$  we get:

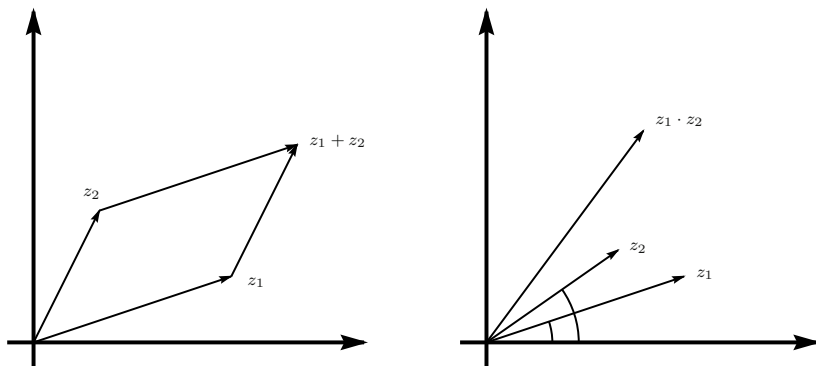
$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + i \cdot b_1) \cdot (a_2 + i \cdot b_2) \\ &= a_1 a_2 + i \cdot b_1 a_2 + i \cdot a_1 b_2 + i^2 \cdot b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i \cdot (b_1 a_2 + a_1 b_2). \end{aligned}$$

At first sight, this formula does not reveal an obvious geometric interpretation. Nevertheless such an interpretation will turn out to be the major key to all our applications of complex numbers to geometry.

To analyze what is going on, we introduce two magnitudes: the *length*  $|a + i \cdot b|$  of a complex number, which is defined by

$$|a + i \cdot b| = \sqrt{a^2 + b^2},$$

and the *angle* of  $z = a + i \cdot b$ , which is the angle between the vector  $(a, b)$  and the vector  $(1, 0)$  on the  $x$ -axis. The length is also sometimes called *absolute value* or *modulus* of  $z$ . The angle is also called *argument* or *phase* of  $z$ . We will



**Fig. 16.3** Geometric interpretation of addition and multiplication.

prefer the geometric terms “length” and “angle.” It is clear that a complex number is completely determined if we know its length  $r$  and its angle  $\psi$ . The corresponding complex number then calculates (as can be seen by simple trigonometry) to

$$z = r \cdot \cos(\psi) + r \cdot i \cdot \sin(\psi).$$

Let us see what happens if we multiply a complex number  $z_1 = a + i \cdot b$  by another number  $z_2$  that is defined by its length  $r$  and its angle  $\psi$ . We get

$$\begin{aligned} z_1 \cdot z_2 &= (a + i \cdot b) \cdot (r \cdot \cos(\psi) + r \cdot i \cdot \sin(\psi)) \\ &= (ar \cdot \cos(\psi) - br \sin(\psi)) + i \cdot (br \cdot \cos(\psi) + ar \sin(\psi)). \end{aligned}$$

If we abbreviate  $z_1 \cdot z_2 = a' + i \cdot b'$ , we can express this formula by a matrix multiplication:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = r \cdot \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

This gives an immediate interpretation in geometric terms: multiplying by a complex number with length  $r$  and angle  $\psi$  results in a rotation around the origin by an angle of  $\psi$  combined with a stretch (or dilatation) by a factor  $r$ . Figure 16.3 illustrates the geometric interpretation of addition and multiplication of two complex numbers.

## 16.4 Euler’s Formula

Expressing complex numbers by trigonometric functions is a nice feature, but it is not the most compact form in which we can write a complex number

given by angle and length. There is a beautiful formula known as *Euler's formula* that closely relates trigonometric functions, complex numbers, and the exponential function. One way to express this formula is

$$e^{ix} = \cos(x) + i \cdot \sin(x).$$

If  $x$  is a real number, this means that the exponential of the purely imaginary number  $ix$  can be expressed as combination of  $\cos(x)$ , which forms the real part, and  $i \cdot \sin(x)$ , which forms the imaginary part. In this form the formula was published by Euler in 1748 (although it was discovered earlier by Roger Cotes in 1714). Since the geometric interpretation of complex numbers was not known at this time, both Cotes and Euler considered this result a purely analytical statement.

When trying to interpret this formula, we encounter an important difficulty: What is  $e^{ix}$ ? How should we define the result of an analytic function applied to a complex argument? The short answer to this goes as follows: *Whenever a function is expressible as a formal power series, one can use this power series to evaluate the function also for complex arguments, as long as it converges.* In particular, the functions  $e^x$ ,  $\sin(x)$ , and  $\cos(x)$  have formal power series that converge for all complex numbers. These power series are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \end{aligned}$$

Comparing the entries of these power series, we observe a striking similarity of the summands of the three power series. Up to sign changes, all summands of  $e^x$  occur in  $\sin(x)$  or  $\cos(x)$ . So, how do we get the signs right? This is where the number  $i$  enters the game. If we expand the function  $e^{ix}$ , we see that depending on the power of the summand, the summand occurs either with a factor  $i$  (for even powers) or not (for odd powers). By the rule  $i^2 = -1$  also the signs are altered according to a very regular pattern. In detail, we get

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i \frac{x^3}{3!} + \frac{x^4}{4!} + i \frac{x^5}{5!} - \frac{x^6}{6!} - i \frac{x^7}{7!} + \cdots, \\ i \sin(x) &= ix - i \frac{x^3}{3!} + i \frac{x^5}{5!} - i \frac{x^7}{7!} + \cdots, \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \end{aligned}$$

We have printed the power series for  $\cos(x)$  and  $i \sin(x)$  for reference purposes. We see that  $e^{ix}$  splits into these two functions. So we have

$$e^{ix} = \cos(x) + i \cdot \sin(x).$$

This formula has many consequences. First of all, we see that if  $x$  is real, the length of  $e^{ix}$  is one, since  $|e^{ix}| = |\cos(x) + i \cdot \sin(x)| = \cos(x)^2 + \sin(x)^2 = 1$ . Thus these numbers all lie on the unit circle in the complex plane. We may think of  $x$  as an angle, and as  $x$  increases, the number  $e^{ix}$  rotates counterclockwise along the unit circle. In particular, we get  $e^{i\pi} = -1$  and  $e^{2i\pi} = 1$ .<sup>3</sup>

If we consider  $\ln(x)$  as the inverse function of  $e^x$ , we must consider  $\ln(x)$  to be a *many-valued* function. If we search, for instance, for a number  $x$  with  $e^x = 1$ , any number in the set

$$\{\dots, -4i\pi, -2i\pi, 0, 2i\pi, 4i\pi, \dots\}$$

will do. So we could say that  $\ln(1)$  could be any of these numbers. In general, the value of  $\ln(x)$  is defined only up to additive constants of the form  $2ki\pi$ , where  $k$  is an integer. Still, usually one prefers to define  $\ln(x)$  as a single-valued function and defines the *principal value* of  $\ln(x)$  to be the value  $a + ib$ , where  $b$  is in the half-open interval  $(-\pi, \pi]$ . We will have to deal with the many-valuedness of  $\ln$  later on. It simply reflects the geometric fact that a rotation by  $360^\circ$  is indistinguishable from the identity.

There is another important application of Euler's formula. It allows us to express a complex number directly by its length and its angle. If  $r$  is the length and  $\psi$  is the angle, we get

$$z = r \cdot e^{i\psi}.$$

This is the so-called polar representation of a complex number. It allows us also immediately to understand how one can calculate the product of two complex numbers using the rule  $e^x \cdot e^y = e^{x+y}$ . We get in polar coordinates

$$z_1 \cdot z_2 = r_1 \cdot e^{i\psi_1} \cdot r_2 \cdot e^{i\psi_2} = r_1 r_2 \cdot e^{i(\psi_1 + \psi_2)}.$$

The angles add and the lengths multiply. Figure 16.4 compares the vector and the polar representations of a complex number  $z = a + i \cdot b = r \cdot e^{i\psi}$ .

Polar representations allow us also to easily describe division by a complex number. If a number  $z \neq 0$  is given by  $z = r \cdot e^{i\psi}$ , then its inverse  $z^{-1}$  is given by  $z^{-1} = \frac{1}{r} \cdot e^{-i\psi}$ , since we have

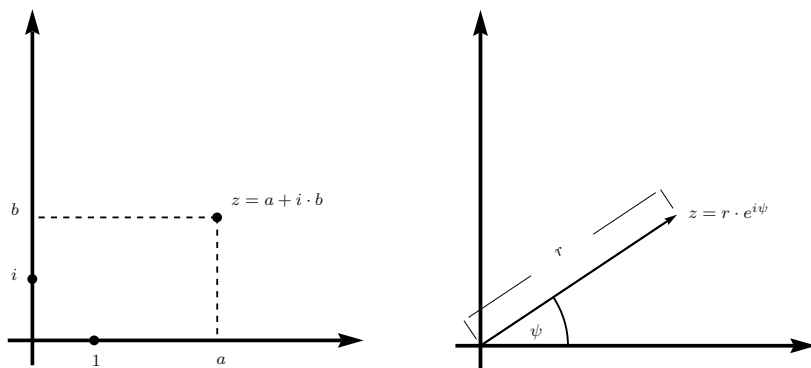
$$r e^{i\psi} \cdot \frac{1}{r} e^{-i\psi} = e^{i\psi - i\psi} = e^0 = 1.$$

In particular, numbers of the form  $e^{i\psi}$  can be considered numbers with *pure angle*  $\psi$ . Thus these numbers can be considered synonymously to angles. Adding angles corresponds to multiplication of these numbers. The behavior of these numbers nicely reflects the fact that angles are usually defined only

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<sup>3</sup> Many consider  $e^{i\pi} + 1 = 0$  to be one of the most beautiful formulas in mathematics, since it connects five very important mathematical constants: 0, 1,  $\pi$ ,  $e$ , and  $i$  and nothing else. Furthermore, this formula involves just four different basic operators: equality, addition, multiplication, and exponentiation.





**Fig. 16.4** Vector and polar representation of a complex number.

modulo  $2\pi$ . Adding two angles of  $270^\circ$  and  $180^\circ$  results in an angle of  $450^\circ$ . For most applications, this angle is equivalent to the angle  $450^\circ - 360^\circ = 90^\circ$ . This is reflected by calculations with numbers of the form  $e^{i\psi}$ . We have

$$e^{\frac{3}{2}i\pi} \cdot e^{i\pi} = e^{\frac{5}{2}i\pi} = e^{(2+\frac{1}{2})i\pi} = e^{2i\pi} \cdot e^{\frac{1}{2}i\pi} = 1 \cdot e^{\frac{1}{2}i\pi} = e^{\frac{1}{2}i\pi}.$$

## 16.5 Complex Conjugation

We have seen that complex addition and complex multiplication can be used nicely to express geometric transformations. Addition corresponds to translation, and multiplication can be used to express scaling and rotation (or a combination of both). There is one Euclidean transformation that is still missing: reflection.

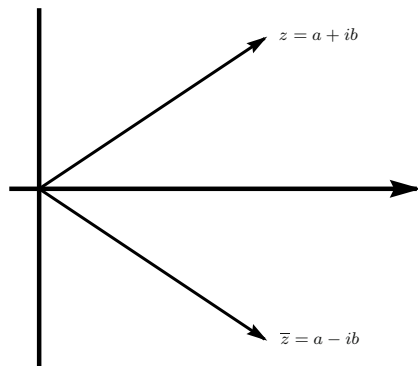
Reflections are closely related to another operation on complex numbers: *conjugation*. Conjugation has no counterpart in the field of real numbers. Conjugation expresses a mirror reflection in the real axis of the complex number plane. The conjugate of  $z = a + i \cdot b$  is denoted by  $\bar{z}$  and is defined by

$$\bar{z} = \overline{a + i \cdot b} = a - i \cdot b.$$

If  $z = r \cdot e^{i\psi}$  is given in polar coordinates, then the conjugate calculates as

$$\bar{z} = \overline{r \cdot e^{i\psi}} = r \cdot e^{-i\psi}.$$

Thus the conjugate of a real number is the number itself. The conjugate of a purely imaginary number is its negative. Complex conjugation is a nontrivial field automorphism. We have



**Fig. 16.5** Complex conjugation.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \quad \text{and} \quad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}.$$

The first equation follows immediately from the vector representation; the second follows immediately from the polar representation. In respect of the fundamental theorem of projective geometry this means that in complex projective spaces we will have harmonic maps or collinearities that do not come from projective transformations. If we consider only continuous automorphisms, complex conjugation is the only nontrivial field automorphism of  $\mathbb{C}$ , so that we do not get too many types of continuous harmonic maps or collineations.

There are a few facts on complex conjugation that should be mentioned, since they will turn out to be useful later. The first couple of facts concern the relation of complex conjugates to the elementary arithmetic operations:

- Adding a number  $z$  to its own conjugate, we obtain twice the real part of the number:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a.$$

- Subtracting  $\bar{z}$  from  $z$ , we obtain twice the imaginary part of the number:

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib.$$

- Multiplying  $z$  by  $\bar{z}$ , we obtain the square of the absolute value of  $z$ :

$$z \cdot \bar{z} = re^{i\psi} \cdot re^{-i\psi} = r^2 e^{i\psi - i\psi} = r^2 e^0 = r^2.$$

- Dividing  $z$  by  $\bar{z}$ , we obtain the number that has length 1 and twice the angle of  $z$ :

$$z/\bar{z} = re^{i\psi}/re^{-i\psi} = e^{i\psi + i\psi} = e^{2i\psi}.$$

It is a remarkable fact that complex conjugates and the four arithmetic operations are so closely related to the parameters of a complex number. We

will make use of this later on. In particular, the third equation implies that one can define the absolute value of a complex number by  $|z| = \sqrt{z \cdot \bar{z}}$ .

Another important fact is that if we have a polynomial  $f(x) = \sum_{i=0}^n a_i x^i$  with real parameters  $a_i$ , then as mentioned before, not all zeros of this polynomial may be real. However, if we have a complex root  $z$  of this polynomial, then  $\bar{z}$  will also be a root. This can be seen easily by evaluating  $f(\bar{z})$ :

$$f(\bar{z}) = \sum_{i=0}^n a_i \bar{z}^i = \sum_{i=0}^n \overline{a_i z^i} = \overline{\sum_{i=0}^n a_i z^i} = \overline{0} = 0.$$

The second equality holds since the  $a_i$  were assumed to be real and hence we have  $\overline{a_i} = a_i$ .

The proof strategy we have used here may be viewed as a special case of a more general concept. If we have any complex function  $f(z_1, z_2, \dots, z_n)$  that is composed only of the four arithmetic operations and complex conjugation, then we will automatically have

$$f(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n) = \overline{f(z_1, z_2, \dots, z_n)}.$$



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# The Complex Projective Line

*The shortest route between two truths in the real domain passes through the complex domain.*

*Jacques Salomon Hadamard (1865–1963)*

Now we will study the simplest case of a complex projective space: the *complex projective line*. We will see that even this case has already very rich geometric interpretations. The close relation of complex arithmetic operations to geometry allows us to express geometric properties by nice algebraic structures. In particular, this case will be the first example of a projective space in which we will be able to properly deal with circles.

## 17.1 $\mathbb{CP}^1$

Let us recall how we introduced the *real* projective line. We took the one-dimensional space  $\mathbb{R}$ , considered it as a Euclidean line, and added one point at infinity. Topologically, we obtained a circle. The best way to express the elements of the real projective line algebraically was to introduce homogeneous coordinates. For this we associated to each number  $x \in \mathbb{R}$  the vector  $(x, 1)^T$  and identified nonzero scalar multiples. Finally, we identified the vector  $(1, 0)^T$  (and all its non-zero multiples) with the point at infinity. We ended up with the space

$$\mathbb{RP}^1 = \frac{\mathbb{R}^2 - \{(0, 0)^T\}}{\mathbb{R} - \{0\}}.$$

We will do exactly the same for the complex numbers! To obtain the *complex projective line* we start with all the numbers in  $\mathbb{C}$ . We associate

every number  $z \in \mathbb{C}$  with the vector  $(z, 1)^T$  and identify nonzero scalar multiples. By this we associate all vectors of the form  $(a, b)^T; b \neq 0$ , to a number in  $\mathbb{C}$ . What is left is the vector  $(1, 0)^T$  and all its nonzero multiples. They will represent a unique *point at infinity*. All in all, we obtain the space  $\mathbb{CP}^1$  defined by

$$\mathbb{CP}^1 = \frac{\mathbb{C}^2 - \{(0, 0)^T\}}{\mathbb{C} - \{0\}}.$$

This space is isomorphic to all complex numbers together with *one* point at infinity.

What is the dimension of this space? In a sense, this depends on the point of view. From the perspective of real numbers, the complex plane  $\mathbb{C}$  is a two-dimensional object, since it requires two real parameters to specify the objects of  $\mathbb{C}$ . Hence one would say that also  $\mathbb{CP}^1$  is a real-two-dimensional object, since it differs from  $\mathbb{C}$  just by one point. On the other hand, from a complex perspective,  $\mathbb{C}$  is just a one-dimensional object. It contains just one (complex) parameter. Since  $\mathbb{R}$  is the *real number line*, one could consider  $\mathbb{C}$  the *complex number line*. Thus  $\mathbb{C}$  is complex-one-dimensional, and so is  $\mathbb{CP}^1$ . This is the reason why we call the space  $\mathbb{CP}^1$  the *complex projective line*!

There is another issue important to mention in this context. Identifying vectors that differ only by a nonzero scalar this time also includes multiplication by complex numbers. Thus  $(1, 2)^T$ ,  $(3i, 6i)^T$ ,  $(2 + i, 4 + 2i)^T$  all represent the same point. From every point represented by  $(a, b)^T$  with  $b \neq 0$  we can reconstruct the corresponding number of  $\mathbb{C}$  by multiplication by  $1/b$ . The dehomogenized number is then  $a/b \in \mathbb{C}$ . We will also frequently identify  $\mathbb{CP}^1$  with the space  $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ . As in the case of real numbers, we use the standard rules for arithmetic operations with  $\infty$ :

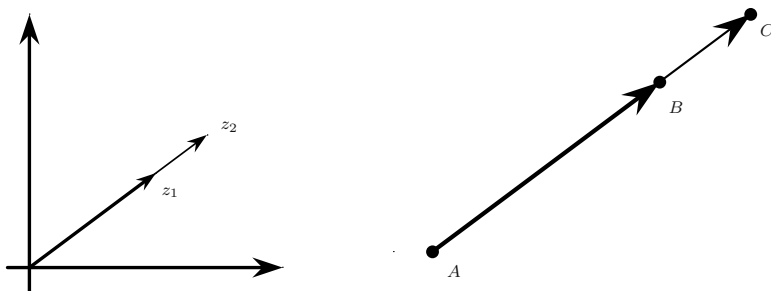
$$1/\infty = 0; \quad 1/0 = \infty; \quad 1 + \infty = \infty; \quad \infty + \infty = \infty.$$

## 17.2 Testing Geometric Properties

We will now identify the finite part of the complex projective line (i.e.,  $\mathbb{C}$  as a part of  $\mathbb{CP}^1$ ) with the Euclidean plane  $\mathbb{R}^2$  and investigate how certain geometric properties can be expressed in terms of algebraic expressions in  $\mathbb{C}$ . Most of these tests will, however, not correspond to projectively invariant properties in  $\mathbb{CP}^1$ .

We will first express the property that two vectors associated to two complex numbers  $z_1$  and  $z_2$  point in the same (or in the opposite) direction. For this we simply have to calculate the quotient  $z_1/z_2$ . Using polar coordinates, we get (if  $z_2 \neq 0$ )

$$z_1/z_2 = (r_1 e^{i\psi_1} / r_2 e^{i\psi_2}) = (r_1/r_2) e^{i(\psi_1 - \psi_2)}.$$



**Fig. 17.1** Testing simple geometric properties.

If the two vectors point in the same direction, we have  $\psi_1 = \psi_2$ , and the above expression is a positive real number. If they point in opposite directions we have  $\psi_1 = \pi + \psi_2$ , and (since  $e^{-i\pi} = -1$ ) the above number is real and negative. In other words, we could say that in the complex plane 0,  $z_1$  and  $z_2$  are collinear if the quotient  $z_1/z_2$  is real (provided  $z_2 \neq 0$ ). Using complex conjugation we can even turn this into an equality, since  $z$  is real if and only if  $z = \bar{z}$ . We get

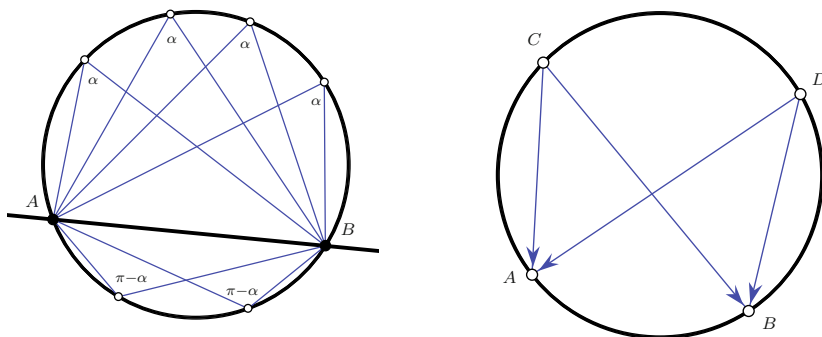
$$z_1 \text{ and } z_2 \text{ point in the same or opposite direction} \iff z_1/z_2 = \overline{z_1/z_2}.$$

We can use a similar test to decide whether three arbitrary distinct numbers correspond to collinear points. Let  $A$ ,  $B$ , and  $C$  be three points in the complex plane. We represent these points by their corresponding complex numbers. Then these three points are collinear if the quotient  $(B-A)/(C-A)$  is real (provided the denominator does not vanish). This is an immediate consequence of our previous considerations, since this quotient simply compares the directions of the vectors  $B-A$  and  $C-A$ . Analogously to the last statement, we get

$$A, B, C \text{ are collinear} \iff (B-A)/(C-A) = \overline{(B-A)/(C-A)}.$$

Unfortunately, the last two expressions do not fit into our concepts of projective invariants. The next one, however, will. We will describe whether four points lie commonly on a circle.

For this we first need a well-known theorem of elementary geometry: the *peripheral angle theorem*. This theorem states that if we have a circle and a secant from  $A$  to  $B$  on this circle, then all points on the circle on one side of the secant “see” the secant under the same angle. The two (invariant) angles on the left and on the right side of the secant sum to an angle of  $\pi$ . These two cases completely characterize cocircularity. Figure 17.2 illustrates this theorem. We omit a proof here (it can be found in many textbooks of elementary geometry), but we formulate the theorem on the level of complex



**Fig. 17.2** The peripheral angle theorem.

numbers. For this we denote by  $\angle_A(B, C)$  the counterclockwise angle between the vectors  $B - A$  and  $C - A$ . We can summarize both cases of the peripheral angle theorem by measuring angle differences modulo multiples of  $\pi$ . In such a version the peripheral angle theorem reads as follows

**Theorem 17.1.** *Let  $A, B, C, D$  be four points on a circle embedded in the complex plane. Then the angle difference  $\angle_C(A, B) - \angle_D(A, B)$  is a multiple of  $\pi$ .*

The “multiple of  $\pi$ ” comes from the fact that if  $C$  and  $D$  are on the same side of the secant through  $A$  and  $B$ , then both angles are equal and the difference is  $0 \cdot \pi$ . If they are on opposite sides of the secant, the difference is  $+\pi$  or  $-\pi$  depending on the order. Rather than proving this theorem, we will use this elementary geometric fact as a basis to derive a projective characterization of cocircularity in  $\mathbb{CP}^1$ .

We will interpret this theorem in the light of complex numbers. The angle  $\angle_C(A, B)$  can be calculated as the angle  $\psi_1$  of the following complex number:

$$\frac{C - A}{C - B} = r_1 e^{i\psi_1}.$$

Similarly, the angle  $\angle_D(A, B)$  can be calculated as angle  $\psi_2$  of

$$\frac{D - A}{D - B} = r_2 e^{i\psi_2}.$$

We can get the difference of the angles simply by dividing these two numbers. We get

$$\frac{C - A}{C - B} \bigg/ \frac{D - A}{D - B} = (r_1/r_2) e^{i(\psi_1 - \psi_2)}.$$



Since the angle difference is a multiple of  $\pi$  by the peripheral angle theorem, this number must be *real*. Now, the amazing fact is this: the expression on the left is nothing but a cross-ratio in the complex projective plane. Thus we can say that if the four points are on a circle, this cross-ratio  $(A, B; C, D)$  is a real number. The converse is also “almost” true; we have only to include the special case that the circle may have infinite radius and degenerate to a line. It is easy to check that if  $A, B, C, D$  are collinear, the cross-ratio is real as well. As usual, in our considerations we have to consider  $\infty$  as a real number as well. The cross-ratio assumes this value if either  $C = B$  or  $D = A$ . All in all, we obtain the following beautiful theorem, which highlights the close relationship between complex projective geometry and the geometry of circles.

**Theorem 17.2.** *Four points in  $\mathbb{CP}^1$  are cocircular or collinear if and only if the cross-ratio  $(A, B; C, D)$  is in  $\mathbb{R} \cup \{\infty\}$ .*

We can even be more specific about the sign of the cross-ratio:

**Theorem 17.3.** *If for four noncoinciding cocircular points in  $\mathbb{CP}^1$  the pair  $(A, B)$  cyclically separates the pair  $(C, D)$ , then  $(A, B; C, D) < 0$ . If  $(A, B)$  does not separate  $(C, D)$ , then  $(A, B; C, D) > 0$ .*

*Proof.* We have seen that the cross-ratio calculates as

$$\frac{C - A}{C - B} \bigg/ \frac{D - A}{D - B} = (r_1/r_2)e^{i(\psi_1 - \psi_2)},$$

where  $\psi_1$  and  $\psi_2$  are the angles under which  $C$  and  $D$  see the segment  $A, B$ . If  $C$  and  $D$  are on the same side of the segment  $A, B$ , then these angles are identical and we get  $e^{i(\psi_1 - \psi_2)} = e^0 = 1$ , which implies a positive cross-ratio. If  $C$  and  $D$  are on different sides, then the angles differ by  $\pi$  or by  $-\pi$ . In both cases this gives  $e^{i(\psi_1 - \psi_2)} = e^{\pm i\pi} = -1$ , which implies a negative cross-ratio.  $\square$

## 17.3 Projective Transformations

A projective transformation in  $\mathbb{CP}^1$  can (as in the real case) be expressed by a matrix multiplication. If the vector  $(z_1, z_2)^T \in \mathbb{C}^2$  represents a point in  $\mathbb{CP}^1$  by homogeneous coordinates, then a projective transformation can be expressed as

$$\begin{aligned} \tau: \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1, \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

As usual, the matrix must be nondegenerate. All entries can be complex. Matrices differing only by a nonzero scalar multiple represent the same projective

transformation. Since the matrix has four complex parameters and scalar multiples will be identified, we have three complex degrees of freedom (or six *real* degrees of freedom, if one prefers). A complex projective transformation is uniquely determined by fixing three pairs of images and preimages. The same method as introduced in the proof of Theorem 3.4 can be used to obtain the concrete matrix if three preimages and images are given.

Like all projective transformations, the projective transformations of  $\mathbb{CP}^1$  leave cross-ratios invariant. In particular if a cross-ratio of four points  $A, B, C, D$  is real, then the cross-ratio of the image points  $\tau(A), \tau(B), \tau(C), \tau(D)$  will be real again. Combining this fact with Theorem 16.2, we obtain the following remarkable fact, which by projective transformations circles and lines are transformed to circles and lines.

**Theorem 17.4.** *Let  $\tau$  be a projective transformation of  $\mathbb{CP}^1$ . Let  $A, B, C, D$  be four points on a circle or a line. Then the images  $\tau(A), \tau(B), \tau(C), \tau(D)$  will also be on a circle or on a line.*

*Proof.* The proof is immediate, since the fact that  $A, B, C, D$  are on a circle or on a line is characterized by  $(A, B; C, D) \in \mathbb{R} \cup \{\infty\}$ . After applying the projective transformation, the cross-ratio is again in  $\mathbb{R} \cup \{\infty\}$ .  $\square$

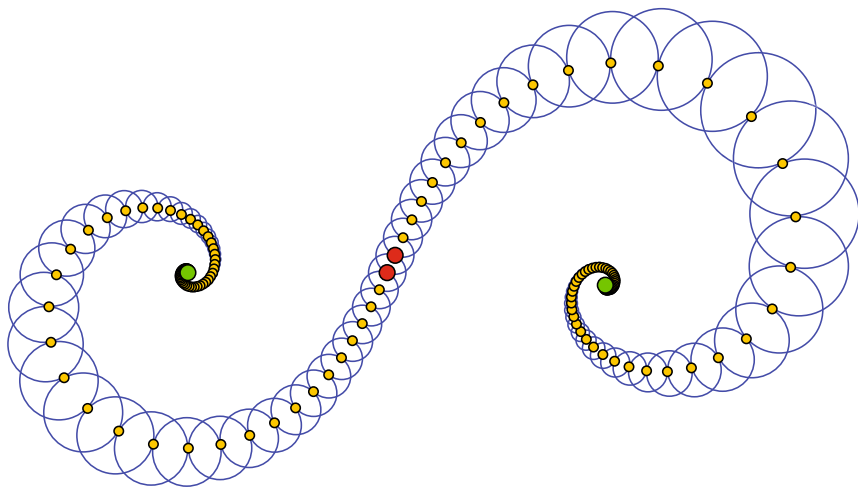
For reasons of convenience we will consider lines to be very large circles. One may think of lines as circles with infinite radius or alternatively as circles that pass through the point at infinity  $\infty$ . With this convention we may summarize the previous theorem in the simple statement that a circle is mapped by a projective transformation  $\tau: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  again to a circle.

If we identify  $\mathbb{CP}^1$  with  $\mathbb{C} \cup \{\infty\}$ , we may also express a projective transformation by a simple rational expression in  $z \in \mathbb{C} \cup \{\infty\}$ . A projective transformation represented by the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  leads via a sequence of homogenization/transformation/dehomogenization to the following rational mapping:

$$z \mapsto \begin{pmatrix} z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \mapsto \frac{az + b}{cz + d}.$$

Special care has to be taken if  $\infty$  is involved, since  $\infty$  is mapped to  $a/c$  and if the denominator of the ratio is zero, then the result should be considered infinite. Such a rational mapping  $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is called a *Möbius transformation*. Möbius transformations are extensively studied in complex function theory. Our considerations show that they are nothing but complex projective transformations in  $\mathbb{CP}^1$ .

We will briefly study different types of Möbius transformations. We will illustrate them by pictures in the finite part of  $\mathbb{CP}^1$ . We will go from the most general to more special transformations. Fixed points of projective transformations correspond to the eigenvectors of the transformation matrices. In contrast to the real case, in the complex case the fixed points will always be



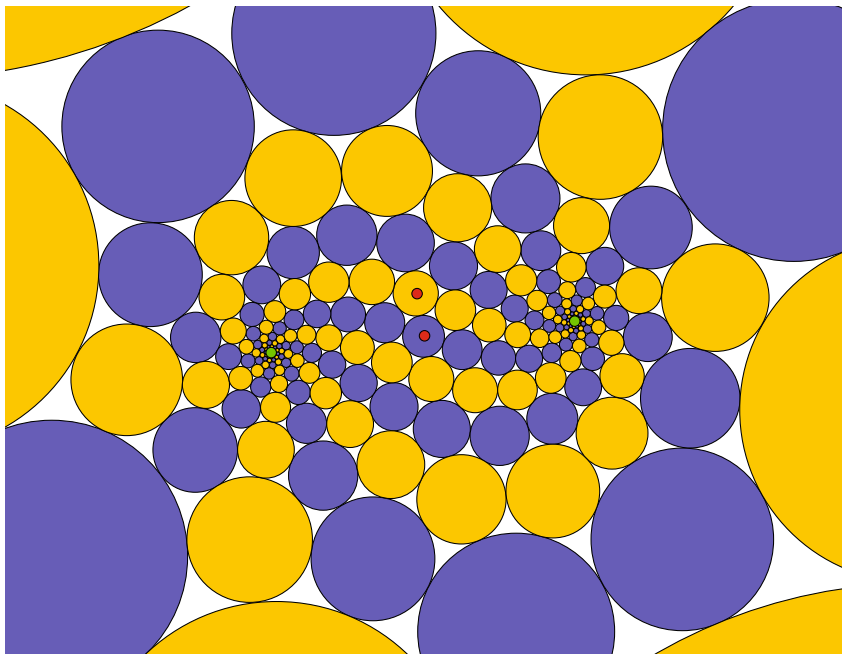
**Fig. 17.3** Iterated application of a Möbius transformation.

elements of  $\mathbb{CP}^1$  (in the real case it might have happened that a fixed point is complex and therefore not part of the real plane). If the two fixed points are distinct we may specify a Möbius transformation by the position of the two fixed points and another point and its image.

Figure 17.3 illustrates a most general Möbius transformation. The transformation  $\tau$  is defined by the two fixed points (the green points) and the two red points: one red point is mapped by  $\tau$  to the other. The picture shows the iterated application of  $\tau$  to one of the red points and the iterated application of  $\tau^{-1}$  to one of the red points. The yellow points represent these images. The iterated application of  $\tau$  and  $\tau^{-1}$  converges to the two green fixed points, respectively. Also the iterated images of a circle are shown. Observe that the images are again circles.

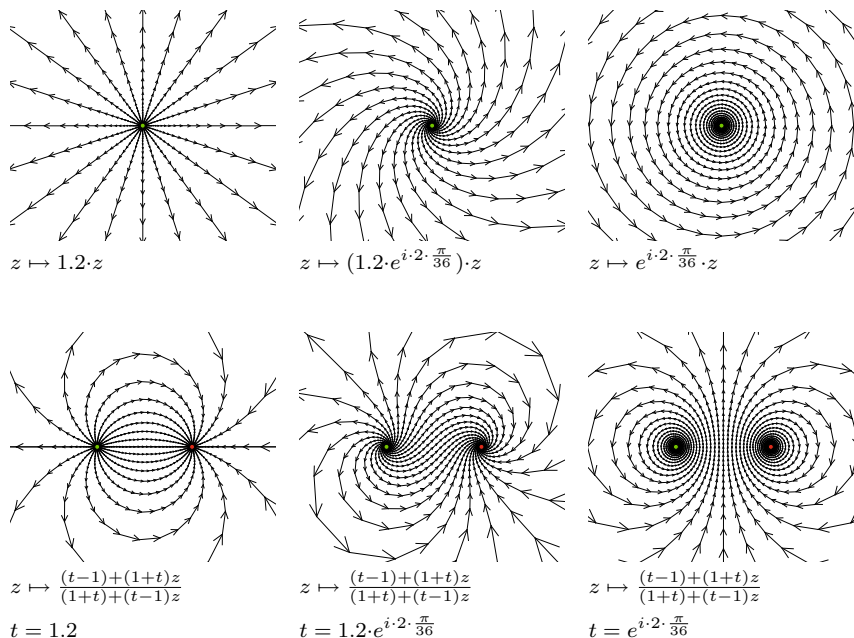
Iterated application of a Möbius transformation may generate pictures of mind-twisting beauty. Figure 17.4 shows an example of another Möbius transformation (and its inverse) applied to a circle. As before, the transformation is defined by two fixed points and another pair of points. The circles produced by the iterated application of the transformation are colored blue and yellow alternately. A careful choice of the parameters produces interesting circle-packing patterns with several spiraling structures.

Let us have a closer look at different kinds of Möbius transformations that may occur. We will express the transformations in the form  $z \mapsto \frac{az+b}{cz+d}$  with nonvanishing determinant  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ . In particular, if  $c = 0$  and  $d = 1$ , the transformation assumes the simple form  $az + b$ . Thus in particular, the linear



**Fig. 17.4** Iterated application of a Möbius transformation.

transformations are Möbius transformations. If furthermore  $a = 1$ , we have a simple shift along a translation vector that corresponds to  $b$  considered as a vector. If  $b = 0$  and  $a$  is a real number, then  $z \mapsto a \cdot z$  is a simple scaling around the origin. If, on the other hand,  $b = 0$  and  $a = e^{i\psi}$ ,  $\psi \in \mathbb{R}$ , is a number on the unit circle, then the transformation represents a rotation around the origin by an angle  $\psi$ . If  $a$  is neither real nor on the unit circle, the transformation results in scaling around the origin combined with a rotation. If we iterate this process, we produce spiral traces. The first row of Figure 17.5 represents the cases “ $a$  is real,” “ $a$  is neither real nor on the unit circle,” “ $a$  is on the unit circle.” Each arrow indicates the relationship between a particular point and its image under the transformation. It is interesting to study the fixed points for the simple mapping  $z \mapsto a \cdot z$ . The corresponding matrix is  $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ . The eigenvectors are  $(0, 1)^T$  and  $(1, 0)^T$ . Thus these operations leave the point 0 and the point  $\infty$  of  $\widehat{\mathbb{C}}$  fixed. The second row of Figure 17.5 shows essentially the same transformations. However, the fixed points have been moved to other positions (namely  $-1$  and  $1$ ). A projective transformation that maps 0 to  $-1$  and maps  $\infty$  to  $+1$  is given, for instance, by



**Fig. 17.5** Prototypes of Möbius transformations.

$$g(z) = \frac{-1 + z}{1 + z}.$$

The inverse of this map is given by

$$g^{-1}(z) = \frac{-1 - z}{-1 + z}.$$

Calculating  $z \mapsto g(t \cdot g^{-1}(z))$  we obtain a transformation of the form

$$z \mapsto \frac{(t-1) + (1+t)z}{(1+t) + (t-1)z}.$$

Such a transformation leaves  $-1$  and  $+1$  invariant and furthermore transforms multiplication by the factor  $t$  to the situation with these new fixed points. It is not an accident that the first and last pictures of the second row somehow look like flux pictures from electrodynamics. Möbius transformations play an essential role in the field theory of electrostatic or magnetic charges.

## 17.4 Inversions and Möbius Reflections

There is one subtle but important point in the theory of transformations in  $\mathbb{CP}^1$  that we have neglected so far. Our considerations of Chapter 5 differentiated between *harmonic maps* and *projective transformations*. Over  $\mathbb{RP}^1$  these two concepts coincide. The fundamental theorem of projective geometry, however, states that harmonic maps are the more general concept. Every harmonic map can be expressed by a field automorphism followed by a projective transformation. In fact, unlike  $\mathbb{R}$ , the field  $\mathbb{C}$  possesses a nontrivial field automorphism: *complex conjugation*. And this is the only one. Thus in addition to projective transformations, complex conjugation  $z \mapsto \bar{z}$  also leaves the set of circles invariant. Geometrically, this is not very surprising, since complex conjugation just resembles a mirror reflection in the real axis. The surprising fact is that such simple operations like reflection are not at all covered by projective transformations in  $\mathbb{CP}^1$ . In a certain sense all Möbius transformations will be *orientation-preserving*.

The characteristic magnitude preserved by Möbius transformations can be considered a kind of “circle sidedness predicate.” To make this precise we define what we mean by the *positive side* of a circle in  $\mathbb{CP}^1$ . To be precise here we must consider oriented circles defined by three points  $A, B, C$ . We denote such an oriented circle by  $\bigcirc(A, B, C)$ . Such an oriented circle consists of the circle through  $A, B, C$  together with some orientation information. One may think of this orientation as a rotational sense on the boundary such that we traverse the three points in the order  $A, B, C$  (if the circle degenerates into a line, it may happen that one has to pass through infinity when one traverses  $A, B, C$  in this order). At every point of the circle’s boundary one may think of an arrow indicating the rotational sense. Now we rotate these arrows counterclockwise by  $90^\circ$ . The rotated arrows point to the side of the circle that we call its “positive side.” It is important to notice that by this definition the positive side may be either the interior or the exterior of the circle, depending on the order of  $A, B, C$ . If the circle degenerates to a line, then one of the two half-spaces defined by the line becomes positive, and the other one becomes negative, again depending on the order of  $A, B, C$ .

There is also an easy algebraic characterization of the positive side of the circle. We will take this as the formal definition.

**Definition 17.1.** Let  $A, B, C$  be three points in  $\mathbb{CP}^1$ . The positive side of the circle  $\bigcirc(A, B, C)$  defined by  $A, B, C$  is the set

$$\{ p \in \mathbb{CP}^1 \mid \text{the imaginary part of } (A, B; C, p) \text{ is positive} \}.$$

Thus the imaginary part of the cross-ratio specifies the sides of the circle. It is positive on the positive side, zero on the boundary, and negative on the negative side. The reader may convince himself that this definition agrees with our geometric definition. Now it is immediate to see that we have the following theorem:

**Theorem 17.5.** *Let  $A, B, C$  be three points of  $\mathbb{CP}^1$  and let  $\tau$  be a projective transformation in  $\mathbb{CP}^1$ . Then  $\tau$  maps the positive side of  $\bigcirc(A, B, C)$  to the positive side of  $\bigcirc(\tau(A), \tau(B), \tau(C))$ .*

*Proof.* The proof is immediate. Since  $\tau$  is a projective transformation, in particular it preserves the sign of the imaginary part of cross-ratios.  $\square$

Thus we may say that projective geometry in  $\mathbb{CP}^1$  deals not only with circles, but with *oriented* circles. Projective transformations map the positive sides of oriented circles to the positive sides of other oriented circles. Hence there cannot be a projective transformation that leaves the real axis point-wise invariant and interchanges its upper and lower half-spaces. Likewise, there cannot be a projective transformation that leaves a circle invariant and interchanges its interior and its exterior.

Complex conjugation  $z \mapsto \bar{z}$  helps to add these kinds of transformations to our geometric system. It leaves the real axis (which is a special “circle” with infinite radius) invariant. But it interchanges its positive and negative sides.

A general harmonic map is a field automorphism followed by a projective transformation. Hence we may express those harmonic maps that are not projective transformations in the form

$$z \mapsto \frac{a \cdot \bar{z} + b}{c \cdot \bar{z} + d}; \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

We will call a map of this kind an *anti-Möbius transformation*. Anti-Möbius transformations preserve cocircularity but they interchange sidedness. In fact, every anti-Möbius transformation has a circle that as a whole stays invariant under the transformation. Its interior and its exterior will be interchanged. We will not prove this here. Instead, we will restrict ourselves to a few important examples of such maps with geometric significance.

One of those maps we already encountered,  $z \mapsto \bar{z}$ , is the reflection in the real axis. Another important map of this kind is given by

$$\begin{aligned} \iota: \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1, \\ z &\mapsto \frac{1}{\bar{z}}. \end{aligned}$$

This map is known as *inversion in the unit circle*. It leaves points on the complex unit circle point-wise invariant and interchanges the interior and the exterior of this circle. The map  $\iota$  is an involution, since  $\iota(\iota(z)) = z$ , as an easy computation shows. The origin is mapped to the point  $\infty$ . This operation is sometimes also called *circulereflection*. However, one should be aware that this operation does not represent a map that mimics optical reflection in a circle. Reflections in other circles can also be expressed easily using conjugation with a Möbius transformation  $\tau$ . If  $\tau$  maps the unit circle to another circle  $C$ , then  $z \mapsto \tau(\iota(\tau^{-1}(z)))$  is the inversion in the circle  $C$ .

The area of circle inversion is a very rich geometric field and by our considerations can be considered very closely related to projective geometry. However, we will not go into details here. The interested reader is referred to the vast literature on inversive and circle geometry (for nice treatments see [28, 49]).

## 17.5 Grassmann-Plücker relations

We now return to our main track and investigate what conclusions we may draw from bracket expressions in  $\mathbb{CP}^1$ . We will first investigate Grassmann-Plücker relations. As in the real case, also in  $\mathbb{CP}^1$  the rank-2 three-term Grassmann-Plücker relation holds. For any quadruple of points  $A, B, C, D \in \mathbb{CP}^1$  we have

$$[AB][CD] - [AC][BD] + [AD][BC] = 0.$$

There is an amazing consequence that we can draw from this formula that has a completely Euclidean interpretation. For this, by  $|AB|$  we denote the distance from a point  $A$  to a point  $B$ .

**Theorem 17.6 (Ptolemy's theorem).** *Let  $A, B, C, D$  be four points in the Euclidean plane. Then we have*

$$|AB||CD| + |AD||BC| \geq |AC||BD|.$$

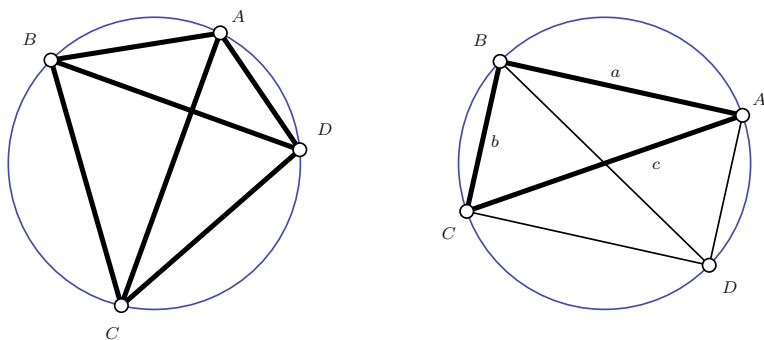
*Equality holds if and only if the four points are on a common circle or line in the cyclic order  $A, B, C, D$ .*

*Proof.* First note the striking resemblance of Ptolemy's formula and the Grassmann-Plücker relations. The brackets simply seem to be replaced by distances. In fact, we will see that Ptolemy's theorem may be considered a kind of "shadow" of the Grassmann-Plücker relation. For this we start with the relation

$$\underbrace{[AB][CD]}_{\alpha} - \underbrace{[AC][BD]}_{\gamma} + \underbrace{[AD][BC]}_{\beta} = 0.$$

We represent each of the three summands by a single greek letter. Thus the relation simply reads  $\alpha - \gamma + \beta = 0$ . Rewriting this, we get  $\alpha + \beta = \gamma$ . These three variables are complex numbers. In polar coordinates they can be written as  $r_\alpha e^{i\psi_\alpha}$ ,  $r_\beta e^{i\psi_\beta}$ ,  $r_\gamma e^{i\psi_\gamma}$ . The length  $r_\alpha$  is just the product of the lengths of  $[AB]$  and  $[CD]$ . If we assume that the points are embedded in the standard homogenization with  $[XY] = X - Y$ , we obtain that  $r_\alpha = |AB||CD|$ . Similarly, we have  $r_\beta = |AD||BC|$  and  $r_\gamma = |AC||BD|$ . By the triangle inequality we get from  $\alpha + \beta = \gamma$  the inequality





**Fig. 17.6** Ptolemy's theorem and the Pythagorean theorem.

$$|\alpha| + |\beta| \geq |\gamma|.$$

This is just Ptolemy's expression.

Equality is obtained in this expression if all three vectors point in the same direction. In this case we have

$$\frac{[AB][CD]}{[AC][BD]} = \frac{\alpha}{\gamma} \in \mathbb{R}^+ \quad \text{and} \quad \frac{[AD][BC]}{[AC][BD]} = \frac{\beta}{\gamma} \in \mathbb{R}^+.$$

Thus in the case of equality these two cross-ratios are real and positive. This is the case if and only if the four points are cocircular in this order.  $\square$

We will briefly have a look at a very special case of this theorem. Assume that  $A, B, C, D$  are the points of a rectangle in this order. The four points of a rectangle are cocircular. Furthermore, opposite sides have same lengths and the diagonals have the same length. Thus the expression

$$|AB||CD| + |AD||BC| = |AC||BD|$$

may be written as

$$|AB|^2 + |BC|^2 = |AC|^2.$$

If we denote the sides of the rectangle by  $a$  and  $b$  and its diagonal by  $c$ , then we get

$$a^2 + b^2 = c^2.$$

This is just the Pythagorean theorem! In other words, the Pythagorean theorem may be considered a shadow of a Grassmann-Plücker relation. Algebraically it has exactly the same shape.

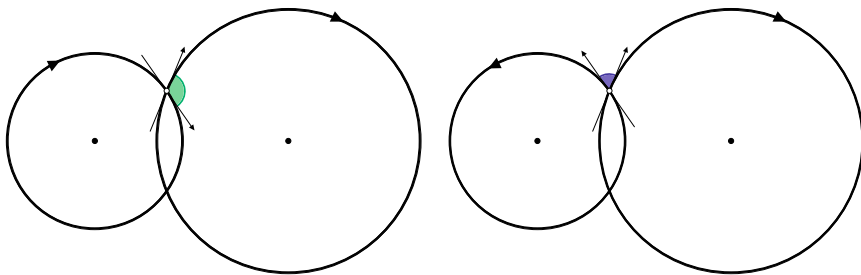
## 17.6 Intersection Angles

There is one more interesting geometric property that is directly related to the cross-ratio: the intersection angle of two oriented circles  $C_1$  and  $C_2$ . For this we consider two intersecting oriented circles in  $\mathbb{CP}^1$  that intersect in two points (see also [111]). At an intersection point we can attach two oriented tangents to the two circles. By the *intersection angle* of these two oriented circles we mean the angle  $\angle(C_1, C_2)$  that is needed to rotate the tangent at  $C_1$  to match the tangent at  $C_2$  in the same orientation. The situation for two different circle orientations is shown in Figure 17.7. The notion of intersection angle is well-defined, since it is independent of the choice of the intersection point at which it is measured.

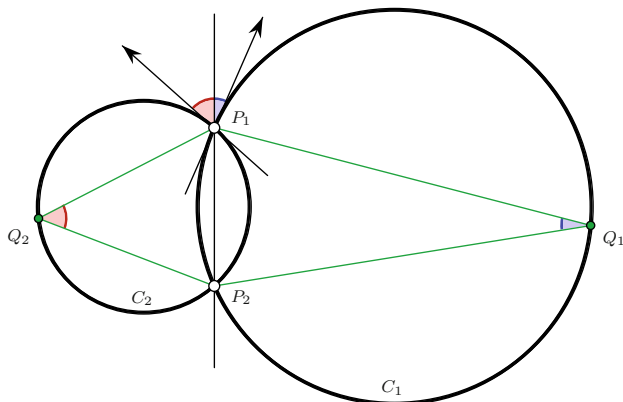
The intersection angle can be directly read off from a suitable cross-ratio. For this assume that  $P_1$  and  $P_2$  are the two intersections of the circles. To specify the orientation of  $C_1$  we choose a point  $Q_1$  on this circle such that the cyclic order  $P_1, Q_1, P_2$  is in agreement with the orientation of the circle (thus the order depends on the relative position of  $Q_1$  with respect to the line spanned by  $P_1, P_2$ ). With the notion of Section 17.4,  $C_1$  is the oriented circle  $\odot(P_1, Q_1, P_2)$ . Similarly, we choose a point  $Q_2$  to specify the orientation of  $C_2 = \odot(P_1, Q_2, P_2)$ . Now with these settings the intersection angle can be simply calculated by evaluating the cross-ratio, as the following theorem shows.

**Theorem 17.7.** *With the settings described above, let  $re^{i\psi} = (Q_1, Q_2; P_1, P_2)$  be the cross-ratio of the four points. Then  $\psi$  is the intersection angle of the oriented circles  $C_1$  and  $C_2$ .*

*Proof.* The quotient  $\frac{Q_1 - P_1}{Q_1 - P_2} = r_1 e^{i\psi_1}$  is a complex number whose angle  $\psi_1$  measures the angle under which  $Q_1$  sees the points  $P_1, P_2$ . In particular, the angle  $\psi_1$  is independent of the concrete choice of  $Q_1$  as long as it stays on the



**Fig. 17.7** Intersection angles of oriented circles.



**Fig. 17.8** Computing the intersection angle.

same side of  $\text{join}(P_1, P_2)$ . A similar statement holds for  $Q_2$  that sees  $P_1, P_2$  under the angle  $\psi_2$ . Thus the angle  $\psi_1 - \psi_2$  of the cross-ratio

$$\frac{Q_1 - P_1}{Q_1 - P_2} \bigg/ \frac{Q_2 - P_1}{Q_2 - P_2} = r_1 e^{i\psi_1} / r_2 e^{i\psi_2} = r e^{i(\psi_1 - \psi_2)}$$

depends only on the two circles and their orientation and not on the particular choice of  $Q_1, Q_2$ . To see that  $\psi_1 - \psi_2$  indeed equals the intersection angle of the two circles we now consider the limit case in which  $Q_1, Q_2$  asymptotically approach  $P_1$ . The cross-ratio may also be written

$$\frac{Q_1 - P_1}{Q_1 - P_2} \cdot \frac{Q_2 - P_2}{Q_2 - P_1} = \underbrace{\frac{Q_1 - P_1}{Q_2 - P_1}}_{=a} \cdot \underbrace{\frac{Q_2 - P_2}{Q_1 - P_2}}_{=b}.$$

In the limit case the term  $a$  in this expression has an angle identical to the intersection angle, since the numerator and the denominator describe complex numbers that have the same directions as the oriented tangents at  $P_1$ . The term  $b$  has an angle of zero, since the point  $P_2$  is distinct from  $P_1$  that is asymptotically approached by both  $Q_1$  and  $Q_2$ . Hence the overall expression of the cross-ratio has an angle equal to the intersection angle.  $\square$

Figure 17.8 demonstrates how the intersection angle is composed from the two angles at  $Q_1$  and  $Q_2$ . Since cross-ratios remain invariant under Möbius transformations, we immediately get the following consequence.

**Corollary 17.1.** *The intersection angle of oriented circles is invariant under Möbius transformations.*

*Remark 17.1.* With a similar argument one can prove that anti-Möbius transformations reverse all intersection angles.

## 17.7 Stereographic Projection

There is one important issue about the complex projective line  $\mathbb{CP}^1$  that we have not mentioned so far: it is topologically equivalent to a 2-sphere. This can be seen nicely by exhibiting a concrete projection that maps every point of  $\mathbb{CP}^1$  to a corresponding point on the 2-sphere. For this we first identify the complex number plane with the  $xy$ -plane of a three-dimensional real vector space via  $x + iy \mapsto (x, y, 0)^T$ . We now consider a sphere sitting tangentially on top of the complex number plane. We may assume that the sphere has radius 1 and touches  $\mathbb{C}$  at the origin. The equation of such a sphere is  $x^2 + y^2 + (z - 1)^2 = 1$ . The north pole of the sphere has coordinates  $N = (0, 0, 2)^T$ . With this point as projection center we project each point of  $\mathbb{C}$  to the sphere. Since every line through  $N$  intersects the sphere in exactly one additional point, this map is one-to-one. We can explicitly calculate this map by taking a point  $p = (x, y, 0)^T$  and finding the point on the line  $\lambda p + (1 - \lambda)N$  that is on the sphere. This leads to the equation

$$\lambda^2 x^2 + \lambda^2 y^2 + ((1 - \lambda) \cdot 2 - 1)^2 = 1,$$

which reduces to

$$\lambda^2 x^2 + \lambda^2 y^2 + 4(\lambda^2 - \lambda) = 0.$$

One of the solutions is  $\lambda = 0$ , which corresponds to the north pole (one of the two intersection points). The other solution is

$$\lambda = \frac{4}{x^2 + y^2 + 4},$$

which leads to the corresponding point

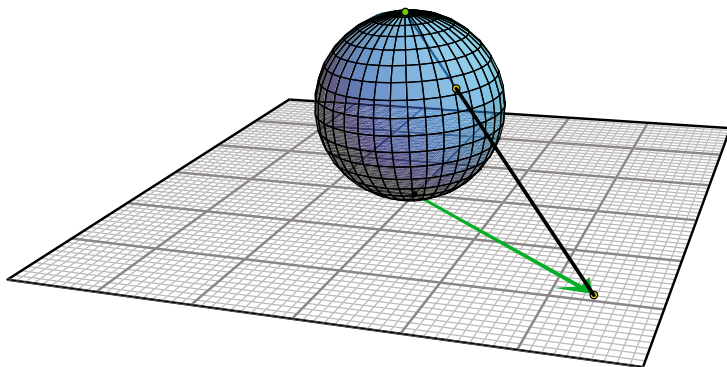
$$\frac{2}{x^2 + y^2 + 4} \begin{pmatrix} 2x \\ 2y \\ x^2 + y^2 \end{pmatrix}. \quad (17.1)$$

Points with large absolute value get mapped closer and closer to the north pole. Thus in the limit, the point  $\infty$  gets mapped to the north pole itself. Thus we get

$$\mathbb{CP}^1 \approx \mathbb{C} \cup \{\infty\} \approx S^2.$$

This bijective map has a few remarkable properties. Without proof, we mention only a few of them.

- Circles in  $\mathbb{CP}^1$  map to circles on  $S^2$  and vice versa.
- The intersection angle of circles is preserved under this map.
- Reflection of the sphere with respect to the north-south equator corresponds to inversion in the unit circle.



**Fig. 17.9** Stereographic projection.

- Möbius transformations can be considered as mapping  $\mathbb{CP}^1$  to the sphere, then rotating the sphere, scaling it, moving it to a new still tangential position, and finally projecting from the north pole of the moved sphere back to  $\mathbb{CP}^1$ .

For a more elaborate treatment of stereographic projection that in particular highlights the relations between projective geometry, complex numbers, matrix groups, and quaternions, we recommend [111].



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## Euclidean Geometry

*Dieses schöne Resultat [...] blieb aber lange unbeachtet, vermutlich, weil sich die Geometer an den Gedanken gewöhnt hatten, daß Metrik und projektive Geometrie in keiner Beziehung zueinander ständen.*

*Felix Klein about Laguerre's formula,  
Vorlesungen über Nicht-Euklidische Geometrie, 1928*

In this chapter we will merge two different worlds:  $\mathbb{CP}^1$  and  $\mathbb{RP}^2$ . Both can be considered as representing a real two-dimensional plane. They have different algebraic structures, and they both represent different compactifications of the Euclidean plane: For  $\mathbb{RP}^2$  we added a *line* at infinity. For  $\mathbb{CP}^1$  we added a *point* at infinity. Both spaces have different weaknesses and strengths. In the first two parts of the book we learned that  $\mathbb{RP}^2$  is very well suited for dealing, for instance, with incidences of lines and points, with conics in their general form, and with cross-ratios. We did not have a proper way to talk about circles, angles, and distances in  $\mathbb{RP}^2$ . The previous two chapters introduced  $\mathbb{CP}^1$ . This space was very good for dealing with cocircularity and also for dealing with angles. However, lines were poorly supported by  $\mathbb{CP}^1$ . They had to be considered circles with infinite radius, and they were not even projectively invariant objects.

We will now introduce an algebraic system that is capable of merging advantages of both worlds. We will end up with a framework in which we express all Euclidean properties by projectively invariant expressions.

## 18.1 The points I and J

The key to expressing Euclidean properties in projective geometry is as simple as it is powerful. We have to introduce two special points. All Euclidean properties will be expressed as projectively invariant expressions in which these two points play a special role. There are several possibilities for the choice of these points. However, there is a special choice for the two points under which all formulas become very simple and elegant. The points are

$$\mathbf{I} = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

Strictly speaking, these points are not even members of  $\mathbb{RP}^2$ , since they have complex coordinates. We will consider them formally as members of  $\mathbb{CP}^2$ . All algebraic calculations will be carried out in  $\mathbb{CP}^2$  (we use homogeneous coordinates with complex coordinate entries). However, the initial elements and the results of our calculations will usually be in  $\mathbb{RP}^2$ . All our considerations will refer to the standard embedding of the Euclidean plane  $\mathbb{R}^2$  into  $\mathbb{RP}^2$  by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Thus as usual, the line at infinity will have coordinates  $l_\infty = (0, 0, 1)^T$ . We will start by collecting a few useful properties of the points I and J.

*I and J span the line at infinity:* The points I and J both are on  $l_\infty$ , since they have a zero in their last entry. Since they are two different points, they even span  $l_\infty$ :

$$\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2i \end{pmatrix} = -2i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

*I and J can transfer finite points from  $\mathbb{RP}^2$  to  $\mathbb{C}$ :* Consider a point  $p$  with homogeneous coordinates  $(a, b, 1)^T$  in  $\mathbb{RP}^2$ . In the Euclidean plane this point would represent the point  $(a, b)^T$ . In the complex plane it would represent the point  $a + ib$ . Now consider the  $3 \times 3$  determinant  $[p, \mathbf{I}, l_\infty]$ . We get

$$[p, \mathbf{I}, l_\infty] = \begin{vmatrix} a - i & 0 \\ b & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = a + ib.$$

Similarly, we get the conjugate  $\overline{a + ib}$ :



$$[p, J, l_\infty] = \begin{vmatrix} a & i & 0 \\ b & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = a - ib.$$

**I** and **J** can express determinants in  $\mathbb{CP}^1$  as determinants in  $\mathbb{RP}^2$ : We consider also the standard embedding in  $\mathbb{CP}^1$ :

$$z \mapsto \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

Now take two points  $p_1$  and  $p_2$  in  $\mathbb{RP}^2$  represented by vectors  $(a_1, b_1, 1)^T$  and  $(a_2, b_2, 1)^T$ . In  $\mathbb{CP}^1$  they would represent points  $\tilde{p}_1 = (a_1 + ib_1, 1)$  and  $\tilde{p}_2 = (a_2 + ib_2, 1)$ . Considering the determinant  $[p_1, p_2, \mathbf{I}]$ , we get

$$\begin{aligned} [p_1, p_2, \mathbf{I}] &= \begin{vmatrix} a_1 & a_2 & -i \\ b_1 & b_2 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ &= a_2 - ib_1 - a_1 + ib_2 \\ &= (a_2 + ib_2) - (a_1 + ib_1) \\ &= [\tilde{p}_2, \tilde{p}_1]. \end{aligned}$$

And similarly we get the conjugate

$$[p_1, p_2, \mathbf{J}] = \overline{[\tilde{p}_2, \tilde{p}_1]}.$$

Thus the use of **I** and **J** allows us to express determinants of  $\mathbb{CP}^1$  (and their conjugates) as determinants of  $\mathbb{RP}^2$  involving **I** and **J**. The fact that we rely on the standard embedding in both worlds will not harm us later on, since in both worlds we will have to deal only with projectively invariant conditions.

The last property is crucial. It is the key to translating projective invariants of  $\mathbb{CP}^1$  to projective invariants of  $\mathbb{RP}^2$ .

## 18.2 Cocircularity

Let us begin applying **I** and **J** to express Euclidean properties. The strategy here will be:

- Express the property in  $\mathbb{CP}^1$  as bracket identity.
- Translate the identity bracket by bracket to  $\mathbb{RP}^2$  (using **I** and **J**).
- Consider the translated identity as a projective invariant in  $\mathbb{RP}^2$ .

One of the most fundamental invariants of  $\mathbb{CP}^1$  is cocircularity. Four points of  $\mathbb{CP}^1$  are cocircular if their cross-ratio is real. Assume that we are given four points  $A, B, C, D$  in  $\mathbb{RP}^2$  (with respect to the standard embedding). We consider their corresponding counterparts in  $\mathbb{CP}^1$ , the complex points  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ . It is easy to express cocircularity in terms of  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ . The points are cocircular if

$$\frac{[\tilde{A}\tilde{C}][\tilde{B}\tilde{D}]}{[\tilde{A}\tilde{D}][\tilde{B}\tilde{C}]} \in \mathbb{R},$$

or equivalently

$$\frac{[\tilde{A}\tilde{C}][\tilde{B}\tilde{D}]}{[\tilde{A}\tilde{D}][\tilde{B}\tilde{C}]} = \overline{\left( \frac{[\tilde{A}\tilde{C}][\tilde{B}\tilde{D}]}{[\tilde{A}\tilde{D}][\tilde{B}\tilde{C}]} \right)}.$$

Expressing the brackets in  $\mathbb{RP}^2$ , we get

$$\frac{[ACI][BDI]}{[ADI][BCI]} = \frac{[ACJ][BDJ]}{[ADJ][BCJ]}.$$

We can also multiply by the denominators to obtain a projectively invariant polynomial equation:

$$[ACI][BDI][ADJ][BCJ] = [ACJ][BDJ][ADI][BCI].$$

Summarizing, we get the following characterization of cocircularity:

**Theorem 18.1.** *The points  $A, B, C, D$  in  $\mathbb{RP}^2$  are cocircular if*

$$[ACI][BDI][ADJ][BCJ] = [ACJ][BDJ][ADI][BCI].$$

Comparing the above bracket expression with the expression derived in Section 10.2 that characterizes whether six points are on a conic, we observe that the expression

$$[ACI][BDI][ADJ][BCJ] = [ACJ][BDJ][ADI][BCI]$$

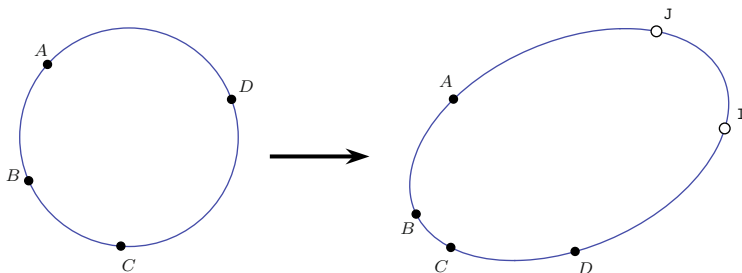
expresses the fact that  $A, B, C, D, I, J$  lie on a common conic (each point occurs quadratically on either side). Thus we can reformulate the previous theorem as follows:

**Theorem 18.2.** *The points  $A, B, C, D$  in  $\mathbb{RP}^2$  are cocircular if  $A, B, C, D, I, J$  are on a common conic.*

Or in other words: *Circles are conics through I and J!*

This last fact can also be derived in a different way. If we consider the (Euclidean) equation of a circle with midpoint  $(m_x, m_y)$  and radius  $r$ ,

$$(x - m_x)^2 + (y - m_y)^2 = r^2,$$



**Fig. 18.1** Projective interpretation of cocircularity.

we can translate this into a quadratic equation in homogeneous coordinates. We obtain

$$x^2 + y^2 - 2m_x \cdot xz - 2m_y \cdot yz + (m_x^2 + m_y^2 - r^2) \cdot z^2 = 0,$$

which is for suitably chosen parameters  $a, b, c$  the special conic

$$x^2 + y^2 + a \cdot xz + b \cdot yz + c \cdot z^2 = 0.$$

Inserting the coordinates of I in this equation, we get

$$(-i)^2 + 1^2 + a \cdot 0 + b \cdot 0 + c \cdot 0 = -1 + 1 = 0.$$

Hence I lies on this arbitrarily chosen circle. A similar calculation also shows that J lies on any circle as well.

Figure 18.1 illustrates the projective interpretation of cocircularity. Projectively, cocircularity of four points has to be considered the coconicality of these four points with I and J. Usually one does not see I and J, since they are complex (and at infinity). This characterization can also be used to calculate the parameters of a circle through three points by applying the method explained in Section 10.1.

## 18.3 The Robustness of the Cross-Ratio

At this point we want to stop a moment with our main stream of thought and collect a few remarkable ways to calculate the same cross-ratio.

**Theorem 18.3.** *Let  $A, B, C, D \in \mathbb{RP}^2$  be four finite points that are cocircular in the standard embedding. Let  $\tilde{A} = (a, 1)^T, \tilde{B} = (b, 1)^T, \tilde{C} = (c, 1)^T, \tilde{D} = (d, 1)^T$  be the corresponding complex points in  $\mathbb{CP}^1$ . We assume that the points lie on a circle  $C$  in the order  $A, C, B, D$ . Then we have*

$$(A, B; C, D)_C = (\tilde{A}, \tilde{B}; \tilde{C}, \tilde{D}) = \frac{|a - c| \cdot |b - d|}{|a - d| \cdot |b - c|}.$$

The first equality also holds for an arbitrary order of the points.

*Proof.* We first prove the first equality. The expression  $(A, B; C, D)_C$  is the well-defined cross-ratio  $(A, B; C, D)_P$  of the four points seen from a point  $P$  on the conic (compare Theorem 10.1). In the proof of Theorem 10.1 the particular choice of the underlying field is irrelevant. Thus we can in particular set  $P = \mathbf{I}$ . Using the relation  $[p_1, p_2, \mathbf{I}] = [\tilde{p}_2, \tilde{p}_1]$  that relates the points in  $\mathbb{RP}^2$  to their complex counterparts, we immediately get  $(A, B; C, D)_C = (A, B; C, D)_\mathbf{I} = (\tilde{A}, \tilde{B}; \tilde{C}, \tilde{D})$ .

Now for the second equality:  $a, b, c, d$  are the complex numbers representing the four points in  $\mathbb{C}$  in the standard embedding of the complex projective line. The difference of two such numbers  $a, c$  has polar coordinates  $a - c = r_{a,c} e^{i\psi_{a,c}}$  with  $r_{a,c} = |a - c|$ . We define the radii and angles of the remaining differences similarly. Then the cross-ratio  $(\tilde{A}, \tilde{B}; \tilde{C}, \tilde{D})$  becomes

$$\frac{[\tilde{A}\tilde{C}][\tilde{B}\tilde{D}]}{[\tilde{A}\tilde{D}][\tilde{B}\tilde{C}]} = \frac{(a - c)(b - d)}{(a - d)(b - c)} = \frac{r_{a,c} r_{b,d}}{r_{a,d} r_{b,c}} \cdot \underbrace{\frac{e^{i\psi_{a,c}} e^{i\psi_{b,d}}}{e^{i\psi_{a,d}} e^{i\psi_{b,c}}}}_{=-1}.$$

The sign  $-1$  of the phase part of this expression is implied by Theorem 17.3 and the assumption that  $A, B$  separates  $C, D$  in the circle. All in all, we get

$$(\tilde{A}, \tilde{B}; \tilde{C}, \tilde{D}) = \frac{[\tilde{A}\tilde{C}] \cdot [\tilde{B}\tilde{D}]}{[\tilde{A}\tilde{D}] \cdot [\tilde{B}\tilde{C}]} = -\frac{r_{a,c} \cdot r_{b,d}}{r_{a,d} \cdot r_{b,c}},$$

which is the claimed equality.  $\square$

This theorem allows us to interpret the cross-ratio of four cocircular points in various ways: as *cross-ratios in the complex projective line*, as *cross-ratios of four points on a circle*, and as *cross-ratios of absolute values of distances*.

## 18.4 Transformations

Before we discuss further examples of expressing Euclidean properties in projective terms we will have a brief look at the “philosophy” behind the approach of Section 18.2. Our way of expressing cocircularity by a projectively invariant expression relied heavily on the standard embedding of Euclidean geometry into projective geometry (as well in  $\mathbb{RP}^2$  as in  $\mathbb{CP}^1$ ). This standard embedding caused the particular choice of the coordinates for  $\mathbf{I}$  and  $\mathbf{J}$ . The crucial property of  $\mathbf{I}$  and  $\mathbf{J}$  is that they, considered as a pair, remain invariant under certain transformations (in particular under Euclidean transformations). At this point we have to be a little careful to specify exactly which

groups of transformations we consider. Roughly speaking, the geometrically relevant groups of transformations in  $\mathbb{RP}^2$  form a hierarchical system. In order of generality the groups relevant for us are *projective transformations*, *affine transformations*, *similarity transformations*, *Euclidean transformations*. The following table lists transformations and invariant properties that belong to these different subgroups of the projective transformations.

	projective	affine	similarity	Euclidean
<b>Transformations :</b>				
general projective	•			
shearing	•	•		
scaling	•	•	•	
rotation	•	•	•	•
reflection	•	•	•	•
translation	•	•	•	•
<b>Invariants :</b>				
cross-ratio	•	•	•	•
ratios of lengths		•	•	•
angles			•	•
distances				•

We speak of *projective geometry* if we consider only properties that remain invariant under projective transformations. We speak of *affine geometry* if we consider only properties that remain invariant under affine transformations, and so on. Thus *Euclidean geometry* deals with properties that remain invariant under rotation, translation, and reflection. Our considerations will now deal with similarity geometry. In addition to the Euclidean transformations rotation, translation, and reflection, scaling is also allowed. Thus lengths are not an intrinsic concept of similarity geometry, but angles and circles are. Ratios of lengths are also a concept of similarity geometry.

In fact, in the context of geometric theorems, talking about *similarity geometry* is usually more appropriate than talking about Euclidean geometry. The only difference between the two geometries is that in Euclidean geometry we can actually measure a length absolutely, whereas in similarity geometry we can only compare lengths. Theorems of Euclidean geometry, however, are most often not formulated on the level of concrete lengths. Usually they compare lengths only relative to each other (you would not start any reasonable theorem with a sentence like: “Take a segment of length 3 cm ...”). So all the theorems we normally consider as Euclidean theorems are in fact theorems of similarity geometry. We will see that this is exactly the class of theorems governed by I and J.

In Section 3.6, when we introduced projective transformations, we first started by expressing rotations and translations and scalings (with respect

to the standard embedding) as multiplication by a  $3 \times 3$  matrix. There we obtained the following forms for each of the transformations, respectively:

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A matrix that performs an orientation-preserving similarity transformation (a combined rotation, translation, and scaling) has the general form

$$\begin{pmatrix} c & s & a \\ -s & c & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, a general orientation-reversing similarity transformation has the form

$$\begin{pmatrix} c & s & a \\ s & -c & b \\ 0 & 0 & 1 \end{pmatrix}.$$

In both cases we must require  $c^2 + s^2 \neq 0$  for having a nonzero determinant.

**Theorem 18.4.** *With respect to the standard embedding, orientation-preserving similarity transformations are exactly those matrices that leave  $\mathbf{I}$  and  $\mathbf{J}$  invariant. Orientation-reversing similarity transformations are exactly those matrices that interchange  $\mathbf{I}$  and  $\mathbf{J}$ .*

*Proof.* Applying an orientation-preserving similarity transformation  $S$  to  $\mathbf{I}$ , we get

$$S \cdot \mathbf{I} = \begin{pmatrix} c & s & a \\ -s & c & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -ic + s \\ is + c \\ 0 \end{pmatrix} = (c + is) \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = (c + is)\mathbf{I}.$$

The scalar  $c + is$  is nonzero, since  $c^2 + s^2 \neq 0$ . Similarly, we obtain that  $S \cdot \mathbf{J} = (c + is) \cdot \mathbf{J}$ . For an orientation-reversing similarity transformation  $R$  the calculation is similar. We get

$$R \cdot \mathbf{I} = \begin{pmatrix} c & s & a \\ s & -c & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -ic + s \\ -is - c \\ 0 \end{pmatrix} = (-c - is) \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = (-c - is)\mathbf{J}.$$

Again similarly we get  $R \cdot \mathbf{J} = (-c - is)\mathbf{I}$ .

Now we conversely assume that  $M$  is a matrix that leaves  $\mathbf{I}$  invariant. Since  $M$  is assumed to be a matrix that represents a projective transformation of  $\mathbb{RP}^2$ , we may assume that  $M$  has real entries. We then have  $M \cdot \mathbf{I} = \lambda \mathbf{I}$ . We now successively determine constraints on the entries of  $M$ . We first consider the first two entries of the last row of  $M$ :

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ x & y & \bullet \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

We get  $-ix + y = 0$ . Since the entries of the matrix must be real, these two entries must be zero. Since the matrix must have a nonvanishing determinant, the last entry of the last row must be nonzero. We may assume that it is 1 by rescaling the matrix. We now focus on the upper left  $2 \times 2$  matrix. We have

$$\begin{pmatrix} u & v & \bullet \\ w & x & \bullet \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

Thus we have  $-ui + v = -i\lambda$  and  $-wi + x = \lambda$ . Subtracting  $i$  times the first equation from the second, we get  $-u - iv - wi + x = 0$ . Since the matrix entries are real, we must have  $u = x$  and  $v = -w$ . The remaining two matrix entries can be arbitrary, since they are multiplied by 0. All in all, our matrix has the form

$$\begin{pmatrix} c & s & a \\ -s & c & b \\ 0 & 0 & 1 \end{pmatrix},$$

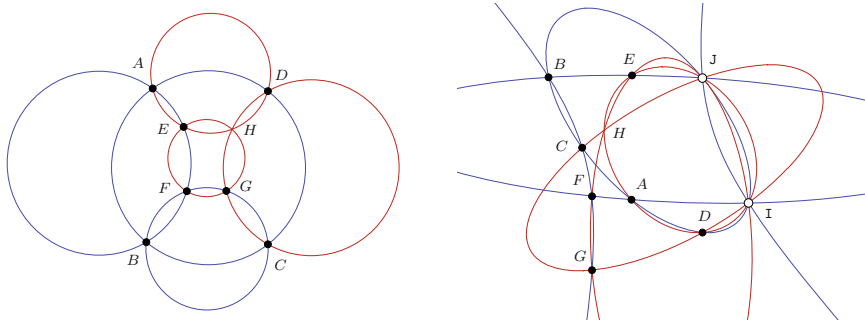
an orientation-preserving similarity transformation. A similar calculation shows that if  $M$  interchanges  $I$  and  $J$  we get an orientation-reversing similarity transformation.  $\square$

The last theorem states that we can characterize similarity transformations by the property that the pair of points  $\{I, J\}$  is left-invariant. Thus we can, in a sense, reverse our point of view and turn  $I$  and  $J$  into *first-class citizens* and use them to define what “similarity geometry” means. We will now provide a detailed example of how one can introduce the concepts of similarity geometry entirely based on two special points  $I$  and  $J$ .

If  $I$  and  $J$  are fixed, we can *define* similarity transformations to be those projective transformations that leave the pair  $\{I, J\}$  invariant. A property that is invariant under similarity transformations is called a *similarity property*. It will be our aim to base invariant properties and constructions of similarity geometry directly on  $I$  and  $J$  without the detour via similarity transformations.

In the previous section we already achieved this goal for one specific similarity property: *cocircularity*. We expressed cocircularity of four points  $A, B, C, D$  as a purely projective condition involving these points and the pair  $\{I, J\}$ . Now we again change the point of view, make  $I$  and  $J$  the primary objects, and *define* cocircularity by the property of Theorem 18.2. Thus we have:

- Similarity transformations are those that fix the pair  $\{I, J\}$ .
- $A, B, C, D$  are cocircular if  $A, B, C, D, I, J$  lie on a conic.



**Fig. 18.2** Miquel's theorem and its projective interpretation.

Based on these definitions we can derive the statement

- Cocircularity is invariant under similarity transformations.

*Proof.* A proof of this fact would look as follows: Assume that  $A, B, C, D$  are cocircular (thus  $A, B, C, D, I, J$  lie on a conic) and assume that  $\tau: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$  is a similarity transformation. We will prove that  $\tau(A), \tau(B), \tau(C), \tau(D)$  are cocircular as well. Since  $\tau$  is a projective transformation and *being on a conic* is a projectively invariant property, the six points  $\tau(A), \tau(B), \tau(C), \tau(D), \tau(I)$ , and  $\tau(J)$  are on a conic as well.

Since  $\tau$  is a similarity transformation either we have  $\tau(I) = I$  and  $\tau(J) = J$  or we have  $\tau(I) = J$  and  $\tau(J) = I$ . In either case this implies that  $\tau(A), \tau(B), \tau(C), \tau(D), I$ , and  $J$  are on a conic, which means that  $\tau(A), \tau(B), \tau(C), \tau(D)$  are cocircular.  $\square$

At first sight such reasoning may seem to be unnaturally complicated. However, it is conceptually very nice, since we reduced the concept of cocircularity entirely to the introduction of two special points  $I$  and  $J$  and to projective invariants, without even referring to the particular coordinates of  $I$  and  $J$ . If we choose special coordinates  $I = (-i, 1, 0)^T$  and  $J = (i, 1, 0)^T$ , then (with respect to the standard embedding) this setup specializes to our usual picture of similarity transformations and cocircularity.

## 18.5 Translating Theorems

The considerations of the last section show that behind every theorem of similarity geometry (or Euclidean geometry) there lies a projective truth. We will exemplify this by a nice theorem that needs cocircularity as the only predicate.



**Theorem 18.5 (Miquel's theorem).** *Let  $A, B, C, D, E, F, G, H$  be eight distinct points in the Euclidean plane such that the following quadruples are cocircular:  $(A, B, C, D)$ ,  $(A, B, E, F)$ ,  $(B, C, F, G)$ ,  $(C, D, G, H)$ ,  $(D, A, H, E)$ . Then  $(E, F, G, H)$  will be cocircular as well.*

*Proof.* We will present a proof of Miquel's theorem based on projective invariants. The hypotheses of the theorem imply that the following sextuples lie on common conics:  $(A, B, C, D, I, J)$ ,  $(A, B, E, F, I, J)$ ,  $(B, C, F, G, I, J)$ ,  $(C, D, G, H, I, J)$ ,  $(D, A, H, E, I, J)$ . This produces the following five bracket equations:

$$\begin{aligned} [CDJ][ABJ][BCI][ADI] &= [ABI][CDI][ADJ][BCJ], \\ [ABI][AEJ][BFJ][EFI] &= [ABJ][BFI][AEI][EFJ], \\ [BCJ][BFI][CGI][FGJ] &= [BCI][BFJ][CGJ][FGI], \\ [CDI][CGJ][GHI][DHJ] &= [CDJ][CGI][GHJ][DHI], \\ [ADJ][AEI][EHJ][DHI] &= [ADI][AEJ][EHI][DHJ]. \end{aligned}$$

All brackets in these expressions will be nonzero, since they always involve two distinct finite points and either I or J. Multiplying all left sides and all rights sides and canceling brackets that appear on both sides, we derive the equation

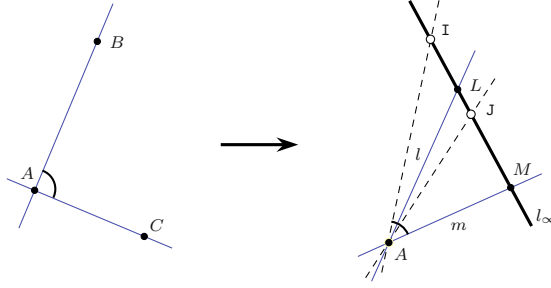
$$[EFI][FGJ][EHJ][GHI] = [EFJ][FGI][EHI][GHJ].$$

This expression is exactly the condition that also  $(E, F, G, H, I, J)$  are on a conic. And this implies the cocircularity of  $(E, F, G, H)$ .  $\square$

Our proof did not refer (except for nondegeneracy assumptions) to the coordinates of I and J. The bracket argument also proves a corresponding purely projective theorem about eight points and six conics, where I and J are located at real and finite positions. The corresponding theorem is shown in Figure 18.2 on the right. The labeling is consistent with the labeling of Miquel's theorem on the left.

## 18.6 More Geometric Properties

Our next aim is to derive projective conditions for other concepts of similarity geometry. We start with *perpendicularity*. We want to characterize the property that for three points  $A, B, C$  the lines  $\overline{AB}$  and  $\overline{AC}$  are perpendicular to each other. If we associate the three points with the corresponding points of the complex number plane  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{C}$ , then we can represent the above relation as an algebraic condition. For this we have to certify that the angle between the complex numbers  $\tilde{A} - \tilde{B}$  and  $\tilde{A} - \tilde{C}$  is  $90^\circ$ . This is the case if and only if



**Fig. 18.3** Projective interpretation of orthogonality.

$$\frac{\tilde{A} - \tilde{B}}{\tilde{A} - \tilde{C}} \in i\mathbb{R}.$$

This in turn translates to the equation

$$\frac{\tilde{A} - \tilde{B}}{\tilde{A} - \tilde{C}} = -\overline{\left(\frac{\tilde{A} - \tilde{B}}{\tilde{A} - \tilde{C}}\right)}.$$

As before, we reinterpret this equation in  $\mathbb{RP}^2$  with the help of  $I$  and  $J$ . We get

$$\frac{[ABI]}{[ACI]} = -\frac{[ABJ]}{[ACJ]}.$$

Slightly reordering the terms, we get

$$-1 = \frac{[ABI][ACJ]}{[ACI][ABJ]} = (B, C; I, J)_A.$$

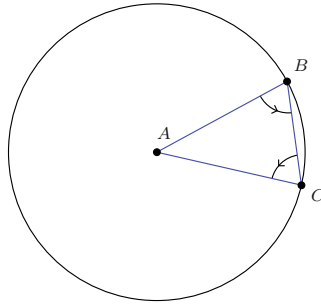
We can formulate this characterization in the following result.

**Theorem 18.6.** *The lines  $\overline{AB}$  and  $\overline{AC}$  are orthogonal if and only if the pairs of lines  $(\overline{AB}, \overline{AC})$  and  $(\overline{AI}, \overline{AJ})$  are in harmonic position.*

Alternatively, we can speak directly of orthogonal lines by considering their intersections with the line at infinity in relation to  $I$  and  $J$ :

**Theorem 18.7.** *Two lines  $l$  and  $m$  are orthogonal if and only if their intersections with the line at infinity  $L = l \times l_\infty$  and  $M = m \times l_\infty$  are in harmonic position with  $I$  and  $J$ .*

*Proof.* Assume that  $l$  and  $m$  are orthogonal and that  $A$  is their intersection. Let  $B$  be a point on  $l$  and let  $C$  be a point on  $m$ . The cross-ratio  $(B, C; I, J)_A$  is the same as the cross-ratio  $(L, M; I, J)$ . Hence by Theorem 18.6, the point pairs  $\{L, M\}$  and  $\{I, J\}$  are in harmonic position.



**Fig. 18.4** Encoding same distance by equal angles.

Conversely, assume that  $\{L, M\}$  and  $\{I, J\}$  are in harmonic position. Then  $l$  and  $m$  cannot be parallel, since otherwise, the cross-ratio of the four infinite points would be 1. Thus we may assume that  $l$  and  $m$  have a finite intersection  $A$ . After again introducing two auxiliary points  $B$  and  $C$  on the two lines, Theorem 18.6 proves the claim.  $\square$

The next geometric property we want to translate is closely related to characterizing circles by their midpoint and a point on the circle boundary.

**Theorem 18.8.** *If the distance from  $A$  to  $B$  equals the distance from  $A$  to  $C$ , then  $[AB I][AC I][CB J]^2 = [AB J][AC J][CB I]^2$ .*

*Proof.* Again we first consider the situation realized in the complex number plane  $\mathbb{C}$ . If  $|AB| = |AC|$ , then the three points  $A, B, C$  form an isosceles triangle. In this case the angle  $\angle_B(A, C)$  is the same as the angle  $\angle_C(B, A)$ . This means that we have

$$\frac{\tilde{A} - \tilde{B}}{\tilde{C} - \tilde{B}} \bigg/ \frac{\tilde{B} - \tilde{C}}{\tilde{A} - \tilde{C}} \in \mathbb{R}.$$

This is equivalent to

$$\frac{(\tilde{A} - \tilde{B})(\tilde{A} - \tilde{C})}{(\tilde{C} - \tilde{B})(\tilde{B} - \tilde{C})} = \overline{\left( \frac{(\tilde{A} - \tilde{B})(\tilde{A} - \tilde{C})}{(\tilde{C} - \tilde{B})(\tilde{B} - \tilde{C})} \right)}.$$

Translated to  $\mathbb{RP}^2$ , this reads as

$$[AB I][AC I][CB J]^2 = [AB J][AC J][CB I]^2,$$

which is the desired equation.  $\square$

Observe that also the characterization in the last theorem turns out to be a projectively invariant expression. However, it is only a *necessary condition*

for  $|AB| = |AC|$ . The point  $A$  occurs quadratically in this expression. Thus we may expect a conic section as locus for  $A$  if  $B$  and  $C$  are fixed. In fact, this conic consists of two lines. One is the median of  $B$  and  $C$ ; the other is the join of  $B$  and  $C$ . The equation is satisfied if  $A$  is on one of these lines. Thus the case of  $A$  being the midpoint of a circle through  $B$  and  $C$  is only one situation in which the above expression vanishes. The other is a being on the join of  $B$  and  $C$ . Later in this section we will also learn about a necessary and sufficient characterization of  $|AB| = |AC|$ .

## 18.7 Laguerre's Formula

So far we have used  $I$  and  $J$  only to express *properties* that are invariant under similarity transformations. We can even go one step further and perform *measurements* using  $I$  and  $J$ .

In Section 4 we learned that the simplest way to extract projectively invariant data from point configurations is by calculating cross-ratios. We will now use cross-ratios in combination with  $I$  and  $J$  to mimic measurements in a projective setup. The key result (which has also many beautiful generalizations, as we will see later) is *Laguerre's formula*. It was found in 1851 by Edmond Laguerre when he was just 19 years old [77, 68]. It allows one to measure the angle between two lines. Before we state Laguerre's formula we will clarify what exactly we mean by the angle between two lines  $l$  and  $m$ . Let us assume first that these two lines intersect in a single finite point  $O$  of the Euclidean plane. By the angle  $\angle(l, m)$  from  $l$  to  $m$  we mean the angle by which  $l$  has to be rotated counterclockwise around  $O$  until it coincides with  $m$ . Thereby the angle between two distinct lines will always lie in the open interval between 0 and  $\pi$ . If lines coincide or are parallel, then the angle between them is 0. Alternatively one could say that the angle between  $l$  and  $m$  corresponds to the angle about which  $l$  has to be rotated so that its direction coincides with that of  $m$ . One might expect that angles should be measured between 0 and  $2\pi$ . However, this makes no sense for unoriented lines. Now we are ready to state Laguerre's formula.

**Theorem 18.9.** *Let  $l$  and  $m$  be two finite lines of  $\mathbb{RP}^2$  and let  $L = l \times l_\infty$  and  $M = m \times l_\infty$  be the corresponding intersections with the line at infinity. Then the angle between  $l$  and  $m$  is (modulo  $\pi$ )*

$$\frac{1}{2i} \cdot \ln((M, L; I, J)).$$

*Proof.* The proof of this surprising result is straightforward using the method of transferring geometric properties from  $\mathbb{RP}^2$  to  $\mathbb{C}$ . Let  $l = (l_1, l_2, l_3)^T$  and  $m = (m_1, m_2, m_3)^T$  be the homogeneous coordinates of the lines. Then  $L$

and  $M$  have the coordinates  $L = (l_2, -l_1, 0)^T$  and  $M = (m_2, -m_1, 0)^T$ . The first two entries of these vectors are normal vectors to the two lines. The angle of these normal vectors (modulo  $\pi$ ) is exactly the angle between the two lines. We translate the normal vectors to two corresponding complex numbers

$$z_l = l_2 - i \cdot l_1 = r_l \cdot e^{i\psi_l} \quad \text{and} \quad z_m = m_2 - i \cdot m_1 = r_m \cdot e^{i\psi_m}.$$

The angle  $\psi_m - \psi_l$  modulo  $\pi$  is exactly the angle we are looking for. We can extract this angle by forming the following ratio:

$$\frac{z_m \cdot \overline{z_l}}{\overline{z_m} \cdot z_l} = \frac{r_m \cdot e^{i\psi_m} \cdot r_l \cdot e^{-i\psi_l}}{r_m \cdot e^{-i\psi_m} \cdot r_l \cdot e^{i\psi_l}} = \frac{e^{i\psi_m} \cdot e^{-i\psi_l}}{e^{-i\psi_m} \cdot e^{i\psi_l}} = e^{2i(\psi_m - \psi_l)}.$$

In the last expression the absolute values of  $z_l$  and  $z_m$  cancel, since each number (or its conjugate) is used in the numerator and in the denominator. Our considerations of Section 18.1 show that we have

$$z_l = [L, \mathbf{I}, l_\infty]; \quad \overline{z_l} = [L, \mathbf{J}, l_\infty]; \quad z_m = [M, \mathbf{I}, l_\infty]; \quad \overline{z_m} = [M, \mathbf{J}, l_\infty].$$

Using these determinants to express the above expression, we get

$$(M, L; \mathbf{I}, \mathbf{J}) = \frac{[M, \mathbf{I}, l_\infty][L, \mathbf{J}, l_\infty]}{[M, \mathbf{J}, l_\infty][L, \mathbf{I}, l_\infty]} = \frac{z_m \cdot \overline{z_l}}{\overline{z_m} \cdot z_l} = e^{2i(\psi_m - \psi_l)}.$$

Resolving for the desired angle, we get

$$\psi_m - \psi_l = \frac{1}{2i} \cdot \ln((M, L; \mathbf{I}, \mathbf{J})),$$

which is exactly Laguerre's formula.  $\square$

Laguerre's formula relates the properties of angles to the properties of cross-ratios in a surprising way. We will collect a few of these properties:

**Measurement modulo  $\pi$ :** The fact that angles between lines are measured modulo  $\pi$  is reflected in the fact that the natural logarithm function is unique only modulo a factor of  $2\pi i$  (we have  $e^a = e^{a+2\pi i}$ ) together with the factor  $\frac{1}{2i}$  in Laguerre's formula.

**Real lines generate real angles:** At first it is surprising that Laguerre's formula indeed produces only real values. In fact, if the lines have real coordinates, then the numerator and the denominator of  $(M, L; \mathbf{I}, \mathbf{J}) = \frac{[M, \mathbf{I}, l_\infty][L, \mathbf{J}, l_\infty]}{[M, \mathbf{J}, l_\infty][L, \mathbf{I}, l_\infty]}$  are complex conjugates. Dividing two conjugate numbers produces a complex number on the unit circle. Its logarithm is purely imaginary. This is compensated by the factor  $\frac{1}{2i}$ .

**Interchange of lines reverses angle:** We must have  $\angle(l, m) = -\angle(m, l)$  modulo  $\pi$ . Interchanging  $L$  and  $M$  transforms the cross-ratio to its inverse:

$(L, M; I, J) = 1/(M, L; I, J)$ . Since  $\ln(a) = -\ln(1/a)$  modulo  $2\pi i$ , this produces the desired sign change.

**Angles are additive:** If we have three lines  $h, l, m$  we must have  $\angle(l, m) + \angle(m, h) = \angle(l, h)$  modulo  $\pi$ . This formula expresses the multiplicativity of the cross-ratio. We have

$$\begin{aligned} (M, L; I, J) \cdot (H, M; I, J) &= \frac{[M, I, l_\infty][L, J, l_\infty]}{[M, J, l_\infty][L, I, l_\infty]} \cdot \frac{[H, I, l_\infty][M, J, l_\infty]}{[H, J, l_\infty][M, I, l_\infty]} \\ &= \frac{[H, I, l_\infty][L, J, l_\infty]}{[H, J, l_\infty][L, I, l_\infty]} \\ &= (H, L; I, J). \end{aligned}$$

By  $\ln(a \cdot b) = \ln(a) + \ln(b)$  modulo  $2\pi i$  this translates to additivity of angles.

**Orientation-preserving similarity transformations leave angles invariant:** If  $\tau$  is an orientation-preserving similarity transformation, we must have  $\angle(l, m) = \angle(\tau(l), \tau(m))$  (here we interpret  $\tau(l)$  as the corresponding action of  $\tau$  on a line by multiplication of the inverse transformation matrix). To see this identity, we calculate

$$\begin{aligned} (M, K, I, J) &= (\tau(M), \tau(K); \tau(I), \tau(J)) \\ &= (\tau(M), \tau(K); I, J). \end{aligned}$$

Hence the angle is the same after the transformation.

**Orientation-reversing similarity transformations reverse angles:** If  $\tau$  is an orientation-reversing similarity transformation, we must have  $\angle(l, m) = -\angle(\tau(l), \tau(m))$ . We get

$$\begin{aligned} (M, K, I, J) &= (\tau(M), \tau(K); \tau(I), \tau(J)) \\ &= (\tau(M), \tau(K); J, I) \\ &= 1/(\tau(M), \tau(K); I, J), \end{aligned}$$

which produces (via the logarithm) the reversed angle.

One should also observe that Laguerre's formula contains as a special case the characterization of right angles. If  $l$  and  $m$  are orthogonal, then  $\angle(l, m) = \frac{\pi}{2}$ . Since  $e^{i\pi} = -1$ , this translates via Laguerre's formula to the condition  $(M, L; I, J) = -1$ . This is exactly our characterization given in Theorem 18.7.

## 18.8 Distances

Using  $I$  and  $J$  we can also express distances between two points  $P$  and  $Q$  in Euclidean geometry. This formula is a little bit “tricky,” and we will present it here without a strictly formal proof. First of all, we cannot expect to express the distance purely as a projectively invariant formula only involving  $P$ ,  $Q$ ,  $I$ , and  $J$ , since *distance* is not an invariant of *similarity geometry*. The only thing we can hope for is that we obtain a formula that compares the distance between  $P$  and  $Q$  to the distance between two reference points  $A$  and  $B$ . Thus we will compute a formula for  $\frac{|P,Q|}{|A,B|}$ . If the distance  $|A,B|$  is normalized to be the unit length, we will have a formula for the distance of two arbitrary points. We will finally have an invariant expression in the six points  $A, B, P, Q, I, J$ .

For two points  $P = (p_1, p_2)$  and  $Q = (q_1, q_2)$  in the Euclidean plane  $\mathbb{R}^2$ , we usually calculate the distance via Pythagorean theorem:

$$|P, Q| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$

We will first express this formula in terms of determinants and then enlarge this formula to get a projectively invariant expression. Again we assume that  $P$  and  $Q$  have homogeneous coordinates with respect to the standard embedding. Thus we have  $P = (p_1, p_2, 1)^T$  and  $Q = (q_1, q_2, 1)^T$ . We now consider the expression  $\sqrt{[P, Q, I][P, Q, J]}$ . Expanding this term, we get

$$\begin{aligned} \sqrt{\begin{vmatrix} p_1 & q_1 & -i \\ p_2 & q_2 & 1 \\ 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} p_1 & q_1 & i \\ p_2 & q_2 & 1 \\ 1 & 1 & 0 \end{vmatrix}} &= \sqrt{((q_1 - p_1) + i(q_2 - p_2)) \cdot ((q_1 - p_1) - i(q_2 - p_2))} \\ &= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \\ &= |P, Q|. \end{aligned}$$

This is exactly the desired distance. Unfortunately, the expression

$$\sqrt{[P, Q, I][P, Q, J]}$$

is not at all a projective invariant. The following expression, however, is:

$$\frac{\sqrt{[P, Q, I][P, Q, J][A, I, J][B, I, J]}}{\sqrt{[A, B, I][A, B, J][P, I, J][Q, I, J]}} = |PQ|.$$

This expression indeed is a projective invariant (each letter occurs with the same power in numerator and denominator). Furthermore, for the case of the standard embedding and  $|A, B| = 1$  we get exactly the right distance function (as the following comparison of terms shows):

$$\frac{\overbrace{\sqrt{[P, Q, I][P, Q, J]}}^{|P, Q|} \overbrace{[A, I, J]}^{-2i} \overbrace{[B, I, J]}^{-2i}}{\underbrace{\sqrt{[A, B, I][A, B, J]}}_1 \underbrace{[P, I, J]}_{-2i} \underbrace{[Q, I, J]}_{-2i}} = |P, Q|.$$

All in all we get the following result:

**Theorem 18.10.** *The Euclidean distance  $|P, Q|$  between two points  $P$  and  $Q$  can be calculated by*

$$\frac{\sqrt{[P, Q, I][P, Q, J]}[A, I, J][B, I, J]}{\sqrt{[A, B, I][A, B, J]}[P, I, J][Q, I, J]},$$

if  $|A, B| = 1$  is a reference length.

It is also instructive to inspect this formula more closely. The occurrence of the square-root function expresses that it makes sense to speak of distances only up to a global sign. We can also compare the last result to the formula derived in Theorem 18.8, where we expressed a necessary condition for the property  $|AB| = |AC|$ . Theorem 18.10 contains this situation as special case. If we set  $P = A$  and  $Q = C$  and square the expression of Theorem 18.10, we get

$$\begin{aligned} 1 &= \left( \frac{\sqrt{[A, C, I][A, C, J]}[A, I, J][B, I, J]}{\sqrt{[A, B, I][A, B, J]}[A, I, J][C, I, J]} \right)^2 \\ &= \frac{[A, C, I][A, C, J][A, I, J]^2[B, I, J]^2}{[A, B, I][A, B, J][A, I, J]^2[C, I, J]^2} \\ &= \frac{[A, C, I][A, C, J][B, I, J]^2}{[A, B, I][A, B, J][C, I, J]^2}. \end{aligned}$$

This is also a necessary condition for  $|AB| = |AC|$ . In the formula

$$1 = \frac{[A, C, I][A, C, J][B, I, J]^2}{[A, B, I][A, B, J][C, I, J]^2} \quad (18.1)$$

the points  $A$ ,  $B$ , and  $C$  occur quadratically. If we fix two of the points, then the locus for the remaining one must be a conic. The conic for  $B$  is the circle with center  $A$  and perimeter point  $C$ . Similarly, the conic for  $B$  is the circle with center  $A$  and perimeter point  $B$ . The conic for  $A$  turns out to consist of two lines. One of the lines is the median of  $B$  and  $C$ ; the other is the line at infinity (as a simple calculation shows). If we combine the equation

$$[A, C, I][A, C, J][B, I, J]^2 = [A, B, I][A, B, J][C, I, J]^2$$



with the equation of Theorem 18.8,

$$[A, B, \mathbf{I}][A, C, \mathbf{I}][C, B, \mathbf{J}]^2 = [A, B, \mathbf{J}][A, C, \mathbf{J}][C, B, \mathbf{I}]^2,$$

by multiplying left and right sides we get

$$[A, C, \mathbf{I}]^2[C, B, \mathbf{J}]^2[B, \mathbf{I}, \mathbf{J}]^2 = [A, B, \mathbf{J}]^2[C, B, \mathbf{I}]^2[C, \mathbf{I}, \mathbf{J}]^2.$$

Taking the square root on both sides, we arrive at

$$[A, C, \mathbf{I}][C, B, \mathbf{J}][B, \mathbf{I}, \mathbf{J}] = \pm[A, B, \mathbf{J}][C, B, \mathbf{I}][C, \mathbf{I}, \mathbf{J}].$$

After choosing the right sign in this expression, it turns out to be a necessary and sufficient condition for  $|AB| = |AC|$ . The formula that characterizes this case is

$$[A, C, \mathbf{I}][C, B, \mathbf{J}][B, \mathbf{I}, \mathbf{J}] = -[A, B, \mathbf{J}][C, B, \mathbf{I}][C, \mathbf{I}, \mathbf{J}].$$

Observe that  $A$  occurs linearly in this formula, and both  $B$  and  $C$  occur quadratically.



## Euclidean Structures from a Projective Perspective

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*Projective geometry does not start where elementary Geometry leaves off; that is to say, it does not presuppose any of the results of elementary Geometry. It stands by itself, and is developed logically from its own initial propositions. The reader will find, however, that the two subjects are not entirely unconnected, for it will appear that elementary Geometry is a particular case of Projective Geometry. As a consequence of the fact that it is not dependent on elementary Geometry he must not expect to find that the initial propositions are familiar to him from what he already knows. Indeed, it is only at the end of the development that he will see elementary Geometry emerging. But it must not be supposed that the only aim of Projective Geometry is to establish the results of elementary Geometry; it does this incidentally, but at the same time it shows them in their true perspective, for it shows clearly what places in the hierarchy of Geometry this and other Geometries occupy.*

*C.W. O'Hara and D.R. Ward,  
An Introduction to Projective Geometry, 1937*

In the previous chapter we laid the foundations for representing Euclidean concepts (transformations, angles, distances, orthogonality, cocircularity ...) in a projective framework. In this chapter we will apply these concepts. We will give a loose and by no means complete collection of interesting constructions/calculations/theorems in Euclidean geometry that can be nicely carried out in our projective framework. Here we will strictly follow the concept that

*Euclidean geometry is projective geometry together with I and J.*

In all our considerations we will use only a projective framework and the two special points  $I$  and  $J$ . Euclidean definitions and properties will always strictly be expressed using these two points in special position.

Our examples will cover three different aspects. We will consider elementary geometric constructions, calculations, and geometric theorems. The constructions will essentially be based on constructive primitive operations such as *join*, *meet*, *intersection of conics and lines*, *tangents to conics*. All these constructions were introduced algebraically earlier in this book. We will show how these operations together with  $I$  and  $J$  can be used to derive elegant constructions for typical geometric problems. For instance, we will see that a mirror image of a point can be constructed using only a few join and meet operations.

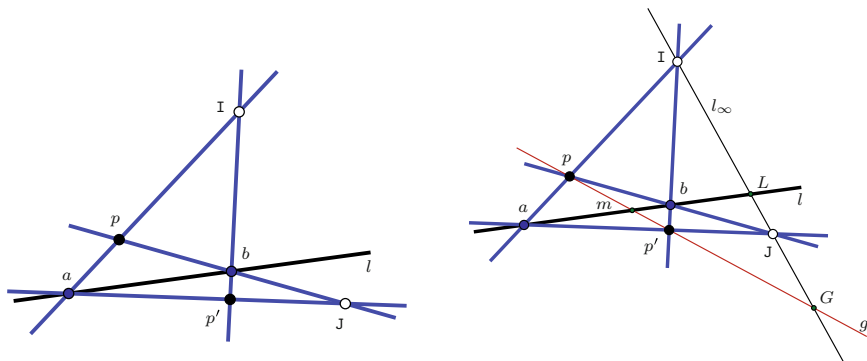
We will also extend our sampler of algebraic “implementations” for geometric primitive operations and provide approaches for typical Euclidean problems. In particular, we will derive nice formulas for angle bisectors, and for conics for which the foci are given. Finally, we will demonstrate how bracket algebra can be used to derive nice and short proofs for facts from elementary geometry, such as the typical triangle theorems. Derivations in a similar spirit may be found in the beautiful book on projective geometry by W. Blaschke [6] and in the book on hyperbolic geometry by M.J. Greenberg [49].

## 19.1 Mirror Images

We begin with the geometric construction (and characterization) of the mirror image of a point with respect to a line. Given the point  $p$  and the reflection line  $l$ , the following seven-step construction constructs the Euclidean mirror image  $p'$  of  $p$  with respect to  $l$ :

- 1:  $l_{p,I} = \mathbf{join}(p, I)$ ;
- 2:  $l_{p,J} = \mathbf{join}(p, J)$ ;
- 3:  $a = \mathbf{meet}(l_{p,I}, l)$ ;
- 4:  $b = \mathbf{meet}(l_{p,J}, l)$ ;
- 5:  $l_{p',J} = \mathbf{join}(a, J)$ ;
- 6:  $l_{p',I} = \mathbf{join}(b, I)$ ;
- 7:  $p' = \mathbf{meet}(l_{p',I}, l_{p',J})$ .

The situation is illustrated in Figure 19.1 on the left. There the points  $I$  and  $J$  have, as usual for our pictures, been moved to a finite position. The correctness of the construction can be proven by verifying two characteristic properties of the Euclidean mirror image: The line  $g = \mathbf{join}(p, p')$  must be orthogonal to  $l$  and the intersection  $m$  of these two points must be the midpoint of  $p$  and  $p'$ .



**Fig. 19.1** Constructing the mirror image of a point.

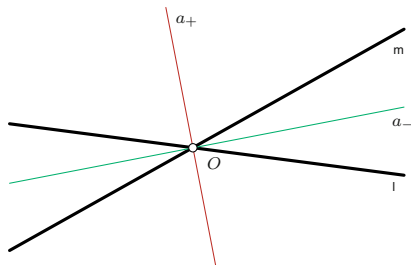
To verify the first property we refer to our characterization of orthogonality from Section 18.6. Let  $l_\infty$  be the line at infinity. Let  $L = \text{meet}(l, l_\infty)$  and  $G = \text{meet}(g, l_\infty)$  be the infinite points of  $l$  and  $g$ . We have to show that the point pairs  $(L, G)$  and  $(I, J)$  form a harmonic point set. This can easily be seen from the drawing in Figure 19.1 (right), where a few auxiliary points and lines have been added to the construction. There are exactly six lines in addition to the line at infinity. These six lines form our witness construction (compare Figure 5.1) that the point pairs  $(L, G)$  and  $(I, J)$  indeed form a harmonic quadruple.

The second part of the statement is that  $m$  forms the midpoint of  $p$  and  $p'$ . This can be shown by considering four points on the line  $g$ . If the pairs  $(p, p')$  and  $(m, G)$  form a harmonic set, then  $m$  is the midpoint of  $p$  and  $p'$  (recall that  $G$  was an infinite point). Also this harmonic relation can be directly read off from the picture. All lines except for  $g$  form the witness for the harmonic set construction.

## 19.2 Angle Bisectors

Our next example will demonstrate how to calculate the pair of angle bisectors of a given pair of lines. It is a good exercise in the kind of projective/Euclidean thinking we propagate in this chapter. All calculations are straightforward; nevertheless, the result is somewhat surprising and remarkably simple.

Let  $l$  and  $m$  be two lines and let  $a$  be an angle bisector of the two lines. We assume that  $l$  and  $m$  are not parallel, i.e., their intersection  $O$  does not lie on the line at infinity. Theorem 18.9 taught us that the angle between



**Fig. 19.2** Angle bisectors of two lines.

two lines can be calculated as  $\frac{1}{2i} \cdot \ln((L, M; I, J))$  with  $L$  and  $M$  being the intersections of  $l$  and  $m$  with the line at infinity  $l_\infty$ . Let  $A$  be the intersection of  $a$  with  $l_\infty$ . Then  $a$  is an angle bisector if and only if

$$(L, A; I, J) = (A, M; I, J). \quad (19.1)$$

Finding the correct position for  $A$  solves the problem of calculating the angle bisector: it is then simply **join**( $A, O$ ).

Now let  $P$  be an arbitrary point not on  $l_\infty$  that we can use to express the cross-ratios in terms of  $3 \times 3$  determinants. Then on the bracket level, (19.1) reads

$$\frac{[P, L, I][P, A, J]}{[P, L, J][P, A, I]} = \frac{[P, A, I][P, M, J]}{[P, A, J][P, M, I]},$$

or equivalently,

$$[P, L, I][P, M, I][P, A, J]^2 = [P, M, J][P, L, J][P, A, I]^2.$$

The point  $A$  is on  $l_\infty = \mathbf{join}(I, J)$ . Thus for suitable  $\lambda$  and  $\mu$  we have  $A = \lambda I + \mu J$ . Inserting this into the bracket expression, we get

$$[P, L, I][P, M, I][P, \lambda I + \mu J, J]^2 = [P, M, J][P, L, J][P, \lambda I + \mu J, I]^2.$$

And after expanding and removing brackets with identical letters, we have

$$[P, L, I][P, M, I][P, \lambda I, J]^2 = [P, M, J][P, L, J][P, \mu J, I]^2.$$

Extracting  $\lambda$  and  $\mu$  leads to

$$\lambda^2 [P, L, I][P, M, I][P, I, J]^2 = \mu^2 [P, M, J][P, L, J][P, J, I]^2.$$

Thus after canceling  $[P, J, I]^2$ , we get

$$\lambda^2 [P, L, I][P, M, I] = \mu^2 [P, M, J][P, L, J],$$

and a suitable choice for  $\lambda$  and  $\mu$  is

$$\lambda = \pm \sqrt{[P, L, J][P, M, J]} \quad \text{and} \quad \mu = \pm \sqrt{[P, L, I][P, M, I]}.$$

Inserting this into  $A = \lambda I + \mu J$ , we get the nicely symmetric formula

$$A_{\pm} = \sqrt{[P, L, J][P, M, J]} \cdot I \pm \sqrt{[P, L, I][P, M, I]} \cdot J.$$

The choice of the sign corresponds to the fact that two lines in general have two angle bisectors. It is also easy to check that the two possible angle bisectors are orthogonal to each other, since for an arbitrary pair of points of the forms  $A_+ = \lambda I + \mu J$  and  $A_- = \lambda I - \mu J$  we get

$$\begin{aligned} (A_+, A_-; I, J)_P &= \frac{[PA_+, I][PA_-, J]}{[PA_+, J][PA_-, I]} \\ &= \frac{[P\lambda I + \mu J, I][P\lambda I - \mu J, J]}{[P\lambda I + \mu J, J][P\lambda I - \mu J, I]} \\ &= \frac{[P\mu J, I][P\lambda I, J]}{[P\lambda I, J][P, -\mu J, I]} \\ &= -1. \end{aligned}$$

Summarizing these results, we have the following theorem:

**Theorem 19.1.** *With the notation as above, the angle bisectors of  $l$  and  $m$  may be calculated by  $\text{join}(A, O)$  with*

$$A_{\pm} = \sqrt{[P, L, J][P, M, J]} \cdot I \pm \sqrt{[P, L, I][P, M, I]} \cdot J.$$

*The two angle bisectors are perpendicular to each other.*

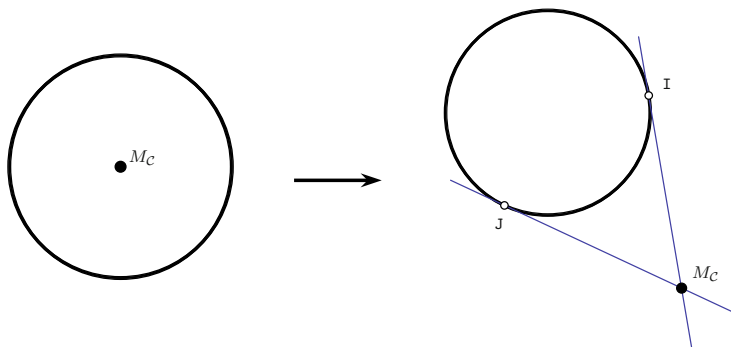
It is also interesting to see what being an angle bisector means algebraically. We will consider the slightly more general case in which we want to test whether for four lines  $l, m, a, b$  the angle enclosed between  $l$  and  $a$  equals the angle enclosed between  $b$  and  $m$ . Setting  $a = b$ , we obtain the special case of an angle bisector. We again let  $L, M, A, B$  be the corresponding intersections of the lines with  $l_{\infty}$ . From Laguerre's formula we obtain the condition

$$(L, A; I, J) = (B, M; I, J).$$

We may permute the entries on the left and right side, of this equation consistently and still obtain the same condition. For instance, we get

$$(L, I; A, J) = (B, I; M, J).$$

Again using a suitably generic point  $P$  distinct from  $l_{\infty}$ , we may write the above equation on the level of brackets as



**Fig. 19.3** Construction of the center of a circle.

$$\frac{[P, L, A][P, I, J]}{[P, L, J][P, I, A]} = \frac{[P, B, M][P, I, J]}{[P, B, J][P, I, M]},$$

or after cancellation of  $[P, I, J]$  and shuffling denominators,

$$[P, L, A][P, B, J][P, I, M] = [P, L, J][P, B, M][P, I, A].$$

Comparing this condition with the characterizing equation for quadrilateral sets from Section 8.2, we see that on a projective level we get the condition that  $(L, M; B, A; I, J)$  forms a quadrilateral set.

### 19.3 Center of a Circle

We now deal with the problem of constructing (or calculating) the center of a given circle in projective terms. The construction is of striking simplicity and we will give a geometric as well as an algebraic proof of it.

**Theorem 19.2.** *Let  $\mathcal{C}$  be a circle. Then the tangents to  $\mathcal{C}$  at the points  $I$  and  $J$  intersect in the center of  $\mathcal{C}$ .*

Before presenting the proofs we will highlight a few fine points of this construction. First of all it must be mentioned that since  $\mathcal{C}$  is a circle, the points  $I$  and  $J$  are incident with it. Thus it is totally feasible to speak of the tangents at the points  $I$  and  $J$ . Secondly, both tangents turn out to be completely complex objects (one would never “see” them in a real picture). However, they are complex conjugates such that their intersection is again a real point. Figure 19.3 illustrates the construction with  $I$  and  $J$  located at finite real positions.



Algebraically, this construction can also be translated into a very simple calculation. Let  $Q$  be the matrix of the quadratic form that represents  $\mathcal{C}$ . The tangents at  $I$  and  $J$  can be calculated by  $Q \cdot I$  and  $Q \cdot J$ . Their intersection is simply

$$M_{\mathcal{C}} = (Q \cdot I) \times (Q \cdot J).$$

*Algebraic proof:* A simple calculation verifies that  $M_{\mathcal{C}}$  is indeed the center of the circle. We assume that  $\mathcal{C}$  has a center with Euclidean coordinates  $(m_x, m_y)^T$ . The circle equation then is of the form

$$(x - m_x)^2 + (y - m_y)^2 = r^2.$$

The corresponding matrix of the quadratic form is

$$Q = \begin{pmatrix} 1 & 0 & -m_x \\ 0 & 1 & -m_y \\ -m_x & -m_y & \alpha \end{pmatrix}$$

for a suitable  $\alpha$ . By expanding  $(QI) \times (QJ)$ , we get

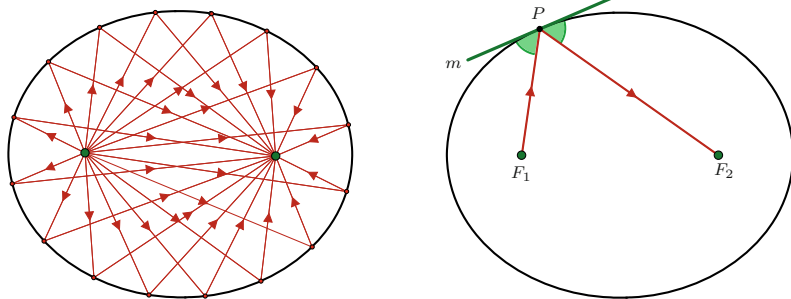
$$\begin{aligned} (QI) \times (QJ) &= \begin{pmatrix} 1 & 0 & -m_x \\ 0 & 1 & -m_y \\ -m_x & -m_y & \alpha \end{pmatrix} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & -m_x \\ 0 & 1 & -m_y \\ -m_x & -m_y & \alpha \end{pmatrix} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -i \\ 1 \\ im_x - m_y \end{pmatrix} \times \begin{pmatrix} i \\ 1 \\ -im_x - m_y \end{pmatrix} \\ &= \begin{pmatrix} -2im_x \\ -2im_y \\ -2i \end{pmatrix} \\ &= -2i \begin{pmatrix} m_x \\ m_y \\ 1 \end{pmatrix}. \end{aligned}$$

This proves that this point is the center of the circle.  $\square$

An even simpler geometric argument can be used as well to show the midpoint property. For this recall that for three collinear points  $A, B, M$  on a line  $l$  we can characterize the fact that  $M$  is the midpoint of  $A$  and  $B$  by the condition that  $(A, B; M, L)$  are in harmonic position (here  $L$  is again the intersection of  $l$  and  $l_{\infty}$ ). Thus the center  $M$  of a circle  $\mathcal{C}$  is characterized by the following property. For a line  $l$  that passes through  $M$  let  $A$  and  $B$  be the intersections of  $l$  and  $\mathcal{C}$  and let  $L$  be the intersection of  $l$  and  $l_{\infty}$ . The point  $M$  is the center of the circle if for any such line,  $(A, B; M, L)$  are in harmonic position.

*Geometric proof:* Let  $M$  be the intersection of the two tangents at  $I$  and  $J$ . The only thing we have to prove is that for an arbitrary line  $l$  through  $M$

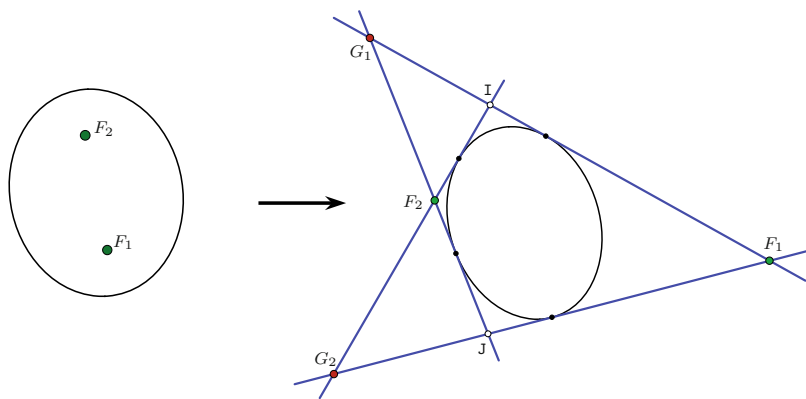




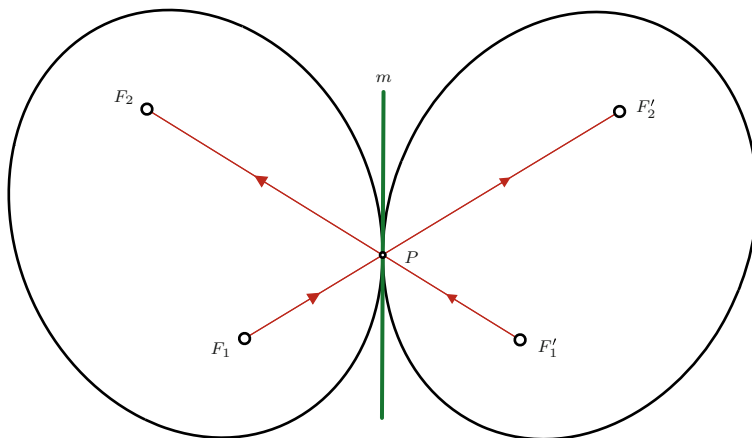
**Fig. 19.5** Definition of the foci of a conic.

mirror. Thus the supporting lines of the original ray and the reflected one are mirror symmetric with respect to this tangent. A similar characterization also holds for a hyperbola. A hyperbola also has two foci, with the property that if a ray is emitted from  $F_1$ , then the supporting line of the reflected ray is incident with  $F_2$ . In the case of a circle the two foci will coincide with the center.

We will now give a projective construction involving  $I$  and  $J$  that creates focal points with exactly the property described above. In order to get a nondegenerate construction we will assume (for the moment) that the conic under consideration  $\mathcal{C}$  is not a circle. We will see later on that the construction also covers this case. If  $\mathcal{C}$  is not a circle, then  $I$  and  $J$  are not coincident with it. Thus we can draw two tangents  $i_1$  and  $i_2$  from  $I$  at the conic  $\mathcal{C}$ . Similarly, we can draw two tangents  $j_1$  and  $j_2$  from  $J$ . The intersections of these tangents turn out to be foci (see Figure 19.6).



**Fig. 19.6** Construction of foci using  $I$  and  $J$ .



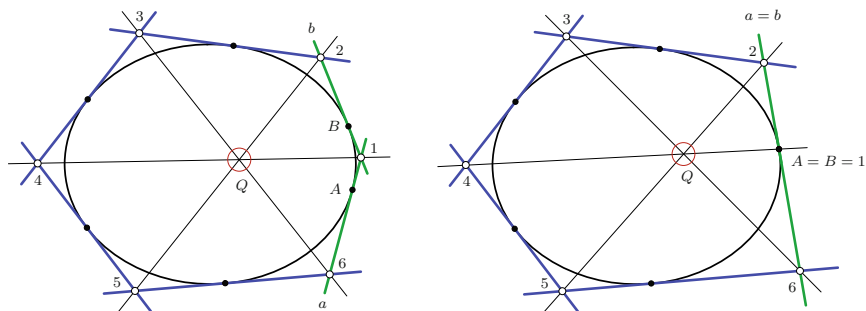
**Fig. 19.7** Reflection property of foci.

**Theorem 19.3.** *The pairs of points  $(F_1, F_2) = (\text{meet}(i_1, j_1), \text{meet}(i_2, j_2))$  and  $(G_1, G_2) = (\text{meet}(i_1, j_2), \text{meet}(i_2, j_1))$  form two pairs of foci of the conic  $\mathcal{C}$ .*

*Proof.* The proof is nontrivial in the sense that it needs several constructions and facts that we encountered before and requires some preparation. It relies on the characterization of the mirror image of a point given in Section 19.1. Furthermore, we need a degenerate version of Brianchon's theorem (the dual of Pascal's theorem) that we introduced in Section 10.6. And we need a reasonably simple incidence-theoretic characterization of the focus property.

Let us start with the latter. For this consider once more the situation in Figure 19.5 on the right. The reflection property states that a ray emitted from  $F_1$  and reflected by the focus hits  $F_2$ . One can reinterpret this in the following way. Take the tangent  $m$  at point  $P$  where the ray hits the conic. Reflecting the two foci at the tangent  $m$  produces two images  $F_1'$  and  $F_2'$ . The reflection property is equivalent to saying that  $F_1$ ,  $P$ , and  $F_2'$  are collinear independently of the choice of the specific ray. Figure 19.7 illustrates this property. Thus one can ensure that  $F_1$  and  $F_2$  are foci by verifying the following property: take an arbitrary point  $P$  on the conic and the tangent  $m$  at this point. Reflect  $F_2$  at this tangent to obtain the mirror image  $F_2'$ . Then  $F_1$  and  $F_2$  are foci if independently of the choice of  $P$ , the points  $F_1, F_2, P$  are collinear.

As a second ingredient we need Brianchon's theorem. A picture of this theorem is given in Figure 19.8 on the left. This theorem states that if we draw six tangents  $t_1, \dots, t_6$  to a conic and cyclically (indices modulo 6) intersect  $t_i$  and  $t_{i+1}$ , we get points  $1, \dots, 6$  with the property that the lines **join**(1, 4), **join**(2, 5), and **join**(3, 6) are concurrent. We will need a limit case of this theorem for which two adjacent tangents (green in the picture) coincide. This



**Fig. 19.8** Brianchon's theorem, and a limit case of it.

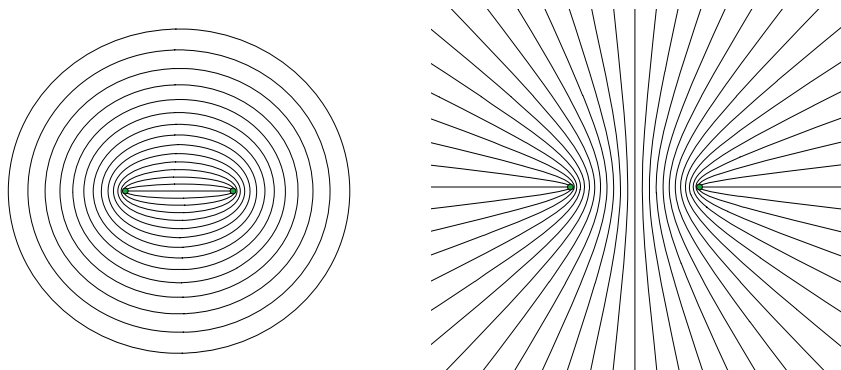
situation is shown in Figure 19.8 on the right. Notice that the corresponding intersection point of the two tangents in the limit case becomes the point at which the tangent touches the conic.

Now we are almost done. Consider Figure 19.9. There you see the construction of the presumed foci by intersecting the tangents of I and J to the conic. Let us focus on one pair of foci, namely  $F_1$  and  $F_2$ . Now consider an arbitrary point  $P$  on the conic and the corresponding tangent  $m$ . Using the construction of Section 19.1, we construct the reflection of  $F_2$  of one of the foci with respect to the line  $m$ . For this we connect I and J with  $F_2$  (the lines are already there) in order to get the two intersections  $X$  and  $Y$  of these lines with  $m$ . Then we connect  $X$  and  $Y$  crosswise with I and J. The intersection of these two lines is the reflected focus  $F'_2$ . Inspecting the drawing, we see that we have just constructed the hypotheses of the degenerate version of Brianchon's theorem. The theorem tells us that  $F_1$ ,  $F'_2$ , and  $P$  are collinear. This proves that  $F_1$  and  $F_2$  are foci. By symmetry of the construction a similar argument also holds for the pair  $G_1$  and  $G_2$ .  $\square$

Again a few subtle matters deserve to be mentioned.

- First of all it is clear that all our drawings used in the proof are just *real* counterparts of configurations that mainly consist of *complex* objects. If I and J are at their proper positions, almost everything in the construction becomes complex. Nevertheless, we can argue with the real pictures, since the theorems we used (in particular Brianchon's theorem) hold also for projective planes over the complex numbers.
- The second and perhaps most striking fact is the problem that our construction generates *four* foci, whereas commonly one observes only two foci for an ellipse or a hyperbola. The resolution of this problem again lies in the relation of complex to real elements. The tangents  $j_1$  and  $j_2$  related to J are the complex conjugates to the tangents  $i_1$  and  $i_2$  related to I. If we intersect complex conjugates, we get a real meet. Thus out





**Fig. 19.10** Confocal ellipses and confocal hyperbolas.

general, there are two different solutions to the above construction problem. This is also justified if we look at the projective characterization of foci. Assume that a pair of foci  $F_1, F_2$  is given. For any conic with these two foci, tangents through  $I$  and  $J$  pass through  $F_1$  and  $F_2$ . We can explicitly construct these tangents without knowing a specific conic. We simply have to construct

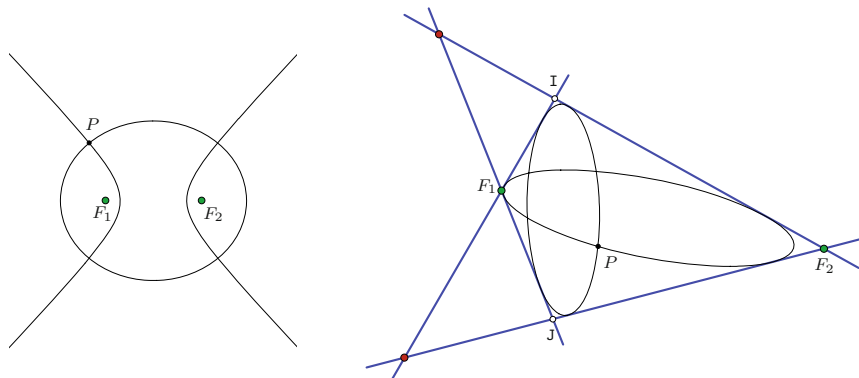
$$t_1 = \text{join}(F_1, I), \quad t_2 = \text{join}(F_1, J), \quad t_3 = \text{join}(F_2, I), \quad t_4 = \text{join}(F_2, J).$$

Now the problem reduces to finding a conic that is simultaneously tangent to  $t_1, \dots, t_4$  and passes through the given point  $P$ . This problem, however, is dual to the problem of constructing a conic through four points and tangent to a line that we attacked in Section 11.6. There we showed that this problem leads to a quadratic equation and derived a method to explicitly give the two solutions. Figure 19.11 illustrates the existence of the two solutions to the construction problem.

All in all, the construction problem may then be expressed in the following (mainly algebraic) algorithm:

1. Construct the four lines  $t_1, \dots, t_4$  as described above.
2. Consider the homogeneous coordinates  $t_1, \dots, t_4$  as *point* coordinates and consider the homogeneous coordinate  $P$  as *line* coordinate. Use them as input for the method of Section 11.6 to find matrices  $A_1, A_2$  that describe two conics through  $t_1, \dots, t_4$  and tangent to  $P$ .
3. Invert the matrices  $A_1$  and  $A_2$  to get two matrices describing conics with the desired properties.

This method also works for the limit case in which the two foci  $F_1$  and  $F_2$  coincide. In this case one of the solutions turns out to be a circle with center  $F_1 = F_2$ . Clearly, there are simpler ways to calculate a circle with prescribed center and boundary point. Nevertheless, it is nice to see how this fits into the



**Fig. 19.11** The two conics with foci  $F_1, F_2$  through a point  $P$ .

more general framework. Later on we will return to the problem of calculating such circles in the even broader context of non-Euclidean geometries.

## 19.6 Triangle Theorems

So far we have used the projective setup for Euclidean geometry mainly for solving construction problems. We will now consider a few typical Euclidean theorems, demonstrate how they translate to a projective framework, and show how they may be proved projectively. One of the main messages of this section is that every Euclidean theorem can be translated into a projective theorem. In the translation process every relation involving angles or distances must simply be translated into the corresponding projective terms. Very often, the translation process sheds new light on the original problem. Sometimes it unifies a bunch of theorems and shows that they are actually birds of a feather. Conversely, there are often several different ways to find Euclidean interpretations for projective theorems, depending on which points are playing the roles of  $I$  and  $J$ . We already encountered one example of the projective explanation of a Euclidean theorem in Section 18.2, where we proved Miquel's theorem on six circles by proving a corresponding theorem for six conics that meet in a pair of points.

In this section we will focus on extremely simple theorems about triangles: those Euclidean theorems that are already treatable on the level of secondary school mathematics. Nevertheless, the projective viewpoint gives some new perspectives on them and very often leads to interesting generalizations or interesting proofs. (The reader should not expect these proofs to be simpler than the secondary-school-level proofs, but he/she may appreciate how



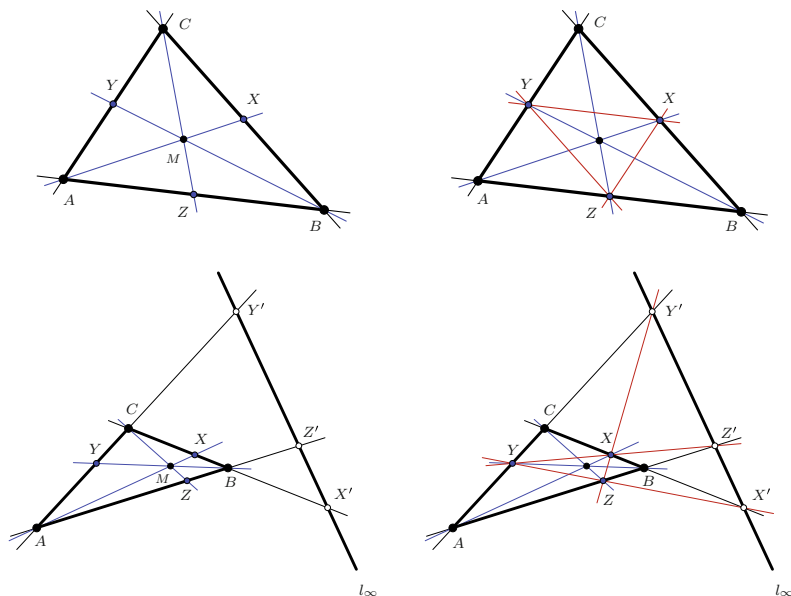


Fig. 19.12 Medians—Euclidean and projective.

everything fits together into our general setup.) As throughout the entire book, we will try to present a variety of proof methods and connect them to different parts of earlier chapters.

**The medians of a triangle meet in a point:** What does this theorem mean on a projective level? The midpoint of a segment  $(A, B)$ <sup>1</sup> is the point  $M$  such that  $(A, B; M, L)$  are in harmonic relation. Here  $L$  is the infinite point of the line supporting the segment (compare Lemma 5.3). Thus the theorem that states that *the medians of a triangle meet in a point* is (from a projective perspective) a theorem about harmonic relations. It translates to the following fact.

**Theorem 19.4.** *Let  $A, B, C$ , be three points in the projective plane and let  $l$  be a line not incident with any of these points. Furthermore, let  $Z', X', Y'$  be the intersections of the lines  $(A, B)$ ,  $(B, C)$ ,  $(C, A)$  with  $l$  and let  $X, Y, Z$  be such that  $(A, B; Z, Z')$ ,  $(B, C; X, X')$ , and  $(C, A; Y, Y')$  are harmonic quadruples. Then the lines  $(A, X)$ ,  $(B, Y)$ , and  $(C, Z)$  are concurrent.*

*Proof.* One elegant way to formulate the proof is to set it in the context of the Ceva and Menelaus configurations we dealt with in Section 15.4 and 15.5.

<sup>1</sup> In what follows we will frequently write  $(A, B)$  as shorthand for the segment from  $A$  to  $B$ . Whenever no confusion can arise we will use  $(A, B)$  as shorthand for the supporting line  $\text{join}(A, B)$  of this segment.

For this we consider the triangle  $A, B, C$  and points  $X, Y, Z$  on its edges (with incidences as in the theorem). The points  $X', Y', Z'$  are the corresponding harmonic points on the triangle edges. The theorem states that if  $X', Y', Z'$  are the edge points of a Menelaus configuration (they are all on  $l_\infty$ ), then  $X, Y, Z$  are the edge points of a Ceva configuration;  $X', Y', Z'$  being the edge points of a Menelaus configuration implies (with oriented lengths along each of the triangle edges)

$$\frac{|A, Z'|}{|Z', B|} \cdot \frac{|B, X'|}{|X', C|} \cdot \frac{|C, Y'|}{|Y', A|} = -1.$$

The harmonic conditions imply

$$\frac{|A, Z'|}{|A, Z|} \cdot \frac{|B, Z|}{|B, Z'|} = -1, \quad \frac{|B, X'|}{|B, X|} \cdot \frac{|C, X|}{|C, X'|} = -1, \quad \frac{|C, Y'|}{|C, Y|} \cdot \frac{|A, Y|}{|A, Y'|} = -1.$$

Multiplying all four expressions and canceling lengths if possible yields

$$\frac{|A, Z|}{|Z, B|} \cdot \frac{|B, X|}{|X, C|} \cdot \frac{|C, Y|}{|Y, A|} = -1,$$

the algebraic condition for  $X, Y, Z$  being the edge points of a Menelaus configuration.  $\square$

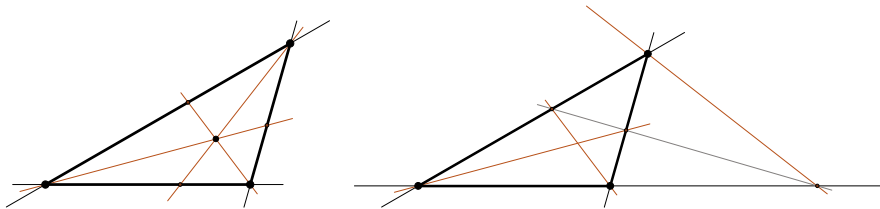
Figure 19.12 on the left top row illustrates the usual Euclidean situation. The left bottom row shows the corresponding projective situation. The line incident to  $X', Y', Z'$  plays the role of the line at infinity  $l_\infty$ .

*Remark 19.1.* It is interesting to notice that the argument in the proof works as well the other way around. If the lines  $(A, X)$ ,  $(A, Y)$ ,  $(C, Z)$  are concurrent, then the corresponding harmonic points  $X', Y', Z'$  are automatically collinear. By this (projectively speaking), for every fixed choice of  $A, B, C$ , and  $M$  in the projective plane we get a kind of relative line at infinity with respect to which  $M$  plays the role of the intersection point of the median.

*Remark 19.2.* There is another interesting fact that has an immediate projective interpretation. The connection of two midpoints in the Euclidean drawing is parallel to the triangle side that is not incident with either of them. In the projective interpretation this means that the point triples  $(X, Y, Z')$ ,  $(Y, Z, X')$ , and  $(Z, X, Y')$  are collinear (see right column of the picture). To prove this on a projective level, observe that the three proposed collinearities are part of witness configurations for the harmonic relations along the sides. The reader is invited to derive a geometric proof of Theorem 19.4 based on this observation.

It should also be mentioned that the configuration is projectively unique. As soon as the vertices of the triangle and the intersection of the medians  $M$  are fixed, the points





**Fig. 19.14** Two cases for angle bisectors.

$$\begin{aligned}
 [\mathbf{I}, R][J, U] &= -[\mathbf{I}, U][J, R] \\
 [\mathbf{I}, S][J, V] &= -[\mathbf{I}, V][J, S] \\
 [\mathbf{I}, T][J, W] &= -[\mathbf{I}, W][J, T]
 \end{aligned}
 \tag{19.3}$$

Furthermore, all right angles are equal to each other. At the end of Section 19.2 we saw that the angle between  $g$  and  $h$  is identical to the angle between  $l$  and  $m$  if for the corresponding infinite points  $G, H, L, M$  the collection  $(G, L; H, M; I, J)$  is a quadrilateral set. Applying this to the altitudes in a triangle, we in addition get the quadrilateral sets  $(\mathbf{I}, J; R, S; U, V)$ ,  $(\mathbf{I}, J; S, T; V, W)$  and  $(\mathbf{I}, J; T, R; W, U)$ . This implies the equations

$$\begin{aligned}
 [\mathbf{I}, V][R, J][U, S] &= [\mathbf{I}, S][R, V][U, J] \\
 [\mathbf{I}, W][S, J][V, T] &= [\mathbf{I}, T][S, W][V, J] \\
 [\mathbf{I}, U][T, J][W, R] &= [\mathbf{I}, R][T, U][W, J]
 \end{aligned}
 \tag{19.4}$$

Multiplying all left and right sides of the equations in (19.3) and (19.4), taking care of the alternating determinant law, and canceling terms that occur on both sides, we obtain (19.2). This proves the altitude theorem projectively.

We briefly mention another projective approach to the theorem from a transformational point of view (we already met this approach in Section 8.6). Rotating a Euclidean line by  $90^\circ$  induces a map on the line at infinity. We associate each infinite point  $A$  of a line  $a$  to the infinite point  $\tau(A) = B$  of the line rotated by  $90^\circ$ . This map is well-defined. In fact, it is a projective transformation on  $l_\infty$  and it is an involution, since  $\tau(\tau(A)) = A$ . In Theorem 8.2 we showed that if  $R, S, T$  are three distinct points on  $l_\infty$  and  $U = \tau(R), V = \tau(S), W = \tau(T)$  are their images under a projective involution, then  $(R, U; S, V; T, W)$  forms a quadrilateral set. From this the altitudes theorem follows easily. Moreover, a simple calculation shows that  $\mathbf{I}$  and  $\mathbf{J}$  are the fixed points of  $\tau$ . Theorem 8.4 tells us that (as we know)  $(\mathbf{I}, J; R, U)$ ,  $(\mathbf{I}, J; S, V)$ ,  $(\mathbf{I}, J; T, W)$  are in harmonic position.

**The angle bisectors of a triangle meet in a point:** Also the angle bisectors in a triangle meet in a point. Again in the projective interpretations the points  $\mathbf{I}$  and  $\mathbf{J}$  must play a role, since angles are involved. As before, we

encode the relevant line slopes by the intersection points with the line at infinity. For the labeling we again refer to Figure 19.13. If line  $W$  is an angle bisector of the lines  $r$  and  $s$ , this implies (according to Section 19.2) that  $(I, J; R, S; W, W)$  is a quadrilateral set. Applying this fact to all three angle bisectors, we obtain the three conditions

$$\begin{aligned} [I, W][R, J][W, S] &= [I, S][R, W][W, J] \\ [I, U][S, J][U, T] &= [I, T][S, U][U, J] \\ [I, V][T, J][V, R] &= [I, R][T, V][V, J] \end{aligned} \quad (19.5)$$

One might expect that multiplying left and right sides and canceling as usual gives the desired result. Unfortunately, not a single bracket cancels. However, there is a nice trick that provides three more equations that make things cancel. The roles of  $I$  and  $J$  are interchangeable in the quadrilateral set relations. This implies that we also get the following three equations:

$$\begin{aligned} [J, W][R, I][W, S] &= [J, S][R, W][W, I] \\ [J, U][S, I][U, T] &= [J, T][S, U][U, I] \\ [J, V][T, I][V, R] &= [J, R][T, V][V, I] \end{aligned} \quad (19.6)$$

Multiplying left and right sides of (19.5) and (19.6) and canceling brackets now leaves us with the equation

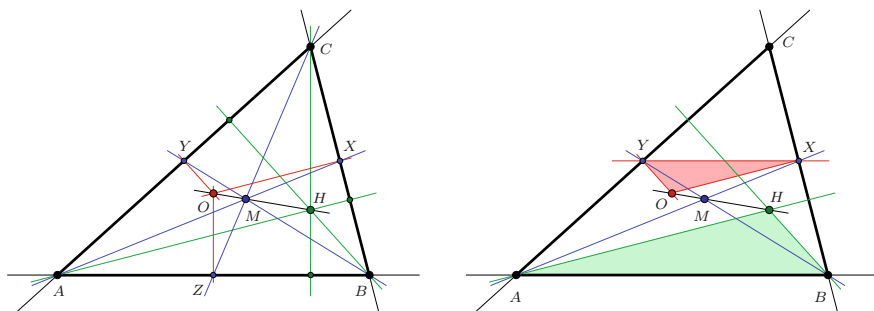
$$([W, S][U, T][V, R])^2 = ([R, W][S, U][T, V])^2.$$

Thus we can conclude that either

$$\begin{aligned} [R, V][S, W][T, U] &= +[R, W][S, U][T, V] \quad \text{or} \\ [R, V][S, W][T, U] &= -[R, W][S, U][T, V] \end{aligned}$$

is true. The first equation describes the quadrilateral set equation we are looking for. The second equation encodes the fact that the intersections  $\mathbf{meet}(r, u)$ ,  $\mathbf{meet}(s, v)$ , and  $\mathbf{meet}(t, w)$  are collinear. In fact, with the given hypotheses this is the best we could hope for, since any pair of lines has *two* angle bisectors. If we made a proper choice, they are collinear; otherwise (in half of the possible cases), we get the collinearity condition. Figure 19.14 illustrates the two possible cases for one given triangle.

**The Euler line:** The intersection of the medians  $M$ , the intersection of the altitudes  $H$  (called the *orthocenter*), and the circumcenter  $O$  are always collinear. They lie on the so-called Euler line (see Figure 19.15 (left)). From a projective viewpoint it turns out that this theorem is a direct consequence of Desargues's theorem. To see this, first observe that for the Euler line theorem it is not necessary to draw all *three* altitudes, all *three* medians, and all *three* perpendicular bisectors. Two of each suffice to determine the three intersection points  $O$ ,  $M$ , and  $H$ . Figure 19.15 (right) shows a drawing in



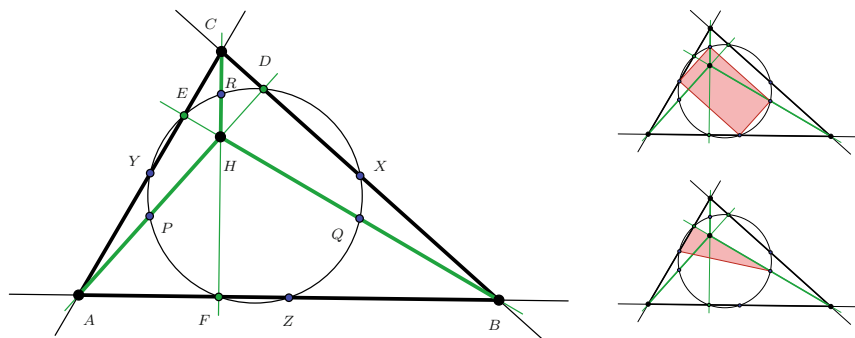
**Fig. 19.15** The Euler line.

which only two of each type of lines are drawn. Furthermore, a line between the two midpoints  $X$  and  $Y$  is drawn. From our considerations about medians (Remark 19.2) we know that this line is parallel to  $A, B$ . Also, the perpendicular bisectors are parallel to the sides of the altitudes associated to the same side of the triangle. Hence, the three sides of the red triangle are parallel to the three sides of the green triangle. So they meet at three points that all lie on the line at infinity. Desargues's theorem now tells us that in this case the two triangles must be perspective with respect to a point. Thus the line  $\text{join}(O, H)$  must pass through the intersection of  $\text{join}(A, X)$  and  $\text{join}(B, Y)$ .

## 19.7 Hybrid Thinking

The world is neither Euclidean nor projective—it simply *is*. We may look at it only through certain *filters of perception*. In the final section of this chapter we want to demonstrate how a kind of *hybrid thinking*, where one takes a projective or Euclidean viewpoint, whichever seems to be more appropriate, can lead to interesting proofs and generalizations of theorems. Again, one could fill a whole book with this. We here will limit ourselves to one example only, namely the *nine-point circle*, which is one of the more advanced theorems in elementary triangle geometry.<sup>2</sup> The theorem is kind of surprising. However,

<sup>2</sup> In fact, I have to admit that I personally hate these overcrowded elementary geometric drawings that show up in many textbooks dealing with advanced triangle geometry. When I started to think about this section I thought, *It would be nice to include also something about the nine-point circle of a triangle*. Starting thinking on this, I was trapped! The mutual relations and correspondences are so nice, surprising, and overwhelming that I continued to produce exactly the same kind of overcrowded pictures over and over on my computer. The situation got even worse when I started to think about possible projectivizations of these relations. New theorems and relations, beautiful configurations, and nice relaxations showed up every minute, many more than would fit into single chapter



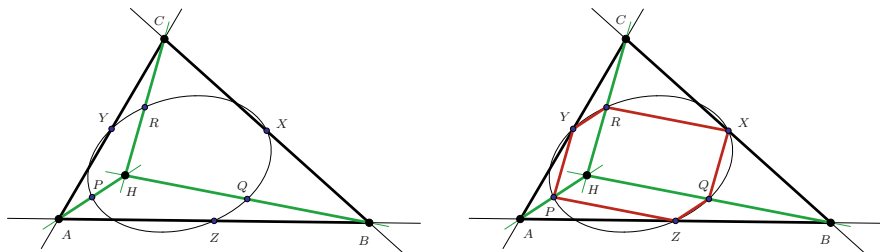
**Fig. 19.16** Nine-point circle.

it is not too difficult to prove. It states that in any triangle the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments that join the orthocenter  $H$  with each vertex all lie on a common circle (see Figure 19.16). A standard (Euclidean) proof technique of this result goes along the following lines: The crucial observation is that the line that joins the midpoints in a triangle is parallel to the third side of the triangle. (We will need this fact frequently later on and simply refer to it as  $(\mathcal{M})$ . This implies that  $(Y, R)$  as well as  $(Z, Q)$  is parallel to  $(A, H)$ . Also  $(R, Q)$  is parallel to  $(B, C)$  and hence orthogonal to  $(Y, R)$ . Taking these statements together, one can conclude that  $(Y, R, Q, Z)$  is a rectangle. Thus its vertices lie on a circle. Similarly, one can prove that  $(X, Q, P, Y)$  and  $(Z, P, R, X)$  are rectangles. This implies that all midpoints (the blue points in the drawing)  $P, Q, R, X, Y, Z$  are on a circle. Since  $(Y, R, Q, Z)$  is a rectangle, the segment  $(Y, Q)$  is a diameter of the circle. Thales' theorem implies that also the foot point  $E$  (a green point) must be on the circle. Similarly,  $F$  and  $D$  are on the circle. The right side of Figure 19.16 illustrates the two ingredients necessary for this proof.

At first sight this argument is far from being projective. However, every single step can be carefully translated to a projective argument (try it), and we end up with a rather lengthy projective proof. This is not what we are aiming for. Instead, this time we are aiming for a closely related, though different, projective theorem. This theorem will show that many of the coincidences of the nine-point circle are still true after the hypotheses of the theorem are somehow relaxed. By this we want to demonstrate how one could approach a theorem with a *projective eye* and how this may lead to interesting Euclidean and projective variants, relaxations, and generalizations (for even more variants see also [95]). We try to use arguments, based on our (so far) well-understood projective concepts like *Desargues's theorem* (Section 15.1),

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of this book. Here I will present a small collection of some of them. I tried to separate different statements into several hopefully not so overcrowded pictures.



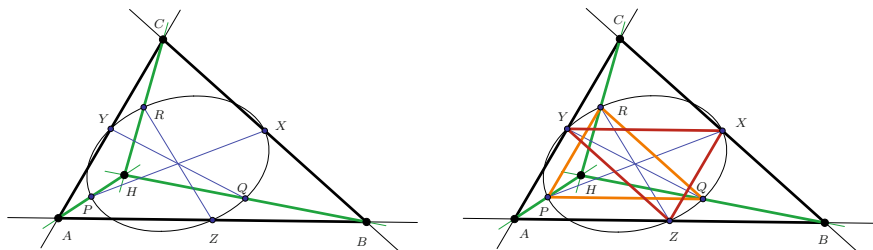
**Fig. 19.17** The midpoints are on a conic.

*Pascal's theorem* (Section 10.6), and *Hesse's transfer principle* (Section 10.5). Our fundamental relaxation is that the points  $H$  will no longer be required to be the orthocenter—it may just be an arbitrary point that is not on a line supporting the triangle's edges.

**Observation 1: Midpoints are on a conic.** From a projective point of view the first interesting relation occurs already on the level of the midpoints  $X, Y, Z, P, Q, R$  in the configuration. Constructing a midpoint is not dependent on the points  $I$  and  $J$ . It is an affine but not a genuinely Euclidean operation. Thus if we start with the points  $A, B, C$ , and  $H$ , we may suspect that even then the six midpoints of pairs of such points lie on a conic. Here  $H$  is assumed to be an arbitrary point and not necessarily the orthocenter. (We may not expect that the midpoints still lie on a circle, since  $I$  and  $J$  are not contained in the hypotheses.) In fact, this conjecture turns out to be true and not hard to prove at all. Figure 19.17 (left) shows the theorem and adds a few lines (red lines in the right picture) that immediately provide a proof. (We may think projectively and still draw Euclidean pictures and are aware of the special role of the line at infinity.) By fact  $(\mathcal{M})$  we know that  $(Y, R)$  and  $(A, H)$  are parallel (i.e., they meet at the line at infinity) and that  $(Z, Q)$  and  $(A, H)$  are parallel. Hence  $(Y, R)$  and  $(Z, Q)$  meet at the line at infinity. Similarly, the pair of lines  $(Y, P)$  and  $(X, Q)$  are parallel, as well as  $(Z, P)$  and  $(X, R)$ . In other words, in the drawing the red lines form a hexagon with opposite sides being parallel. This in turn can be used as the hypotheses for Pascal's theorem, which then implies that  $X, Y, Z, P, Q, R$  are on a conic. Observe that this argument used only the parallelism of the red and green lines (the former altitudes) in the picture.

**Observation 2: The hexagon is symmetric.** There is one more property that attracts attention when we look at Figure 19.17 (right). The hexagon is *point-symmetric*. In other words, if one joins opposite points of the hexagon, then the three resulting segments meet in the center of symmetry of the hexagon (see the blue lines in Figure 19.18, left). How can this be proved? Again a projective argument helps immediately. Similarly to what we did before, we can (now using parallelism to the black triangle edges and fact  $(\mathcal{M})$ )



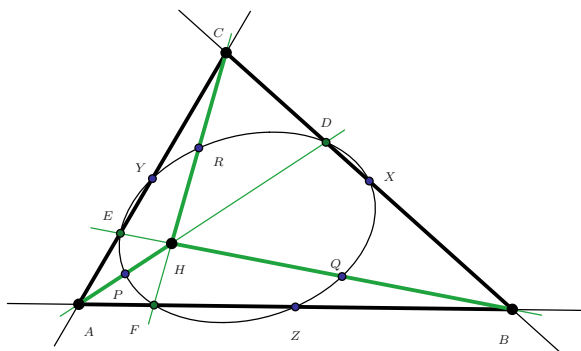


**Fig. 19.18** Segments between opposite midpoints intersect.

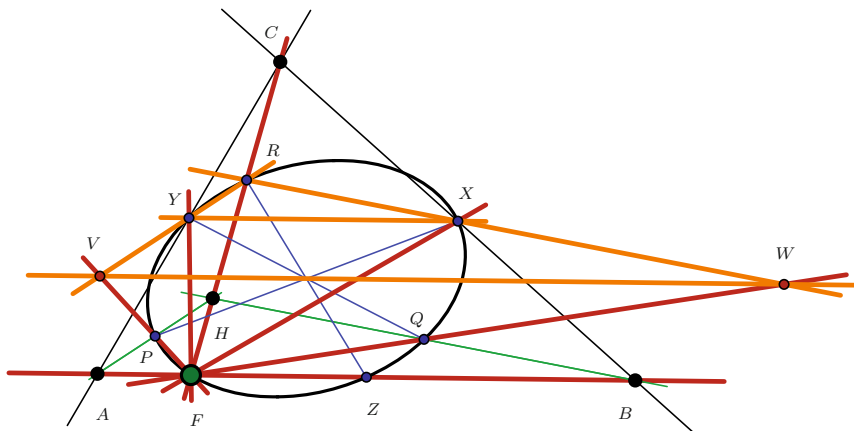
show that in Figure 19.18 (right) the sides of the orange triangle are parallel to the sides of the red triangle. Desargues's theorem implies that they are perspective with respect to each other. This forces that the blue lines meet in the point of perspectivity.

**Observation 3: Also the foot points are on the conic.** Now comes an amazing and remarkable observation. The conic through the six midpoints, passes also through the foot points  $D, E, F$  where the green lines (the former altitudes) hit the triangle edges. Figure 19.19 illustrates the effect. This statement is by far not as easy to prove as the previous ones. In the literature, only proofs involving *loci of polars with respect to pencils of conics* seem to be available (compare [95]). We here want to follow a more incidence-geometric approach and base the proof on Hesse's principle of transfer (see in particular Figure 10.10).

Hesse's principle of transfer states that if we have six points on a conic such that the lines spanned by pairs of them are concurrent, then the condition of a seventh point being on the conic is characterized by the fact that it *sees* the six points in a quadrilateral set relation. We are in the lucky



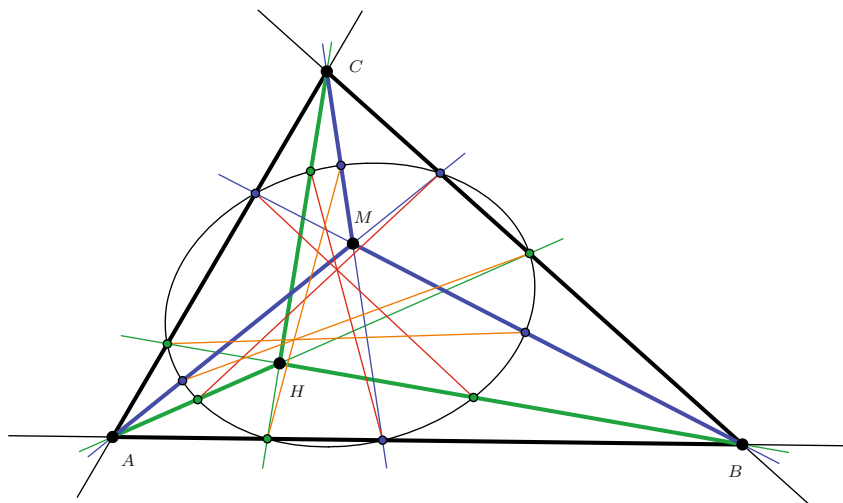
**Fig. 19.19** Foot points are on the conic.



**Fig. 19.20** Foot points are on the conic (proof).

situation that our midpoints  $X, Y, Z, P, Q, R$  are located in a way such that  $(X, P)$ ,  $(Y, Q)$ , and  $(Z, R)$  are concurrent. Thus we can prove that the foot point  $F$  is on the conic by showing that  $F$  sees the midpoints in a quad-set relation  $(X, P; Y, Q; Z, R)_F$ . To prove that this is indeed the case, consider Figure 19.20 (I apologize for the rather crowded picture). The red lines all pass through point  $F$ . They are the joins to the six segment midpoints that lie on the conic. To prove that  $F$  is on the conic, we must show that suitable pairs of them are in a quadrilateral set relation. The four orange lines form a corresponding witness configuration. Such a witness must satisfy the following requirements: *Take one point on each of the lines  $(F, Y)$ ,  $(F, R)$ , and  $(F, X)$ . (We take  $Y, R, X$  themselves.) Show that the intersections  $V = \text{meet}(\text{join}(F, P), \text{join}(Y, R))$ ,  $W = \text{meet}(\text{join}(F, Q), \text{join}(R, Y))$ , and  $U = \text{meet}(\text{join}(F, Z), \text{join}(X, Y))$ , are collinear.* The point  $U$  is located at the line at infinity, since  $(X, Y)$  and  $(A, B)$  are parallel (fact  $(\mathcal{M})$ ). Thus it remains to prove that  $(V, W)$  is parallel to the line  $l$  on which  $A, B, F$ , and  $Z$  lie. We also know that  $(P, Q)$  is parallel to  $l$ . Thus it suffices to prove the parallelism of  $(V, W)$  and  $(P, Q)$ . This, in turn, can be achieved by Desargues's theorem. We apply it to the two triangles  $(V, R, W)$  and  $(P, H, Q)$ , which are perspective to each other (seen from point  $F$ ). Thus the intersections of corresponding sides are collinear. We know that the sides  $(V, R)$ ,  $(P, H)$  are parallel as well as the sides  $(R, W)$  and  $(H, Q)$  (again by fact  $(\mathcal{M})$ ). Desargues's theorem now implies the parallelism of  $(V, W)$  and  $(P, Q)$ .

**What did we prove?** We did not prove the nine-point circle theorem in this way. What we proved is in one respect weaker, in the other, stronger. In our hypotheses we did not assume that  $H$  is the orthocenter. Still under this relaxation we could prove that the relevant nine points lie on a common



**Fig. 19.21** The fully projective version.

conic. If  $H$  is the orthocenter, then in addition this conic becomes a circle. Thus for this special situation we projectively get two additional incidences with  $I$  and  $J$ . For this reason the nine-point circle is sometimes also called the *eleven-point conic*.

Figure 19.21 shows another way to look at what we proved. There we consider a projective transformation of our theorem, in which  $M$  no longer is the (Euclidean) median. Points  $H$  and  $M$  play completely symmetric roles, and thus we may also construct the three additional points that we obtain when  $H$  plays the role of the median and  $M$  plays the role of the orthocenter. Also, these three points must lie on our conic. So we may even speak of a *twelve-point conic* in this purely projective scenario!

*Remark 19.3.* The reader may observe that the points in which the three red and the three orange lines meet, respectively, are also collinear with  $M$  and  $H$ . This is in fact related to a projective generalization of the Euler line.



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## Cayley-Klein Geometries

*It ain't necessarily so.*

*George and Ira Gershwin*

We started out developing projective geometry for two reasons: It was algebraically nice and it helped us to get rid of the treatment of many special situations that are omnipresent in Euclidean geometry. Then, to express Euclidean geometry in a projective setup, we needed the help of complex numbers, our special points I and J, cross-ratios, and Laguerre's formula. We now come to another pivot point in our explanations: We will see that our treatment of Euclidean geometry in a projective framework is only a special case of a variety of other reasonable geometries. One might ask what it means to be a *geometry* in that context. For us it means that there are notions of *points*, *lines*, *incidence*, *distances*, and *angles* with a certain reasonable interplay. Besides Euclidean geometry, among those geometries there are quite a few prominent examples, such as *hyperbolic geometry*, *elliptic geometry*, and *relativistic space-time geometry*.

In all these geometries the fundamental notions of measurement are based on a conic that plays the role of infinitely distant elements. The type of conic determines the type of geometry we get. So the classification of conics we provided in Section 9.5 will play a crucial role in this context.

It is difficult to recommend particular places in the mathematical literature where one can dive into this subject more deeply. Many modern texts are fairly abstract, and the classical texts may be difficult to access. I personally like best the classical book by Klein [68] and his original papers “On the so-called non-Euclidean geometry” from 1872 [66] and 1874 [67]. The first one is translated and reprinted in [125]. More modern treatments may be found in [91, 75]. A very accessible exposition may be found in [3].

## 20.1 I and J Revisited

As we saw in Section 18.7, the measurement of angles in Euclidean geometry is expressed by Laguerre's formula:

$$\angle(l, m) = \frac{1}{2i} \cdot \ln((L, M; I, J)),$$

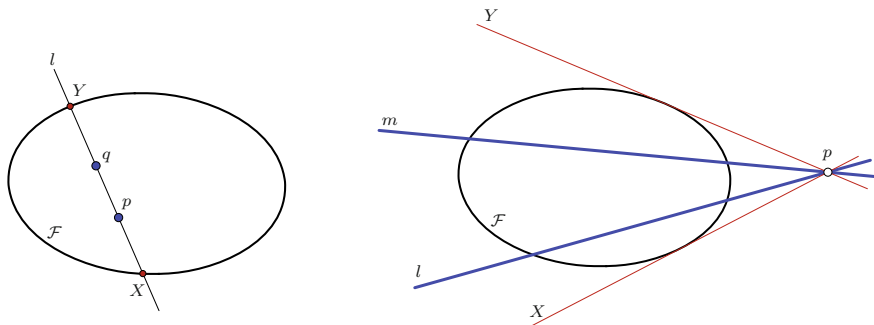
with  $L$  and  $M$  being the infinite points of  $l$  and  $m$ . It is the aim of this chapter to generalize this formula. One might think that this generalization will be carried out by generalizing the way that the elements are arithmetically connected by cross-ratios and logarithms. But actually it will be the elements that enter the formula themselves that are subject to the generalization. For this, observe that we may consider the pair of points  $(I, J)$  a degenerate dual conic consisting of the two complex conjugate points  $I$  and  $J$ . The setup for general Cayley-Klein geometries will be to admit a general conic instead of  $I$  and  $J$ .

To see how this will work, we reexamine  $I$  and  $J$ , now interpreted as a conic. In Chapter 9 we learned that a dual conic may degenerate into a pair of points (either real or complex conjugate). In the case that these two points do not coincide, the corresponding primal conic must be the doubly covered line that connects these two points. In the case of  $I$  and  $J$ , a corresponding primal/dual pair  $\mathcal{C}_{\text{euc}} = (A, B)$  of matrices (in the sense of Definition 9.5) is given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B^\Delta \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here  $B$  is the matrix that describes the degenerate dual quadratic form  $l^T B l = 0$ ;  $A$  describes the primal form  $p^T A p = 0$ . The primal equation (which describes which points are on the conic)  $p^T A p = 0$  with  $p = (x, y, z)^T$  becomes  $z^2 = 0$ . The points that satisfy this equation are all those of the form  $(x, y, 0)$ , i.e., the points on the line at infinity (doubly covered). The solutions of the dual equation (which describes which lines are tangent to the conic)  $l^T B l = 0$  with  $l = (a, b, c)^T$  must satisfy  $a^2 + b^2 = 0$ . Up to a scalar multiple these are all lines of the form  $(-i, 1, \alpha) =: i_\alpha$  and  $(i, 1, \alpha) =: j_\alpha$  with a free parameter  $\alpha$ . These are two bundles of lines through two different points, namely  $I$  and  $J$ . Lines of the form  $i_\alpha$  satisfy  $\langle I, i_\alpha \rangle = (-i) \cdot (-i) + 1 \cdot 1 + 0 \cdot \alpha = 0$  and pass through  $I$ . Lines of the form  $j_\alpha$  satisfy  $\langle J, j_\alpha \rangle = i \cdot i + 1 \cdot 1 + 0 \cdot \alpha = 0$  and pass through  $J$ . The tangents of a point  $p$  to the conic  $\mathcal{C}_{\text{euc}}$  are the two lines  $\mathbf{join}(p, I)$  and  $\mathbf{join}(p, J)$ . Thus we can reinterpret Laguerre's formula for calculating the angle between two lines in the following way:

*Assume that the lines  $l$  and  $m$  meet at a point  $p$ . Construct the two tangents  $t_1$  and  $t_2$  from  $p$  to  $\mathcal{C}_{\text{euc}}$ . The angle between  $l$  and  $m$  is then*



**Fig. 20.1** Definition of distances and angles.

$$\angle(l, m) = \frac{1}{2i} \cdot \ln((l, m; t_1, t_2)).$$

This is our point of departure. We will now base angle measurement (as well as distance measurement) on the relation of objects to an *arbitrary* conic. This conic will be called the *fundamental object* of the geometry.

## 20.2 Measurements in Cayley-Klein Geometries

To define a Cayley-Klein geometry we need three ingredients: two constants  $c_{\text{dist}}$  and  $c_{\text{ang}}$  and a fundamental conic  $\mathcal{F}$  given by a primal/dual pair of matrices. The constants play merely a cosmetic role. They will be chosen in a way such that the measurements are in general real numbers and agree with other common definitions. The really decisive part is the choice of the conic. Since  $\mathcal{F}$  is given as a primal/dual pair, calculating the tangents to  $\mathcal{F}$  from an arbitrary point  $p$  as well as intersecting  $\mathcal{F}$  with an arbitrary line  $l$  are well-defined operations.

A Cayley-Klein geometry is a triple  $\mathcal{K} := (\mathcal{F}, c_{\text{dist}}, c_{\text{ang}})$ . Once these parameters are given, the measurement of distances and angles is described by the following recipe:

**Distance measurement:** To measure the distance between two points  $p$  and  $q$ , take a line  $l$  incident with both of them. Take the two intersection points  $X, Y$  of  $l$  with  $\mathcal{F}$ . The *distance* between  $p$  and  $q$  is defined by

$$\text{dist}_{\mathcal{K}}(p, q) = c_{\text{dist}} \cdot \ln((p, q; X, Y)).$$

Situations with  $p=X=Y$  and  $q=X=Y$  are excluded and called *exceptional*.

The angle measurement is defined in a completely dual fashion:

**Angle measurement:** To measure the angle between two lines  $l$  and  $m$  take a point  $p$  incident with both of them. Take the two tangents  $X, Y$  from  $p$  to  $\mathcal{F}$ . The *angle* between  $l$  and  $m$  is defined by

$$\text{ang}_{\mathcal{K}}(l, m) = c_{\text{ang}} \cdot \ln((l, m; X, Y)).$$

Situations with  $l=X=Y$  and  $m=X=Y$  are excluded and called *exceptional*.

The geometric position of the elements involved is illustrated in Figure 20.1. Let us point out a few subtleties of these two definitions of measurement.

- The distance measurement remains well-defined even if the two points  $p$  and  $q$  coincide. If this happens, any line  $l$  through them will give a measurement of the form  $c_{\text{dist}} \cdot \ln((p, p; X, Y)) = c_{\text{dist}} \cdot \ln(1) = 0$ . Similarly, the angle of coinciding lines is also zero.
- In many practical situations it may make sense to define distance and angles as the absolute values of the measurements defined above. We will make this precise later on. The situation is similar to the situation in the usual Euclidean plane. There along one line it makes sense to speak of oriented distances. However, if one measures distances in the plane without reference to a particular line, it is reasonable to switch to absolute values.
- There are two ambiguities in the definitions of measurement. Firstly, the order of the elements  $X, Y$  is not specified. Interchanging their roles alters the sign of the measurement (one more reason to consider absolute values). Secondly, on a function theoretic level the *logarithm* is a multi-valued function. It is determined only up to multiples of  $2\pi i$ . We will see that for all practical purposes we can deal nicely with these ambiguities.
- Perhaps the most important point is that the elements  $X, Y$  may either be real, coincide, or be complex conjugates. Each of these three possibilities leads to a qualitatively different behavior of the measurement. The pictures in Figure 20.1 represent the situation in which  $X$  and  $Y$  are distinct and real. We get a complex situation for the distance measurement if line  $l$  does not intersect the conic. We get a complex situation for the angle measurement if the lines  $l$  and  $m$  intersect *inside* the conic.
- In the definition we will explicitly allow the fundamental conic to be completely complex. Thus equations like  $x^2 + y^2 + z^2 = 0$  are completely suitable fundamental objects. (In fact, they will lead to very interesting geometries.)
- If  $\mathcal{F}$  is a degenerate conic, then it may happen that  $X$  and  $Y$  coincide for all distance or angle measurements. In that case we get, for instance, a degenerate length measurement of the form  $c_{\text{dist}} \cdot \ln((p, q; X, X)) = c_{\text{dist}} \cdot \ln(1) = 0$ . We will dedicate an entire section to the treatment of such degenerate cases.
- It may happen that in the cross-ratio  $(p, q; X, Y) = [p, X][q, Y]/[p, Y][q, X]$  both the denominator and the numerator become zero and  $p$  and  $q$  do not



coincide (for instance if  $q = X = Y$ ). Then the evaluation of the cross-ratio is of the form  $0/0$ . The corresponding length (resp. angle) is then an *undefined* value. We call such point pairs (or line pairs) *exceptional*.

We will soon see how all these subtleties fit nicely into the picture as a whole. In a sense, each of them makes the whole theory a bit more beautiful.

## 20.3 Nondegenerate Measurements along a Line

We will first study how the above notion of distance applies to the points of *one single line*. Let  $l$  be this line and let  $X$  and  $Y$  be the two intersections of this line with the fundamental conic  $\mathcal{F}$ . While restricting the measurements to the line  $l$  we keep these two points fixed during the entire section. Since the definitions of distance and angle measurement are completely dual, the corresponding dual statements apply analogously for angle measurement.

Since the logarithm has to be considered a multivalued function that is determined only modulo  $2\pi i$ , all equations on distance measurement that follow have to be considered to be true *modulo*  $c_{\text{dist}}2\pi i$ . Analogously, angle measurements are determined only *modulo*  $c_{\text{ang}}2\pi i$ .<sup>1</sup>

**Theorem 20.1.** *Let  $\mathcal{K} := (\mathcal{F}, c_{\text{dist}}, c_{\text{ang}})$  be a Cayley-Klein geometry and let  $l$  be an arbitrary line that intersects  $\mathcal{F}$  in two distinct points  $X, Y$ . For two points  $p$  and  $q$  on  $l$  we set  $\mathbf{dist}_{\mathcal{K}}(p, q) = c_{\text{dist}} \cdot \ln((p, q; X, Y))$ . Then (modulo  $c_{\text{dist}}2\pi i$ ) we get for points  $p, q, r$  on  $l$ ,*

- (i)  $\mathbf{dist}_{\mathcal{K}}(p, p) = 0$ ,
- (ii)  $\mathbf{dist}_{\mathcal{K}}(p, q) = -\mathbf{dist}_{\mathcal{K}}(q, p)$ ,
- (iii)  $\mathbf{dist}_{\mathcal{K}}(p, q) + \mathbf{dist}_{\mathcal{K}}(q, r) = \mathbf{dist}_{\mathcal{K}}(p, r)$ .

*Proof.* (i): For cross-ratios we have  $(p, p; X, Y) = 1$ . This implies

$$\mathbf{dist}_{\mathcal{K}}(p, p) = c_{\text{dist}} \cdot \ln((p, p; X, Y)) = c_{\text{dist}} \cdot \ln(1) = 0.$$

(ii): For cross-ratios the relation  $(p, q; X, Y) = 1/(q, p; X, Y)$  holds. Hence we get

$$\begin{aligned} \mathbf{dist}_{\mathcal{K}}(p, q) &= c_{\text{dist}} \cdot \ln((p, q; X, Y)) \\ &= -c_{\text{dist}} \cdot \ln(1/(q, p; X, Y)) \\ &= -c_{\text{dist}} \cdot \ln((q, p; X, Y)) \\ &= -\mathbf{dist}_{\mathcal{K}}(q, p). \end{aligned}$$

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<sup>1</sup> We know this effect from the usual angle measurement. Given two (unoriented) lines, the angle between them is determined only modulo  $\pi$ . This agrees with Laguerre's formula, where  $c_{\text{ang}} = 1/2i$ .

(iii): For cross-ratios the relation  $(p, q; X, Y) \cdot (q, r; X, Y) = (p, r; X, Y)$  holds. Hence we get

$$\begin{aligned} \mathbf{dist}_{\mathcal{K}}(p, r) &= c_{\text{dist}} \cdot \ln((p, r; X, Y)) \\ &= c_{\text{dist}} \cdot \ln((p, q; X, Y) \cdot (q, r; X, Y)) \\ &= c_{\text{dist}} \cdot \ln((p, q; X, Y)) + c_{\text{dist}} \cdot \ln((q, r; X, Y)) \\ &= \mathbf{dist}_{\mathcal{K}}(p, q) + \mathbf{dist}_{\mathcal{K}}(q, r) \end{aligned}$$

□

We see that the logarithm turns the multiplicative properties of cross-ratios into additive properties of distances. Along a fixed line it makes perfect sense to speak of *oriented distances*, and the above statements in essence introduce a distance scale along such a line. As soon as a point  $p$  on  $l$  is singled out, one can consistently measure the position of all other points on  $l$  with respect to  $p$  by the value of  $\mathbf{dist}_{\mathcal{K}}(p, r)$ . This is in a sense similar to the situation on the real number line, where one can also measure distances as soon as an origin is fixed (in fact, there is a subtle difference between these two pictures, which we will encounter later on). We now want to analyze how this measurement behaves qualitatively. The intersections of  $l$  with the fundamental conic arise as the solutions of a quadratic equation. Thus up to equivalence by real projective transformations we have to consider only three different cases:

- (a)  $X, Y$  are real and distinct,
- (b)  $X, Y$  are complex conjugates,
- (c)  $X, Y$  are identical and real.

We will first focus on the cases (a) and (b).

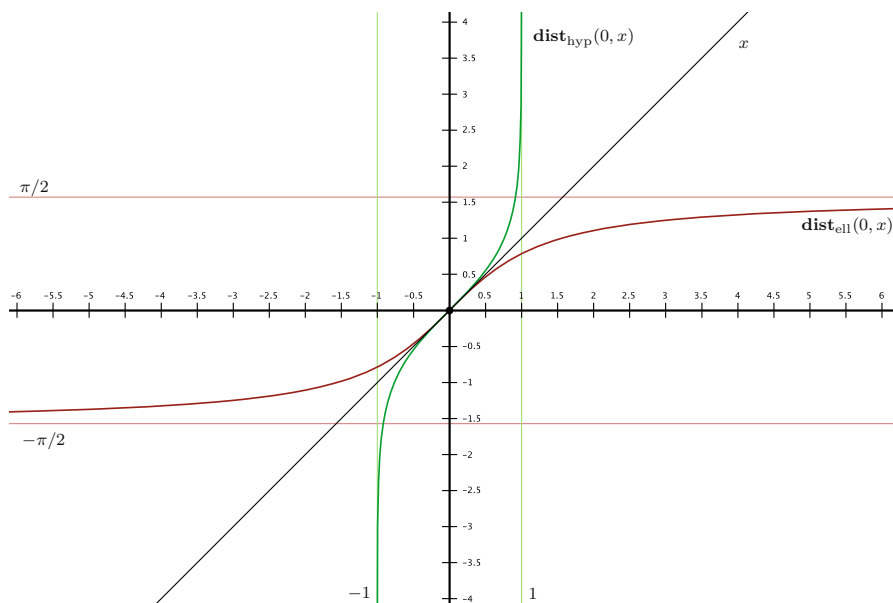
**Case (a) – hyperbolic measurement:** For this we fix two points  $X = (-1, 1)$  and  $Y = (1, 1)$  on the projective line  $l$ . We will study the measurement between points  $(p, 1)$  and  $(q, 1)$ . Since all our considerations will be restricted to the line  $l$ , we may simply identify a point  $(x, 1)$  with the number  $x$  and calculate the cross-ratio by quotients of differences between these numbers. The measurement between two points  $p$  and  $q$  with respect to  $X$  and  $Y$  then becomes

$$\mathbf{dist}(p, q) = c_{\text{dist}} \cdot \ln\left(\frac{(p+1)(q-1)}{(p-1)(q+1)}\right).$$

In particular, if we set  $p = 0$  to be the origin, we get the distance function

$$\mathbf{dist}(0, q) = c_{\text{dist}} \cdot \ln\left(\frac{(1-q)}{(1+q)}\right).$$

The expression in the logarithm is positive whenever  $q$  is in the open interval from  $-1$  to  $1$ . If  $p$  traverses this interval continuously then the logarithm in the above formula will traverse values from  $\infty$  to  $-\infty$ . More generally, the



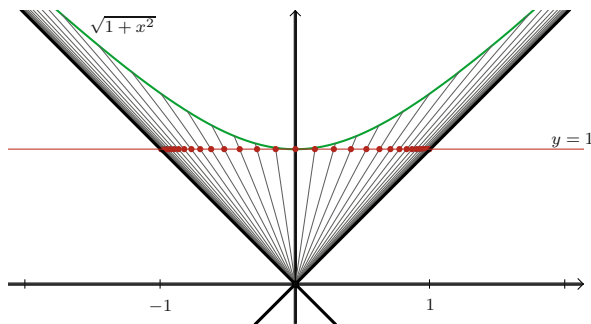
**Fig. 20.2** Distance functions for hyperbolic and elliptic measurement.

function  $\ln\left(\frac{(p+1)(q-1)}{(p-1)(q+1)}\right)$  is real-valued if (along the real projective (!) line  $l$ ) the points  $p$  and  $q$  are not separated by  $-1$  and  $1$ . It is complex or infinite otherwise.

We now come to a particular choice of the constant  $c_{\text{dist}}$ . A reasonable distance measure should behave in a way such that points that are “near” each other (and not separated by  $-1$  or by  $1$ ) have a small *real* distance. Since in such a proximity situation the logarithm is real-valued, we can achieve this by choosing  $c_{\text{dist}}$  to be a *real* number. One particularly nice choice is to choose this number such that locally for very small numbers  $q$  the function  $\mathbf{dist}(0, q)$  is asymptotically close to  $q$  itself (i.e., around  $0$  it approximates the usual Euclidean measurement on the real number line). Considering the derivative

$$\frac{d \mathbf{dist}(0, q)}{d q} = c_{\text{dist}} \cdot \frac{d \ln\left(\frac{(1-q)}{(1+q)}\right)}{d q} = c_{\text{dist}} \cdot \frac{2}{q^2 - 1},$$

we see that by setting  $c_{\text{dist}} = -1/2$  we get the desired local approximation behavior around  $q = 0$ . Although all other constants lead to essentially isomorphic measurements, we will continue with this special setting and define



**Fig. 20.3** Unit steps in hyperbolic measurement.

$$\mathbf{dist}_{\text{hyp}}(p, q) := -\frac{1}{2} \cdot \ln \left( \frac{(p+1)(q-1)}{(p-1)(q+1)} \right).$$

This type of measurement is called *hyperbolic measurement*. The green graph in Figure 20.2 shows the function  $\mathbf{dist}_{\text{hyp}}(0, x)$ . Between  $-1$  and  $1$  it sweeps from  $-\infty$  to  $+\infty$ , approximating the identity around the origin.

The qualitative behavior of this measurement is best understood by calculating a sequence of points between  $-1$  and  $+1$  such that the hyperbolic distance between two consecutive points is constant. This task is in a sense purely projective, since it involves only relations of the cross-ratios of the points. We set  $p_0 = 0$  and  $p_1 = t$ . We are interested in a sequence of points  $\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, \dots$  such that

$$\mathbf{dist}_{\text{hyp}}(p_i, p_{i+1}) = \mathbf{dist}_{\text{hyp}}(p_0, p_1)$$

for all  $i$ . This is equivalent to the requirement that

$$\frac{(p_i + 1)(p_{i+1} - 1)}{(p_i - 1)(p_{i+1} + 1)} = \frac{(0 + 1)(t - 1)}{(0 - 1)(t + 1)} =: \alpha.$$

To resolve this recurrence one first observes that the cross-ratio

$$c := (0, q; 1, -1) = \frac{(0 + 1)(q - 1)}{(0 - 1)(q + 1)}$$

uniquely determines the position of  $q$  by

$$q = -\frac{(c - 1)}{(c + 1)}.$$

On the other hand, we can compute the cross-ratio  $(0, p_i; 1, -1)$  as a telescoping product from the above recurrence relation to be simply  $\alpha^i$ . Thus we get

$$p_i = -\frac{(\alpha^i - 1)}{(\alpha^i + 1)}.$$

Figure 20.3 shows the collection of (red) points  $(p_i, 1)$  in the  $\mathbb{R}^2$  plane for a certain step width. There are also rays that connect these points to the origin. They are elongated until they hit the hyperbola branch given by the function  $\sqrt{1+x^2}$ . An amazing connection of our measurement to this graph (and to the arcsinh function) provides the result that in this drawing all the (almost triangular) regions cut out by consecutive rays and the hyperbola have exactly equal area. Although it does not belong to the core topics of this book, we will give a short proof of this amazing fact. We can equivalently state it in the following way (see Figure 20.4, left).

**Theorem 20.2.** *The function  $\text{dist}_{\text{hyp}}(p, q)$  equals twice the area enclosed by the hyperbola branch  $\sqrt{1+x^2}$  and the two rays connecting the origin to  $(p, 1)$  and  $(q, 1)$ .*

*Proof.* We will provide a proof that does not require an explicit knowledge of the arcsinh function. First we observe that it suffices to prove the theorem for  $p = 0$ . The general case follows from the additivity of area and of the  $\text{dist}_{\text{hyp}}(p, q)$  function.

We will calculate the area of the yellow sector of Figure 20.4. We will reduce the calculation of the area  $F$  of the yellow region plus the area  $G$  of the red region (Figure 20.4, right). The area  $F + G$  can be easily calculated as an integral. First observe that the line through the origin and  $(q, 1)$  hits the hyperbola branch at the point  $(q, 1)/\sqrt{1-q^2}$ . We set  $x = q/\sqrt{1-q^2}$  and get

$$F = \int_0^x \sqrt{t^2 + 1} \, dt - G.$$

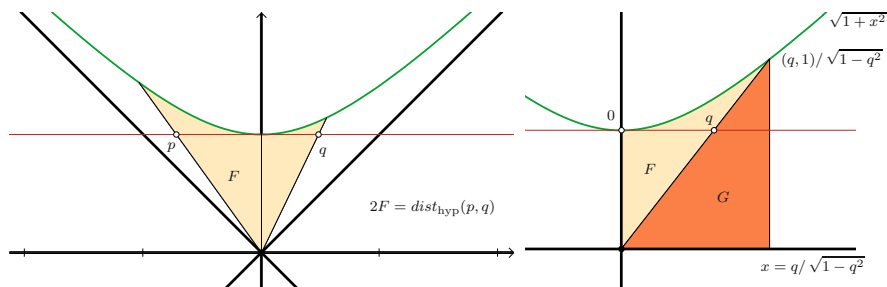
A modest orgy of calculus (or a computer algebra system) shows that

$$\int_0^x \sqrt{t^2 + 1} \, dt = \frac{1}{2} \left( x\sqrt{1+x^2} + \underbrace{\ln(x + \sqrt{1+x^2})}_{=\text{arcsinh}(x)} \right).$$

We furthermore have  $G = (x \cdot \sqrt{1+x^2})/2$  and thus we obtain

$$F = \int_0^x \sqrt{t^2 + 1} \, dt - G = \frac{1}{2} \cdot \ln(x + \sqrt{1+x^2}).$$

We now insert  $x = q/\sqrt{1-q^2}$ , multiply both sides by 2, and get



**Fig. 20.4** Unit steps in hyperbolic and elliptic measurement.

$$\begin{aligned}
 2F &= \ln \left( \frac{q}{\sqrt{1-q^2}} + \sqrt{1 + \frac{q^2}{1-q^2}} \right) \\
 &= \ln \left( \frac{q}{\sqrt{1-q^2}} + \frac{1}{\sqrt{1-q^2}} \right) \\
 &= \ln \left( \frac{q+1}{\sqrt{1-q^2}} \right) \\
 &= \ln \left( \frac{q+1}{\sqrt{1+q}\sqrt{1-q}} \right) \\
 &= \ln \left( \frac{\sqrt{q+1}}{\sqrt{1-q}} \right) \\
 &= -\frac{1}{2} \ln \left( \frac{q-1}{q+1} \right) \\
 &= \mathbf{dist}_{\text{hyp}}(0, q).
 \end{aligned}$$

This proves the claim. □

The distribution of the points in Figure 20.3 tells us a lot about the qualitative behavior of the hyperbolic measurement. Assume that you are some strange one-dimensional being that lives on the line  $l$  between the points  $-1$  and  $1$ . Furthermore, assume that you are able to measure distances only according to the  $\mathbf{dist}_{\text{hyp}}(p, q)$  formula. If you walk in unit steps in one direction (I have no idea where your legs are, but this is your problem), then you are able to walk on and on but you will never reach one of the boundary points, which in your perception are infinitely far away. A being that looks at you from the outside equipped with our usual Euclidean way of measurement will observe that your steps become smaller and smaller, in such a way that

you can never reach the boundary of the interval  $(-1, 1)$ . In the chapter on *hyperbolic geometry* we will investigate this effect more closely.

In the case that one of the points  $p$  and  $q$  is between  $-1$  and  $1$  and the other one is outside (i.e.,  $p$  and  $q$  are projectively separated by  $-1$  and  $1$ ), the cross-ratio inside the logarithm of the distance measurement becomes a negative real number. Taking the logarithm and multiplying by  $c_{\text{dist}}$  leaves us with a number of the form  $a \pm i \cdot \pi/2$ . Thus, in this case we get a complex distance, which indicates that the two points are *unreachably far apart*.

**Case (b) – elliptic measurement:** We now come to the second qualitative possibility for a measurement in which the intersections  $X$  and  $Y$  are complex conjugates. Again, we may, up to a real projective transformation on  $l$ , restrict our considerations to a convenient special case. This time we choose  $X = (-i, 1)$  and  $Y = (i, 1)$ . For the measurement (which is called *elliptic* in contrast to *hyperbolic*) we obtain the formula

$$\mathbf{dist}(p, q) = c_{\text{dist}} \cdot \ln \left( \frac{(p+i)(q-i)}{(p-i)(q+i)} \right).$$

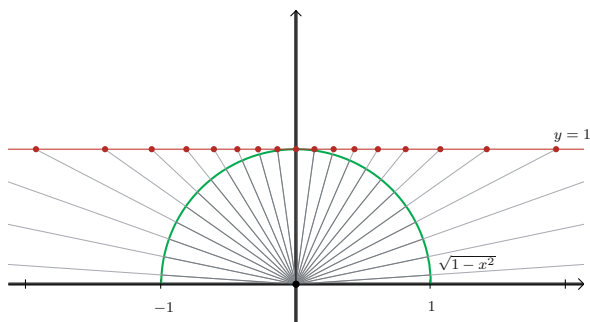
Since  $p$  and  $q$  are assumed to be real numbers, the numerator and denominator inside the logarithm are complex conjugates. The quotient of two complex conjugate numbers is always a complex number on the unit circle  $e^{it}$ ,  $t \in \mathbb{R}$ . Taking the logarithm of such a number results in a number  $it$  that has no real component. The number  $t$  corresponds to the angle that the cross-ratio (as a complex number) forms with the positive part of the real axis. Here it becomes important that the logarithm is determined only up to a factor of  $2\pi i$ , which corresponds to the usual ambiguity of angle measurement. So, this time—in order to get a real measurement—it is reasonable to choose the constant  $c_{\text{dist}}$  as a purely imaginary number. Considering the derivative

$$\frac{d \mathbf{dist}(0, q)}{d q} = c_{\text{dist}} \cdot \frac{d \ln \left( \frac{(i-q)}{(i+q)} \right)}{d q} = c_{\text{dist}} \cdot \frac{2i}{q^2 + 1}$$

shows that setting  $c_{\text{dist}} = 1/2i$  is the choice that makes this function approximate the identity if  $p$  is close to the origin. Similarly to the hyperbolic case we define the *elliptic* distance function on  $l$  to be

$$\mathbf{dist}_{\text{ell}}(p, q) := \frac{1}{2i} \cdot \ln \left( \frac{(p+i)(q-i)}{(p-i)(q+i)} \right).$$

The red graph in Figure 20.2 shows the function  $\mathbf{dist}_{\text{ell}}(0, x)$ . The reader should notice the correspondence to Laguerre's formula, and indeed, if we consider the dual case in which we consider a bundle of lines through a point, then this formula measures the angle enclosed between two lines  $p$  and  $q$ . Figure 20.5 shows a sequence of points on the line  $l$  for which consecutive points all have an identical elliptic distance. This time, the measurement



**Fig. 20.5** Unit steps in elliptic measurement.

is qualitatively different from the hyperbolic case. Starting at  $q = 0$  and proceeding in steps of small but constant elliptic distance, the steps become (from a Euclidean perspective) larger and larger and even so large that they surpass the (Euclidean) infinite point on  $l$  and come back from the other side.

The circle in Figure 20.5 illustrates quantitatively what this measurement corresponds to. If we consider regions that are cut out by rays connecting consecutive points of the sequence to the origin and the unit circle function  $\sqrt{1-x^2}$ , then all these regions have identical area. We only have to be a bit careful when we surpass the infinite point of  $l$ . If one point has already passed the infinite point of  $l$  but the other has not, we have to consider the area as negative. This is nicely compensated by the fact that the measurement is ambiguous modulo  $\pi$ . The reader is invited to check the details.

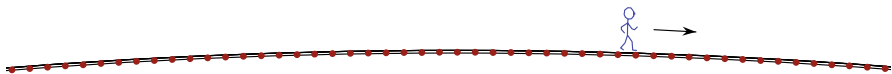
We can also interpret this measurement in a different but equivalent way. By homogenization the points on  $l$  correspond to antipodal point pairs on the unit circle. The distance function  $\mathbf{dist}_{\text{ell}}(p, q)$  measures the (oriented) distance of two such antipodal point pairs along the boundary of the unit circle. It corresponds to the length of the boundary segment that is covered if we start at one representative of  $p$  and proceed clockwise until we meet a representative of  $q$ .

## 20.4 Degenerate Measurements along a Line

So far, we have dealt with the nondegenerate measurements for which  $X \neq Y$ . We still have to explain what happens in the degenerate case.

**Case (c) – parabolic measurement:** We will now deal with the degenerate case  $X = Y$  and both real. Throughout this section we make the general assumption that neither  $p$  nor  $q$  equals the points  $X = Y$ . Thus the cross-ratio is defined and no measurements are exceptional. At first sight there is not much to do. If we calculate





**Fig. 20.6** Columbus1D's journey.

$$\mathbf{dist}(p, q) = c_{\text{dist}} \cdot \ln((p, q; X, X)) = c_{\text{dist}} \cdot \ln(1) = 0,$$

we plainly get *zero*. However, the formula still carries more information than is obvious at first sight. We will see that it is still possible to *compare* distances in a well-defined and reasonable way. We know this effect from our usual Euclidean geometry. Also in this situation there is no absolute way of measuring. We first have to define a reference length (for instance a unit meter) and measure all distances relative to this unit length. We already met this phenomenon in Theorem 18.10, where we explained the distance measurement on the basis of  $I$  and  $J$ . Also there we had to perform the measurement relative to a reference length  $|A, B|$ .

Before we derive formulas for this measurement on the basis of Cayley-Klein geometries we want to relate it to the measurements of Case (a) (hyperbolic) and Case (b) (elliptic) we have dealt with so far. These measurements have a remarkable property. The mere existence of  $X$  and  $Y$  defined an *absolute* scale of measure with respect to which we could declare a distance function. This becomes most transparent in the case of elliptic measurement, for which the total circumference of the unit circle defines a length with respect to which we could compare all other distances. On a qualitative level this effect may be explained as follows. Imagine you are a being living in a one-dimensional space that arises from a sphere with antipodal pairs of points identified (topologically this is actually  $\mathbb{RP}^1$ , but we want to explicitly inherit the arc-length measurement of the sphere). Assume that you are relatively small compared to the sphere. If you are able to move only in your direct neighborhood, you may define a certain unit length (your personal step width) and measure every distance with respect to this unit length. You have a friend whose name is *Columbus1D the great explorer*, who once decided to continue walking in one direction as long as he could. After you said goodbye you never expected to see him again. But—surprise—after about three years he returned to you from the opposite direction with the great news that it took him 13986242 unit steps to make this journey. Now you know that your world is not infinite and there is a kind of absolute measurement of length with respect to the perimeter of your world.<sup>2</sup>

<sup>2</sup> Algebraically, the hyperbolic measurement also possesses an absolute distance measurement, although there is no such nice intuitive story for it.

Intuitively speaking, the degenerate measurement case  $X = Y$  is about the situation in which you have a sphere of infinite radius and Columbus1D will in fact never return. For dealing with the degenerate  $X = Y$  case analytically we will, for the moment, drop our assumption that  $X$  and  $Y$  are fixed. Instead, we will consider them as members of a continuous family of measurements (we blow up the world). For this we consider the quadratic equation  $\alpha x^2 - y^2 = 0$  and assume that  $X, Y$  are the homogeneous coordinates of the solutions of this equation. For  $\alpha = 1$  we get the situation of the hyperbolic case  $X_{\text{hyp}} = (-1, 1)^T$  and  $Y_{\text{hyp}} = (1, 1)^T$ . For  $\alpha = -1$  we get the situation of the elliptic case  $X_{\text{ell}} = (-i, 1)^T$  and  $Y_{\text{ell}} = (i, 1)^T$ . In the general case we get the solutions  $(1, \pm\sqrt{\alpha})^T$ . If the parameter  $\alpha$  is moved continuously from 1 to  $-1$ , we start with the pair of points  $(X_{\text{hyp}}, Y_{\text{hyp}})$ . Then they move away from the origin in opposite directions. They will coincide for  $\alpha = 0$  at the infinite point of  $l$ . After this they continue as complex conjugates, to finally reach the position  $(X_{\text{ell}}, Y_{\text{ell}})$  for  $\alpha = -1$ . We now consider the measurement between two points  $(0, 1)$  and  $(q, 1)$ , both different from  $(1, 0)$ . We consider the behavior of the logarithm

$$\ln \left( \frac{\left| \begin{array}{cc} 0 & 1 \\ 1 & -\sqrt{\alpha} \end{array} \right| \left| \begin{array}{cc} q & 1 \\ 1 & \sqrt{\alpha} \end{array} \right|}{\left| \begin{array}{cc} 0 & 1 \\ 1 & \sqrt{\alpha} \end{array} \right| \left| \begin{array}{cc} q & 1 \\ 1 & -\sqrt{\alpha} \end{array} \right|} \right) = \ln \left( \frac{q\sqrt{\alpha} - 1}{-q\sqrt{\alpha} - 1} \right) = \ln(q\sqrt{\alpha} - 1) - \ln(-q\sqrt{\alpha} - 1).$$

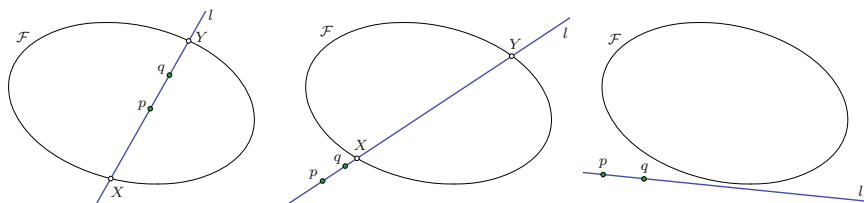
We now want to compare the distance measure generated by this expression with the measurement from  $(0, 1)$  to a fixed point  $(a, 1)$  that will play the role of the unit length. The relation of these two measurements (notice that  $c_{\text{dist}}$  cancels) is

$$f_a(\alpha, q) := \frac{\ln(q\sqrt{\alpha} - 1) - \ln(-q\sqrt{\alpha} - 1)}{\ln(a\sqrt{\alpha} - 1) - \ln(-a\sqrt{\alpha} - 1)}.$$

For  $\alpha = 0$  this gives an undefined expression  $0/0$ . However, we may still consider the limit situation  $\lim_{\alpha \rightarrow 0} f_a(\alpha, q)$ . This limit value can be easily calculated using the L'Hospital's rule. For this we differentiate the numerator and denominator of  $f_a(\alpha, q)$  with respect to  $\alpha$  and divide the two expressions (we omit the boring details of the calculation):

$$\lim_{\alpha \rightarrow 0} f_a(\alpha, q) = \lim_{\alpha \rightarrow 0} \frac{\frac{q}{\sqrt{\alpha}(\alpha q^2 - 1)}}{\frac{a}{\sqrt{\alpha}(\alpha a^2 - 1)}} = \lim_{\alpha \rightarrow 0} \frac{q(\alpha a^2 - 1)}{a(\alpha q^2 - 1)} = \frac{q}{a}.$$

In the limit case the comparison of the two Cayley-Klein measures turns out to be just a comparison of the usual Euclidean distances. In other words, performing a relative measurement with respect to a unit length converges to the usual Euclidean measurement of lengths.



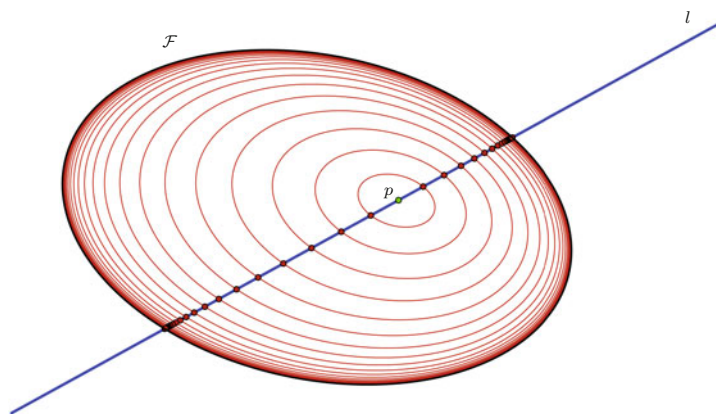
**Fig. 20.7** Situations for distance measurement.

In the literature there are two names for this kind of measurement: *parabolic measurement* to emphasize the fact that the measurement is the limit case between elliptic and hyperbolic, and *Euclidean measurement* to emphasize the fact that it just corresponds to the measurement in usual Euclidean geometry. A word of caution is appropriate here. We obtained the correspondence to Euclidean measurement only under the assumption that  $X = Y$  is the *infinite point* on  $l$ . If  $X = Y$  is at some finite position, then we get a projective scale as was introduced in Section 5.3.

## 20.5 A Planar Cayley-Klein Geometry

We now return to the planar case. Before we give an overview of all possible Cayley-Klein geometries in the next section, we consider one concrete example in which all different types of measurements for angles and distances arise. We study the case of a Cayley-Klein geometry  $\mathcal{K} = (\mathcal{F}, c_{\text{dist}}, c_{\text{ang}})$  that arises from a nondegenerate real conic  $\mathcal{F}$ . We set  $c_{\text{dist}} = -\frac{1}{2}$  and  $c_{\text{ang}} = \frac{1}{2i}$ . The choice of the constants implies that certain measurements of distances and angles become real and others become complex.

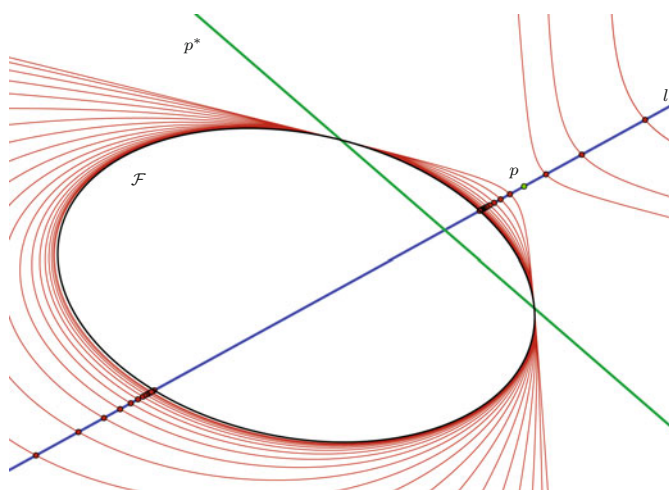
We begin our considerations with distance measurement. For this consider Figure 20.7. What are the distance measurements  $\text{dist}_{\mathcal{K}}(p, q)$  that arise for different positions of  $p$  and  $q$ ? In the leftmost picture, both points are inside the fundamental object  $\mathcal{F}$ . Hence their join  $l$  intersects  $\mathcal{F}$  in two real points  $X$  and  $Y$  and we get a hyperbolic measurement. Since  $p$  and  $q$  are not separated by the pair  $(X, Y)$ , the cross-ratio  $(p, q; X, Y)$  is positive and hence its logarithm is real. With our specific choice of  $c_{\text{dist}} = -\frac{1}{2}$ , this implies that in this case the entire measurement  $\text{dist}_{\mathcal{K}}(p, q)$  is real. Forming a chain of points on  $l$  starting with  $p$  and  $q$  such that two consecutive points have the same distance would (we have a hyperbolic measurement) get closer and closer to  $\mathcal{F}$  but never exceed its boundary. Figure 20.8 illustrates a related effect. There curves of constant real distance from  $p$  are shown. The curve closest to  $p$  has some distance  $d$ , the next curves have distance  $2d, 3d, 4d$ , etc. The sequence of these curves tends to the conic  $\mathcal{F}$  as a limiting object of



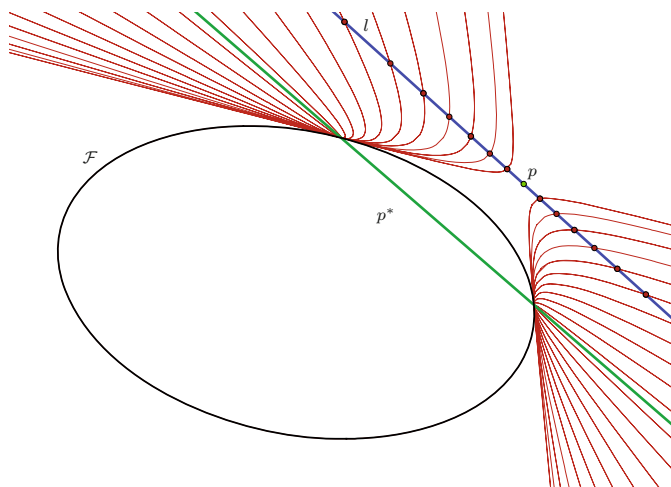
**Fig. 20.8** Curves of constant real distance to  $p$ .

infinite distance. You should observe that all equidistant curves turn out to be conics themselves. We will come to this effect later.

In Figure 20.7 (middle and left) both points are outside the fundamental object  $\mathcal{F}$ . To determine the type of measurement in such a situation the actual position of  $p$  and  $q$  is relevant. If their join still intersects the fundamental object in two points, then the situation is similar to the previous one. We get a real hyperbolic measurement along  $l$ . Starting at  $p$  and proceeding in constant real steps, we approach the fundamental object, this time from the outside. Again, the boundary can never be surpassed. A sequence of



**Fig. 20.9** Curves of constant real distance to  $p$ .

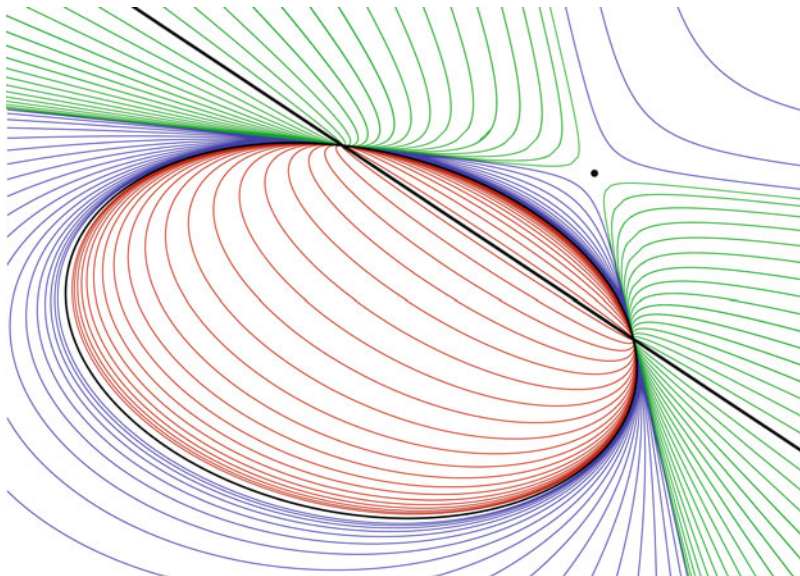


**Fig. 20.10** Curves of constant purely imaginary distance to  $p$ .

curves of constant distance to  $p$  is shown in Figure 20.9. The situation is qualitatively different if  $l$  does not intersect the fundamental object. Then the points  $X$  and  $Y$  are no longer real. They become complex conjugates and we get an elliptic measurement along  $l$ . The cross-ratio  $(p, q; X, Y)$  then will be a complex number on the unit circle, and its logarithm will be of the form  $i \cdot t$ ,  $t \in \mathbb{R}$ . With our choice of the constant  $c_{\text{dist}}$  we get a distance that is purely imaginary. Proceeding in constant (imaginary) steps away from  $p$  will ultimately make a cycle along the *projective* line  $l$ . Qualitatively this is an elliptic measurement as described, for instance, in Figure 20.5. Figure 20.10 shows a sequence of curves of constant imaginary distance from  $p$ .

At this point the reader should observe two important facts (which we will prove later): All curves of constant distance are again conics. These conics have two points of tangency with the fundamental objects. They are the same for every such conic, namely the intersections of the polar of  $p$  with  $\mathcal{F}$  itself. One might wonder what happens to this tangency for the curves shown in Figure 20.8, in which the curves do not touch the fundamental object. Well, they still do! The points of tangency only become complex in this case, so that we do not see them.

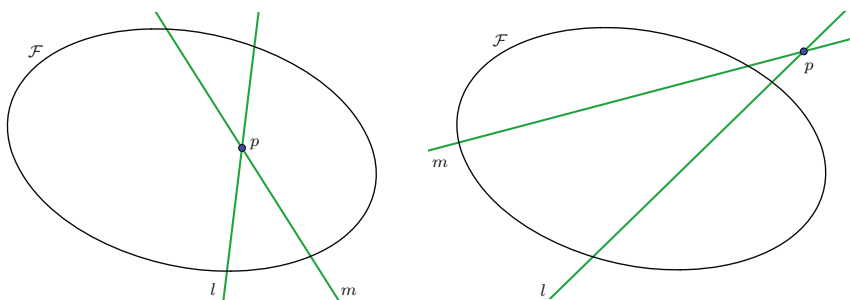
We have left out two interesting cases for the position of  $p$  and  $q$  so far. First, it may happen that one point is inside and the other is outside. In this case the cross-ratio  $(p, q; X, Y)$  is real and *negative*. We then will get a logarithm of the type  $a + i \cdot \pi$ , an entirely complex number, which indicates that  $q$  is unreachable from  $q$  by real steps. It may also happen that  $l$  is tangent to the fundamental object  $\mathcal{F}$ . In this case we get a parabolic measurement on the line  $l$  in the sense of Section 20.5.



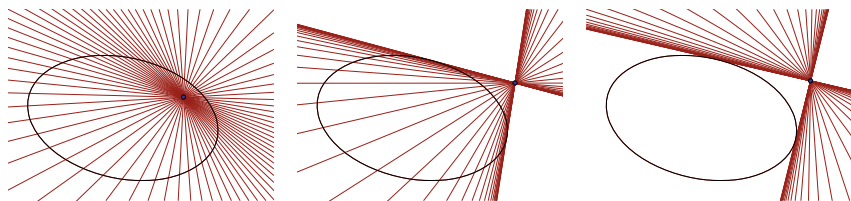
**Fig. 20.11** Curves of constant distances of all kinds.

In Figure 20.11 all types of different curves of constant distance to a point  $p$  outside  $\mathcal{F}$  (the black dot) are shown. With our distance measure, the blue curves have a constant real distance. The green curves have a constant distance of the form  $i \cdot t$ ,  $t \in \mathbb{R}$ . The red curves have a distance of the form  $a + i \cdot \pi/2$ ,  $a \in \mathbb{R}$ .

We next consider the measurement of angles between two lines  $l$  and  $m$ . Recall that we set  $c_{\text{ang}} = \frac{1}{2i}$ . Figure 20.12 shows two qualitatively different situations. In the left picture the lines intersect at a point  $p$  inside the



**Fig. 20.12** Situations for angle measurement.



**Fig. 20.13** Sequences of equiangular lines.

fundamental object. The tangents from  $p$  to  $\mathcal{F}$  are complex conjugates, and we get an elliptic measurement. With our choice of a complex constant  $c_{\text{ang}}$  this results in a real angle measurement. In the other situation on the right the point  $p$  is *outside*  $\mathcal{F}$ . The two tangents are real and distinct. This leads to a hyperbolic type of measurement. With our specific choice of  $c_{\text{ang}}$  this leads to purely imaginary angles. It may also happen that  $p$  lies *on* the fundamental conic. Then we get a parabolic measurement of angles.

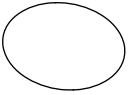

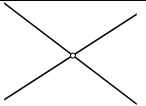

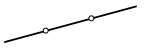
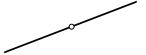
In Figure 20.13 we show three situations of bundles of lines in which each consecutive pair of lines encloses the same angle. In the right situation the angle is real; in the other two pictures the angle is purely imaginary.

## 20.6 A Census of Cayley-Klein Geometries

Our next aim is a systematic study of all possible Cayley-Klein geometries. Clearly, there are infinitely many of such geometries, since there are infinitely many choices of each of the ingredients  $\mathcal{F}$ ,  $c_{\text{dist}}$ , and  $c_{\text{ang}}$ . Changes of the two constants (while leaving  $\mathcal{F}$  invariant) essentially lead to isomorphic geometries. Still, a change here may affect which distances and angles are considered to be real numbers and which are not. We will return to this issue later in this section and for now focus on the influence of  $\mathcal{F}$ . Since the measurements are based on projective operations such as taking joins, meets, tangents, intersections with conics, and cross-ratios, we may focus on a classification modulo projective transformations of  $\mathbb{RP}^2$ . Two Cayley-Klein geometries are equivalent if their fundamental conic differs only by such a projective transformation. At this point it must be emphasized that the projective transformations under consideration have *real* parameters. Admitting also complex transformations can in principle be done. However, then one must consider Cayley-Klein geometries in  $\mathbb{CP}^2$  instead of  $\mathbb{RP}^2$ , which leads to different classification issues.

Up to real projective transformations we already classified primal-dual pairs  $(A, B)$  of conics in Section 9.5. There we saw that the classification essentially depends on the signature of the eigenvalues of the matrices  $A$

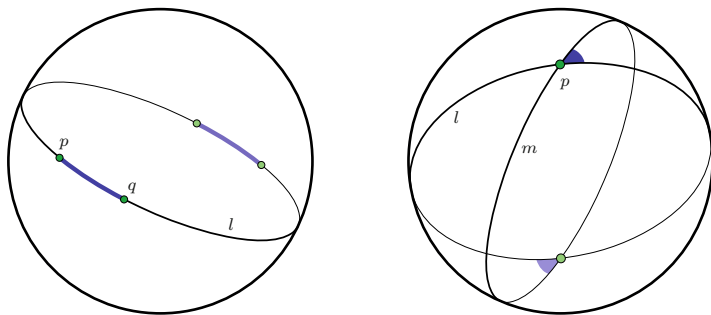
and  $B$ , which is the essential invariant that cannot be changed by a real projective transformation. We repeat the table here, since it essentially gives the classification of possible Cayley-Klein geometries.

	$A$	$B$	type	
I	$(+, +, +)$	$(+, +, +)$	complex nondegenerate conic	
II	$(+, +, -)$	$(+, +, -)$	real nondegenerate conic	
III	$(+, +, 0)$	$(+, 0, 0)$	two complex lines and a real double point on them	
IV	$(+, -, 0)$	$(+, 0, 0)$	two real lines and a double real point on them	
V	$(+, 0, 0)$	$(+, +, 0)$	two complex points and a real double line through them	
VI	$(+, 0, 0)$	$(+, -, 0)$	two real points and a real double line through them	
VII	$(+, 0, 0)$	$(+, 0, 0)$	a real double line and a real double point on it	

Each of these seven cases leads to a genuine Cayley-Klein geometry not isomorphic to the others. We will briefly discuss the different situations and propose appropriate choices of  $c_{\text{dist}}$  and  $c_{\text{ang}}$ .

**Type I:** The fundamental object of type I is a conic whose primal and dual forms are described by an equation of the form  $x^2 + y^2 + z^2 = 0$ . This object has no real points in the projective plane, nor does it admit real tangents. Thus if we measure the distance between two points  $p$  and  $q$ , then the corresponding points  $X$  and  $Y$  are *always* complex conjugates and we end up with an elliptic measurement along every line in  $\mathbb{RP}^2$ . Similarly, for the measurement of angles between lines  $l$  and  $m$  we have to form tangents  $X$  and  $Y$  from the intersection of  $l$  and  $m$  to  $\mathcal{F}$ . Also these tangents turn out to be complex conjugates. Thus we also get an elliptic angle measurement. A good choice for  $c_{\text{dist}}$  and  $c_{\text{ang}}$  is  $c_{\text{dist}} = c_{\text{ang}} = 1/2i$ . If we furthermore choose  $\mathcal{F}$  to be the conic defined by the primal equation  $x^2 + y^2 + z^2 = 0$ ,





**Fig. 20.14** Distance and angle measurement in the spherical model of elliptic geometry.

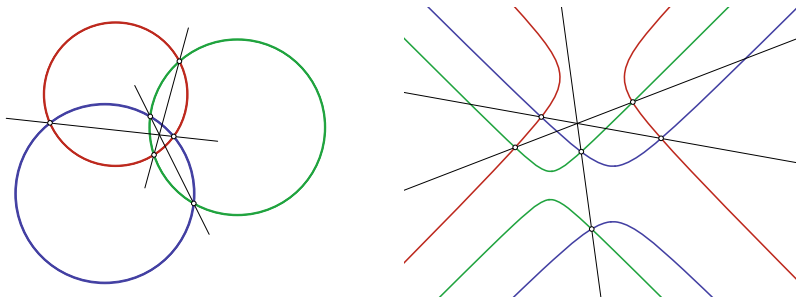
this leads to the following geometric situation: If we interpret the projective plane  $\mathbb{RP}^2$  as the points  $(x, y, z)$  on the unit sphere  $x^2 + y^2 + z^2 = 1$  with antipodal points identified, then the above Cayley-Klein measurement can be interpreted in the following way: The distance between two points  $p$  and  $q$  is the spherical distance between representatives of them on the unit sphere. The angle between two lines is the spherical angle between the two great circles that correspond to the lines. The maximal distance between two points is  $\pi$ , which is the semiperimeter of an equator on the unit ball. The maximal angle between two lines is  $\pi$  as well. This geometry is usually called *elliptic geometry*. It should not be confused with the closely related *spherical geometry* for which antipodal points on the sphere are *not* identified.

**Type II:** Here we have a geometry where (up to isomorphism) the primal fundamental object has the form  $x^2 + y^2 - z^2 = 0$ . This is the real unit circle. In a sense, this is the most complicated Cayley-Klein geometry, since all different kinds of measurements may arise as well for distances as for angles. We dealt with this geometry at length in the previous section. The choice of  $c_{\text{dist}}$  and  $c_{\text{ang}}$  depends on what measurements one wants to be real. We will later see that a particularly interesting situation arises if one limits oneself to the interior of the unit circle only. In this case the distance between two points will always lead to a hyperbolic measurement, and  $c_{\text{dist}} = -1/2$  is a good choice. If one considers only angles of lines that intersect *inside* the unit circle, then angle measurement results in an elliptic situation. Hence,  $c_{\text{ang}} = 1/2i$  is a good choice. From a certain perspective this restriction to the interior is a reasonable choice. Imagine you are a being inside the circle. There is no way for you to get out of there by real step width only. Your universe is the interior, which seems infinite to you. We will dedicate the entire Chapter 25 and Chapter 26 to this situation. It is called *hyperbolic geometry*. However, one should be aware that from the perspective of Cayley-Klein geometries, hyperbolic geometry is only a substructure of a larger ambient situation.

So far, we have dealt with all completely nondegenerate geometries. For types III through VII at least one of the two measurements is necessarily degenerate. An associated Cayley-Klein geometry requires either parabolic distance measurement or parabolic angle measurement or even both. Ignoring the numerical order, we begin with the best-known of these geometries.

**Type V:** This type is our usual *Euclidean geometry*. It has a dual conic consisting of two complex conjugate points, and a primal conic that is a doubly covered line connecting these two points. A good choice for the coordinates of  $\mathcal{F}$  is given by the primal/dual pair given in Section 20.1 (i.e., dual conic  $a^2 + b^2 = 0$  and primal conic  $z^2 = 0$ ). With this choice the complex conjugate points of the dual conic become I and J. The primal conic becomes the doubly covered line at infinity. Cayley-Klein geometry gives us exactly the desired behavior: an elliptic measurement of angles and a degenerate measurement of distances. The choice  $c_{\text{ang}} = 1/2i$  gives the usual angle measurement that we know from Laguerre's formula. The choice of  $c_{\text{dist}}$  does not matter at all, since we have a parabolic distance measurement in which it is not meaningful to speak of absolute lengths. Still, along each line we may compare distances to a given unit length. There is one subtlety about degenerate measurement in the plane. Our considerations in Section 20.4 demonstrated how to compare lengths *along a single line*. It did not explain how measurement with respect to different lines are interrelated. We will not go into this issue in detail here and just mention the final result. With the setup of  $\mathcal{F}$  as above it turns out that a suitable limit process proves that in Euclidean geometry the distance between two points  $p = (p_x, p_y, 1)$  and  $q = (q_x, q_y, 1)$  is, up to a scalar factor, given by the expression  $\sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}$ , the usual Euclidean measurement. This fact is a consequence of the characterization of circles as conics that pass through I and J.

**Type VI:** This is one more geometry with two single points on a double line as fundamental object. Compared to type V, this time the two points are *real*. A way to represent this geometry is given by the dual conic  $a^2 - b^2 = 0$  and the primal conic  $z^2 = 0$ . Thus the role of the points I and J is now played by the two *real infinite points*  $I' = (-1, 1, 0)$  and  $J' = (1, 1, 0)$ . This geometry is often called *pseudo-Euclidean geometry*. All our considerations of Chapter 18 and Chapter 19 apply in an analogous way to this geometry. Distance measurement is again parabolic along each line. However, angle measurement now is hyperbolic. Usually one chooses  $c_{\text{ang}} = -1/2$ . Thus a bundle of lines through a point with equal angles between two consecutive lines now does not wrap around. Instead, the sequence converges to a limit line. Analogously to Euclidean geometry, circles are conics through  $I'$  and  $J'$ . Thus a circle in this geometry is a hyperbola whose asymptotes are parallel to the two lines  $x = y$  and  $x = -y$ . All incidence theorems of Euclidean geometry may be transferred to pseudo-Euclidean geometry analogously. For instance, take the theorem that for three intersecting circles, the lines connecting the intersection points of pairs of circles meet in a point. The same theorem



**Fig. 20.15** A theorem in Euclidean and pseudo-Euclidean geometry.

also holds for pseudo-Euclidean circles (see Figure 20.15). According to the fact that circles are now hyperbolas with prescribed asymptotes, we now get a different distance measure in the limit case (along each line it is still parabolic, but the interrelation of the lines is different). The distance between two points  $p = (p_x, p_y, 1)$  and  $q = (q_x, q_y, 1)$  is up to a scalar factor given by the expression  $\sqrt{(p_x - q_x)^2 - (p_y - q_y)^2}$ . This distance may even become a complex number, which indicates that certain points cannot be connected by finitely many real steps.

The pseudo-Euclidean geometry plays a crucial role in the theory of special relativity. There it is also known under the name *Minkowski geometry*. The physical interpretation of this geometry is as follows. We consider a one-dimensional physical line  $\ell$  on which objects may move in both directions. Interpret the  $x$ -direction of the pseudo-Euclidean plane as time and the  $y$ -direction as location on this line. A straight line in this geometry now corresponds to a uniformly moving object on  $\ell$ . The fact that two directions (the lines that pass through  $I'$  and  $J'$ ) play a special role corresponds to the fact that the speed of light plays a special role in special relativity. The fact that physical addition of speeds may never exceed the speed of light corresponds to the fact that addition of angles tends to a limit line. Distance being an invariant in pseudo-Euclidean geometry corresponds to the fact that there is a relativistic invariant of space-time events. An exhaustive treatment of pseudo-Euclidean geometry may be found in [86].

**Type III:** This is the dual to Euclidean geometry. Here the distance measurement turns out to be elliptic, while the angle measurement is degenerate. We will not consider this geometry in depth here.

**Type VI:** This is the dual to pseudo-Euclidean geometry. Here the distance measurement turns out to be hyperbolic, while the angle measurement is degenerate. We will not consider this geometry in depth here.

**Type VII:** Finally, in this type the angle measurement as well as the distance measurement turns out to be degenerate. The choice of  $c_{\text{dist}}$  and  $c_{\text{ang}}$

does not play a role at all. Instead, we need a unit length and a unit angle as objects of comparison. Also, this geometry is physically relevant, since it describes the limit case from relativistic physics to classical physics, where the speed of light is assumed to be infinite. For this reason it is frequently called *Galilean geometry*. We will have a closer look at this geometry in Section 23.5. A very nice treatment of this geometry may be found in [136].

## 20.7 Coarser and Finer Classifications

There are also other ways to structure the realm of Cayley-Klein geometries. We saw that from an incidence-theoretic point of view, the Euclidean and pseudo-Euclidean geometries are closely related. The reason for this is that the incidence theory depends only on algebraic relations between the objects. These relations are not influenced by a complex projective transformation. Thus any two Cayley-Klein geometries whose fundamental objects are related by a complex projective transformation have identical incidence-theoretic structures. The only thing that may change is the question as to which of the objects are *real* and which are not. One may also be a bit more radical and study complex planar projective geometry, where the question of being real or not does not matter. In this sense we get a coarser classification of Cayley-Klein geometries into four equivalence classes,  $\{I, II\}$ ,  $\{III, IV\}$ ,  $\{V, VI\}$ , and  $\{VII\}$ .

Another way to classify Cayley-Klein geometries depends on the property whether the measurements are elliptic, hyperbolic, or parabolic. There are nine different combinations for the distance and angle measurements. We have seen that types I and III–VII are associated with specific types of measurements. The remaining three combinations may all be found as substructures of type II. The table below associates the different types of measurement with the possible types of geometry.

Distance \ Angle	hyp	ell	par
hyp	$II_a$	$II_b$	VI
ell	$II_c$	I	V
par	VI	III	VII

We will not follow this finer classification here, since in particular, in case of type II it obscures the wholeness and unity of the underlying geometric and algebraic structures.

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## Measurements and Transformations

*Projective geometry is all geometry.*

*Arthur Cayley, as cited by F. Klein*

While in the last chapter we have focused on measurement aspects and related analytic issues, we will return to more qualitative relations of objects in a Cayley-Klein geometry. In this and the next chapter we will mainly study the *projective nature* of Cayley-Klein geometries. This chapter focuses on aspects concerning *transformations*, their (projective) *invariants*, and the behavior of *measurements* under these transformations. The next chapter treats geometric objects and their *elementary geometric properties*, including some elementary geometric theorems. Although Cayley-Klein geometries unify several rather different types of geometric playgrounds in a common projective framework, each *type* of geometry (in the sense of Section 20.6) has its very special properties. This is due to the fact that the degeneracies of the fundamental conic of the geometry lead to degeneracies in the relevant geometric constructions. For this reason we will sometimes have to perform a special case analysis for the different types of Cayley-Klein geometries. Sometimes one general definition covers a certain effect (for instance reflection) in all types of geometries. Still it may be instructive to consider its particular specialization to certain geometries. Since we do not want to become too encyclopedic, we will confine ourselves to highlighting only some of the interesting situations. For any statement and concept we will make clear to which types of geometries it applies and under which circumstances degenerate situations arise. Since again the relevant concepts are quite interwoven, we recommend reading this and the following chapter at least twice. Some relations may become clear only at second reading.

## 21.1 Measurements vs. Oriented Measurements

Before considering transformations and transformation groups, we will clarify how the measurements defined in the last chapter can be interpreted entirely in a projective framework. Since we now want to compare distances (resp. angles) among arbitrary points (resp. lines) in a Cayley-Klein geometry, we will first have to deal with the various ambiguities of the measurements.

Considering the *projective* aspects of Cayley-Klein geometries, one important aspect comes into play: If we consider the distance measurement  $\mathbf{dist}_K(p, q) = c_{\text{dist}} \cdot \ln((p, q; X, Y))$ , each possible value of the cross-ratio  $(p, q; X, Y)$  leads to a different value of  $\mathbf{dist}_K(p, q)$  modulo  $2\pi i c_{\text{dist}}$ . Thus any qualitative relation between distances (like  $\mathbf{dist}_K(p, q) = \mathbf{dist}_K(r, s)$ ) can already be expressed on the level of cross-ratios. A similar statement holds for angle measurements. In a sense, applying the logarithm is a convenient way to translate the (multiplicative) projective world of cross-ratios into the (additive) world of distance and angle measurement we are used to. Moreover, the multiplications by constants are used to make things even more familiar insofar as we get real measurements whenever we are expecting them. Nevertheless, staying directly on the level of the projectively invariant cross-ratios has its advantages. First of all, from a projective viewpoint, taking the logarithm is just an unnecessary “cosmetic isomorphism.” Secondly, taking the logarithm even introduces (unnecessary) ambiguities of the measurement modulo  $2\pi i c_{\text{dist}}$  (resp.  $2\pi i c_{\text{ang}}$ ) that are not essential for the theory.

If we want to operate on the level of cross-ratios instead of measurements, we have to face a formal problem whose solution is essential for making precise statements later on. In our definition of the measurements  $\mathbf{dist}_K(p, q)$ , besides the intrinsic ambiguity of the logarithm there was another serious ambiguity that we did not really care about so far. In Section 20.3 we learned that if we restrict distance measurements to one single line (resp. restrict angle measurement to one single line bundle), it is possible to fix *a priori* the elements  $X$  and  $Y$  and thereby arrive at *oriented* measurements.

This is no longer the case if we do measurements without referring to an *a priori* chosen line (resp. bundle). If we want to measure the distance between two points  $p$  and  $q$  we first have to construct the points  $X$  and  $Y$ . There is no reasonable way to prefer one order of the points  $X$  and  $Y$  over the other. Hence in performing the distance measurement, one might either get

$$d = c_{\text{dist}} \cdot \ln((p, q; X, Y)) \quad \text{or} \quad -d = c_{\text{dist}} \cdot \ln((p, q; Y, X)).$$

One might be tempted to resolve this problem by simply referring instead to the absolute value of a measurement. However, this approach has two problems. Firstly, the result of a measurement may well be a complex number. By taking its absolute value one loses not only the information about its sign but also about its direction, which is unnecessarily coarse. Secondly, we will try to express as many geometric properties as possible directly on the level

of cross-ratios and not on the level of distances (or angles). For the cross-ratio the sign change of the measurement corresponds to the reciprocal connection  $(p, q; X, Y) = 1/(p, q; Y, X)$ . In order to deal with these ambiguities properly we will introduce two special equivalence classes (this is slightly uncommon in the literature on Cayley-Klein geometries, but it resolves the situation most appropriately). We set

$$\{x\} := \{x, -x\} \quad \text{and} \quad \oint x \oint := \{x, 1/x\}.$$

Thus for comparing unoriented distances, an expression like

$$\{\mathbf{dist}_{\mathcal{K}}(p, q)\} = \{\mathbf{dist}_{\mathcal{K}}(r, s)\}$$

is a reasonable formulation. To compare two cross-ratios where the order of  $X, Y$  is undetermined,

$$\oint (p, q; X, Y) \oint = \oint (r, s; X, Y) \oint$$

is a good shorthand. Alternatively, one could use squared distances only and, on the level of cross-ratios, a function that symmetrically uses a cross-ratio and its inverse. We will do this later on, too. However, for the moment we prefer the above notation since, it is a bit closer to the fundamental definition of measurement.

## 21.2 Transformations

Cayley-Klein geometries are projective geometries equipped with certain additional measurements for distances and angles. While a projective transformation is a transformation that leaves fundamental projective properties invariant (incidences, tangencies, etc.), a transformation in a Cayley-Klein geometry in addition preserves distance and angle measurement.

Since Cayley-Klein geometries make it eventually necessary to deal also with complex elements, we will first briefly specify the setup in which objects and transformations are considered in this context. Ultimately, we want to make statements about the *real* projective plane  $\mathbb{RP}^2$  equipped with the measurements of a Cayley-Klein geometry. Thus the only transformations we will consider are projective transformations  $\tau$  of the real projective plane. They are represented by a pair of real nondegenerate matrices  $(T, T^{-1})$  that are used to transform geometric elements. In homogeneous coordinates a point  $p$  is transformed according to  $\tau(p) = T \cdot p$ ; A line  $l$  is transformed by  $\tau(l) = (T^{-1})^T \cdot l$  (compare Section 3.6). A primal quadratic form  $A$  is transformed by  $\tau(A) = (T^{-1})^T A T^{-1}$ , and a dual quadratic form  $B$  is transformed by  $\tau(B) = T B T^T$  (compare Section 10.4). As usual, we must identify nonscalar multiples of matrices and vectors.

In all our operations we will admit the following objects as input elements:

- All points, lines, and primal or dual quadratic forms of  $\mathbb{RP}^2$ .
- The primal and dual fundamental object of the Cayley-Klein geometry.

We admit arbitrary projectively reasonable operations between these objects, no matter whether they result in real or complex elements. This is in the same spirit as our treatment of Euclidean geometry in Chapter 18 and Chapter 19, in which we also had to face the problem that intermediate construction elements may become complex. Whenever we speak of *transformational invariance* of a certain property we limit ourselves to real projective transformation matrices (still they might be applied to complex objects).

We are now going to define motion in a Cayley-Klein geometry  $\mathcal{K} := (\mathcal{F}, c_{\text{dist}}, c_{\text{ang}})$ . Here as usual  $\mathcal{F}$  is given as a primal/dual pair of matrices  $(A, B)$ . There are two possible ways to approach the concept of motion. The first approach asks for those projective transformations that leave the distance and angle measurements invariant. However, in the case of degenerate measurements of both distances and angles (as they appear in Galilean geometries) one would unavoidably have to consider also degenerate measurements, which makes things more complicated. The second approach defines  $\mathcal{K}$ -motions as those projective transformations that leave the fundamental object  $\mathcal{F}$  invariant. As a consequence, such transformations also leave distances and angles invariant. We will follow this second approach.

**Definition 21.1.** Let  $\mathcal{K}$  be a Cayley-Klein geometry whose fundamental object  $\mathcal{F}$  is defined by the primal/dual pair of matrices  $(A, B)$ . Let  $\tau$  be a projective transformation in  $\mathbb{RP}^2$ . We call  $\tau$  a  $\mathcal{K}$ -motion if it leaves both  $A$  and  $B$  invariant.

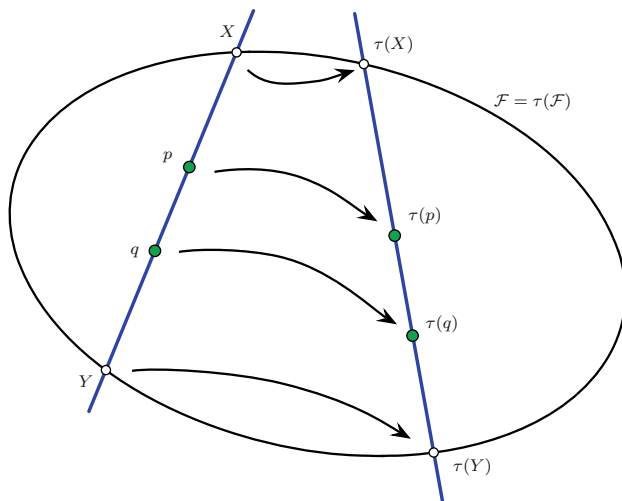
**Theorem 21.1.** Let  $\tau$  be a  $\mathcal{K}$ -motion. Then we have

- (i)  $\{\mathbf{dist}_{\mathcal{K}}(p, q)\} = \{\mathbf{dist}_{\mathcal{K}}(\tau(p), \tau(q))\}$  for all points  $p$  and  $q$ ,
- (ii)  $\{\mathbf{ang}_{\mathcal{K}}(l, m)\} = \{\mathbf{ang}_{\mathcal{K}}(\tau(l), \tau(m))\}$  for all lines  $l$  and  $m$ .

*Proof.* We start with the proof of (i). Without loss of generality we may assume that the distance measurement is not degenerate (i.e., the primal conic is not a double line), since otherwise all distance measurements  $\mathbf{dist}_{\mathcal{K}}(p, q)$  are zero anyway. Let  $p$  and  $q$  be two arbitrary points in  $\mathbb{RP}^2$  and let  $\tau$  be a  $\mathcal{K}$ -motion. Let  $l$  be a line to which both  $p$  and  $q$  are incident. Then  $\tau(l)$  is a line incident to both  $\tau(p)$  and  $\tau(q)$ . Now let  $X$  and  $Y$  be the intersection of  $l$  with the fundamental conic  $\mathcal{F}$  and let  $X'$  and  $Y'$  be the intersection of  $\tau(l)$  with the fundamental conic  $\mathcal{F}$  (in any order). The distance measurement of  $p$  and  $q$  depends on  $(p, q; X, Y)$ , while the distance measurement of  $\tau(p)$  and  $\tau(q)$  depends on  $(\tau(p), \tau(q); X', Y')$ . Since projective transformations preserve incidence relations and  $\mathcal{F}$  is mapped to itself under  $\tau$  we have either

$$X' = \tau(X) \quad \text{and} \quad Y' = \tau(Y)$$





**Fig. 21.1** Invariance of distance measurement.

or

$$X' = \tau(Y) \quad \text{and} \quad Y' = \tau(X).$$

This implies

$$\mathfrak{d}(\tau(p), \tau(q); X', Y') = \mathfrak{d}(\tau(p), \tau(q); \tau(X), \tau(Y)) = \mathfrak{d}(p, q; X, Y).$$

The last equation holds since the cross-ratio is a projective invariant. On the level of distances this gives the desired relation

$$\{\mathbf{dist}_{\mathcal{K}}(p, q)\} = \{\mathbf{dist}_{\mathcal{K}}(\tau(p), \tau(q))\}.$$

This proves the claim for distance measurement. Exactly the dual argument proves the corresponding situation for the angle measurement.  $\square$

*Remark 21.1.* In fact, for all Cayley-Klein geometries except for those of type VII one can prove that the preservation of distances and angles automatically leads to a  $\mathcal{K}$ -motion in the sense of Definition 21.1. Still our definition is preferable for two reasons. On the one hand, it also covers the case of type-VII Cayley-Klein geometries. On the other hand, it is closer to the terms of projective geometry, since it requires the projective invariance of the fundamental object itself and not of a measurement dependent on it.

What is the number of degrees of freedom for defining a  $\mathcal{K}$ -motion? We know that a planar projective transformation is determined uniquely by its action on four points. However, the notion of  $\mathcal{K}$ -motions is by far more restrictive. The degrees of freedom depend on the specific type of the Cayley-Klein

geometry: the more degenerate the fundamental object is, the more degrees of freedom we have. Considering the number of degrees of freedom, we may distinguish three different situations: the nondegenerate cases, the singly degenerate cases, and the doubly degenerate cases.

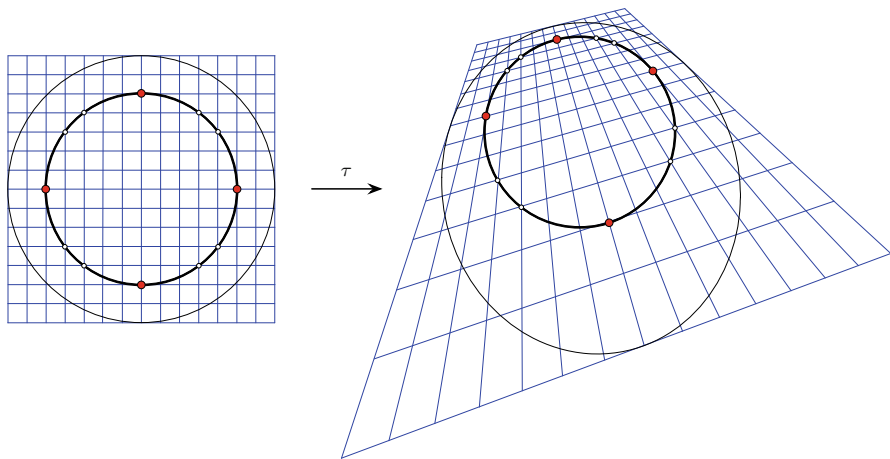
**Nondegenerate cases:** For types I and II, the fundamental object consists of a nondegenerate conic (real or complex). Since in the nondegenerate case the primal conic given by the matrix  $A$  uniquely defines the dual conic  $B = A^{-1}$ , it is sufficient to study those transformations that leave the matrix  $A$  invariant. Without loss of generality we may choose the transformation matrix  $T$  to have determinant  $\det(T) = 1$ . Thus we are looking for transformations  $T \in \text{SL}(\mathbb{R}, 3)$  such that  $(T^{-1})^T A T^{-1} = A$ . One could consider this a purely algebraic problem. At first sight, the problem seems to be identical for both nondegenerate geometries. However, the requirement that the entries of the transformation matrix have to be real-valued results in two different transformation groups. We study the two situations of type I (elliptic case) and type II (hyperbolic case) Cayley-Klein geometries separately, since both of them lead to interesting geometric structures.

*Type I, the elliptic case:* In this situation up to isomorphism the matrix  $A$  may be chosen to be the unit matrix  $E$ . The invariance property then reads

$$(T^{-1})^T T^{-1} = E.$$

This, in turn, is exactly the definition of an orthogonal matrix (the inverse is the transpose). Such a matrix  $T \in \text{O}(3)$  describes either a rotation or a reflection in  $\mathbb{R}^3$ . We can immediately interpret this fact geometrically. We may represent each point  $p \in \mathbb{RP}^2$  by an antipodal pair of points on the unit sphere in  $\mathbb{R}^3$ . Now an elliptic transformation is a rotation or a reflection of this sphere. This is nicely consistent with the fact that elliptic measurement may be interpreted as spherical distances and angles on this sphere (compare Figure 20.13). The number of degrees of freedom for this transformation group is exactly *three*, corresponding to the degrees of freedom of a rotation or reflection in  $\mathbb{R}^3$ . There is an important difference between the elliptic transformation group and the matrix group  $\text{O}(3)$ . In the elliptic transformation group, according to the projective setup, the matrices  $T$  and  $-T$  are identified. Therefore  $\text{O}(3)$  is a double cover of the elliptic transformation group.

*Type II, the hyperbolic case:* The fundamental conic  $\mathcal{F}$  of a Cayley-Klein geometry of type II is a *real* nondegenerate conic. This gives us the possibility to describe these transformations directly in terms of projective geometry. In Section 10.4 we studied projective transformations that leave a given real nondegenerate conic invariant. Recall that the points on the primal fundamental object may be considered an isomorphic image of a one-dimensional projective line under stereographic projection (compare Figure 10.7). The set of projective transformations that leave a nondegenerate conic invariant



**Fig. 21.2** A hyperbolic transformation.

is governed by our results in Theorem 10.3 and Theorem 10.4: a projective transformation in  $\mathbb{RP}^2$  that leaves the fundamental object  $\mathcal{F}$  invariant induces a projective transformation on the primal conic itself (considered a one-dimensional projective line). Conversely, every such one-dimensional transformation of  $\mathcal{F}$  induces a unique projective transformation in  $\mathbb{RP}^2$ . Thus the group of motions of a type-II Cayley-Klein geometry is isomorphic to the group of projective transformations of  $\mathbb{RP}^1$ . It is completely described by the action on the primal conic.

We will call this transformation group the *hyperbolic transformations*, for reasons we will see in the subsequent chapters. We can use the geometric description of the transformations to give a nice algorithm for calculating a hyperbolic transformation after the image/preimage pairs of points on the primal fundamental conic are given. For this assume that  $A, B, C$  are three points on the conic  $\mathcal{F}$ , and that  $A', B', C'$  are their proposed images under a hyperbolic transformation. This implies that these images also lie on  $\mathcal{F}$ . Choose an arbitrary point  $D$  on  $\mathcal{F}$  distinct from  $A, B, C$ . Determine the cross-ratio  $(A, B; C, D)_{\mathcal{F}}$  (recall this is the well-defined cross-ratio under which  $A, B, C, D$  are *seen* from an arbitrary point on  $\mathcal{F}$ ). Determine a point  $D'$  on  $\mathcal{F}$  such that  $\alpha = (A, B; C, D)_{\mathcal{F}} = (A', B'; C', D')_{\mathcal{F}}$ . Find the unique projective transformation  $\tau$  with  $\tau(A) = A', \tau(B) = B', \tau(C) = C', \tau(D) = D'$ . This is the desired hyperbolic transformation.

Figure 21.2 illustrates one such transformation. For better readability the image and preimages have been separated into two different pictures. The black bold circle in the left part of the picture plays the role of the fundamental conic in the preimage space. On this circle four (red) points  $A, \dots, D$  are placed. In the right part of the picture the corresponding circle is shown

again with four points on it. The position of the last point is determined by the above recipe. The underlying grid illustrates the action of the corresponding projective transformation  $\tau$ . A few significant points (white) on the circle are shown, together with their images under  $\tau$ . It is rather amazing that although the projective grid is quite disturbed, the mapped circle is still a circle.

Hyperbolic transformations admit three degrees of freedom. They can be directly associated with the position of the three image points on the boundary.

**Singlydegenerate cases:** For Cayley-Klein geometries of types III–VI either the primal or the dual fundamental conic (but not both) is degenerate. This results in one more degree of freedom for the transformation group. In both cases the degeneration may occur again in two different ways: either a pair of complex conjugate points (lines), or a pair of real noncoinciding points (lines). Thus we get the four different cases:

Type III: *two complex conjugate lines and a double point*

Type IV: *two distinct lines and a double point*

Type V: *two complex conjugate points and a double line*

Type VI: *two distinct real points and a double line*

In Chapter 19 we studied type V in detail. The two complex conjugate points may be considered to be I and J. The double line is the line at infinity. We get four degrees of freedom corresponding to translations (2 DOF), rotations (1 DOF), and scaling (1 DOF). The additional degree of freedom associated to scaling comes from the fact that the distance measurement is degenerate and no absolute length measurement is available in this geometry.

The case of type-VI geometries is the so-called pseudo-Euclidean geometry. Here the role of I and J is played by two real points. For a detailed treatment of its transformation group we recommend [136] and [86]. Also this geometry does not admit an absolute distance measurement.

The cases III and IV are dually isomorphic to the Euclidean and pseudo-Euclidean geometries. Also here we have four degrees of freedom. In these geometries there is an absolute distance measurement. However, the angle measurement becomes degenerate. The fourth degree of freedom corresponds to scaling of angle measurement.

**Doublydegenerate cases:** Finally, case VII corresponds to the situation in which the conic consists of a double point on a double line. In this geometry the measurement is degenerate as well for distances as it is for angles. The transformation group has five degrees of freedom, two of them corresponding to angle scaling and distance scaling. We will not treat this geometry in detail here. A detailed discussion of it is given in [136].

## 21.3 Getting Rid of $X$ and $Y$

For the following derivations in later chapters it is useful to have various equivalent ways to express the cross-ratios  $(p, q; X, Y)$  (resp.  $(l, m; X, Y)$ ) that occur over and over in our measurements. We here consider the case of point measurements only. The line case is just dual to it. Recall that for  $p \neq q$  the points  $X$  and  $Y$  are the intersection of  $l = \mathbf{join}(p, q)$  with the primal fundamental object given by a matrix  $A$ . So in particular, it is helpful to express  $(p, q; X, Y)$  directly in terms of  $p, q$ , and  $A$ . To do so we introduce a local coordinate system on the line  $l$  with points  $p$  and  $q$  as a basis. Each point  $r$  on  $l$  can be represented by parameters  $(\lambda, \mu)^T$  subject to  $r = \lambda p + \mu q$ . In particular,  $X = \lambda_1 p + \mu_1 q$  and  $Y = \lambda_2 p + \mu_2 q$  correspond to the solutions of the quadratic equation  $(\lambda p + \mu q)^T A (\lambda p + \mu q) = 0$  (i.e.,  $X$  and  $Y$  lie on the conic  $A$ ). Expanding this equation yields

$$\lambda^2 p^T A p + 2\lambda\mu p^T A q + \mu^2 q^T A q = 0.$$

For better readability we abbreviate  $\Omega_{pp} = p^T A p$ ,  $\Omega_{pq} = p^T A q$ ,  $\Omega_{qq} = q^T A q$ . Our equation then reads

$$\lambda^2 \Omega_{pp} + 2\lambda\mu \Omega_{pq} + \mu^2 \Omega_{qq} = 0$$

and has (up to scalar multiples) the two solutions

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Omega_{qq} \\ \Omega_{pq} \pm \sqrt{\Omega_{pq}^2 - \Omega_{pp}\Omega_{qq}} \end{pmatrix}. \quad (21.1)$$

The two possible choices of the sign correspond to the points  $X$  and  $Y$ , respectively. It is important to observe that these solutions are homogeneous in each of the involved objects. Hence, scaling of one of the objects just results in a scaling of the representation in the local coordinate system. Within the local coordinate system  $p$  and  $q$  are represented by the unit vectors. Abbreviating  $\Delta_{pq} = \Omega_{pq}^2 - \Omega_{pp}\Omega_{qq}$  to be the discriminant of the quadratic equation, we can calculate the cross-ratio  $(p, q; X, Y)$  as

$$(p, q; X, Y) = \frac{\begin{vmatrix} 1 & \Omega_{qq} \\ 0 & \Omega_{pq} - \sqrt{\Delta_{pq}} \end{vmatrix}}{\begin{vmatrix} 1 & \Omega_{qq} \\ 0 & \Omega_{pq} + \sqrt{\Delta_{pq}} \end{vmatrix}} = \frac{\Omega_{pq} + \sqrt{\Delta_{pq}}}{\Omega_{pq} - \sqrt{\Delta_{pq}}}.$$

When evaluating this formula we have to be slightly careful with the sign of the square root. We understand this formula to mean that for every evaluation of  $\sqrt{\Delta_{pq}}$  we take the *same* branch of the square-root function. Interchanging the roles of  $X$  and  $Y$  corresponds to interchanging the two solutions of the quadratic equations (i.e., the branch of the square-root evaluation) and results

in taking the inverse of the cross-ratio, as expected. The situation  $\Delta_{pq} = 0$  corresponds to the case that  $X$  and  $Y$  coincide. We know that in the case of an exceptional measurement (and only in this case) the cross-ratio (and hence also this expression) assumes the form  $0/0$ . It is instructive to analyze the geometric situation in the exceptional case. Then we have  $X = Y$ , which means that  $\Delta_{pq} = 0$  and thus the join of  $p$  and  $q$  is tangent to the fundamental conic. In addition, either  $p = X$  or  $q = X$ . Without loss of generality we assume that  $p = X$ . Thus the tangent  $\mathbf{join}(p, q)$  is the polar of  $p$ , and  $q$  lies on this polar. This means that  $\Omega_{pq}$  is also zero.

## 21.4 Comparing Measurements

The formula we just derived expresses the cross-ratio  $(p, q; X, Y)$  directly in terms of the quadratic form  $\Omega$ . If one wants to test whether the distance from  $a$  to  $b$  equals the distance from  $p$  to  $q$ , one can do this directly by testing

$$\frac{\Omega_{ab} + \sqrt{\Delta_{ab}}}{\Omega_{ab} - \sqrt{\Delta_{ab}}} = \frac{\Omega_{pq} + \sqrt{\Delta_{pq}}}{\Omega_{pq} - \sqrt{\Delta_{pq}}}$$

without calculating the intersections of the lines  $ab$  and  $pq$  with the fundamental conic. There are several other versions of such comparisons that give additional geometric (and algebraic) insight. We will briefly consider a few of them.

Since the order of  $X$  and  $Y$  in a measurement is not previously specified in the cross-ratio,  $(p, q; X, Y)$  is on an equal footing with its inverse. We can account for that fact by applying a function to

$$\alpha = \frac{\Omega_{pq} + \sqrt{\Delta_{pq}}}{\Omega_{pq} - \sqrt{\Delta_{pq}}}$$

that treats  $\alpha$  and  $1/\alpha$  symmetrically. A particularly interesting choice is

$$\alpha \mapsto \frac{1}{2} \left( \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right) = \frac{\alpha + 1}{2\sqrt{\alpha}} =: f(\alpha).$$

After applying this transformation and a few elementary calculations, we get

$$f \left( \frac{\Omega_{pq} + \sqrt{\Delta_{pq}}}{\Omega_{pq} - \sqrt{\Delta_{pq}}} \right) = \frac{\Omega_{pq}}{\sqrt{\Omega_{pp}\Omega_{qq}}} =: \beta.$$

Still there is some ambiguity in this formula, since we introduced a square root whose sign we did not specify. We will take care of that in a minute. Before this, we will inspect the above formula and discuss its geometric significance.

For this we consider the following formula, which connects the logarithm and the (arc) cosine function:

$$\ln(\alpha) = 2i \arccos\left(\frac{\alpha + 1}{2\sqrt{\alpha}}\right).$$

(This function follows from the Euler formula  $e^{ix} = \cos(x) + i \cdot \sin(x) = \cos(x) + i \cdot \sqrt{1 - (\cos(x))^2}$  and a few elementary calculations.) The expression under the  $\arccos(\dots)$  function exactly corresponds to our transforming function  $f(\alpha)$ . Thus we get (for  $c_{\text{dist}} = 1/2i$ )

$$\mathbf{dist}(p, q) = \frac{1}{2i} \ln((p, q; X, Y)) = \arccos\left(\frac{\Omega_{pq}}{\sqrt{\Omega_{pp}\Omega_{qq}}}\right).$$

This formula is remarkable, since it nicely generalizes a well-known fact to arbitrary Cayley-Klein geometries. The well-known fact is that we can measure the angle between two vectors  $p$  and  $q$  in  $\mathbb{R}^3$  by the following formula:

$$\cos(\angle p, q) := \frac{\langle p, q \rangle}{\sqrt{\langle p, p \rangle \langle q, q \rangle}}.$$

This formula is exactly our general formula applied to elliptic geometry in the standard representation where  $A$  is the unit matrix. In this case we get  $\Omega_{p,q} = p^T A q = p^T q = \langle p, q \rangle$ . The sign ambiguity resembles the fact that there is some ambiguity on the direction in which the angle is measured. A similar (completely dual) statement also holds for angle measurement.

Still there are some other nice representations for the comparison of measurements. The above representation has two major drawbacks. Firstly, the square root in the function introduces a sign ambiguity. Secondly, the formula is usable only for nondegenerate measurements. If  $A$  is a rank-1 matrix ( $A = ll^T$ ) then the expression  $\beta$  always evaluates to 1. This simply means that for degenerate measurements all point pairs have the same degenerate distance zero. There is an amazing trick with which we can use the same (projectively invariant) formula for comparison of measurements for the degenerate and for the nondegenerate case. In principle, this trick encapsulates the limit argument that we made in Section 20.4 by using a suitable replacement for the discriminant  $\Delta_{pq}$ . Let us first deal with the first problem, the sign ambiguity. For this we simply square the expression  $\beta$ . It turns out that it is even better to transform according to

$$\beta \mapsto \beta^2 - 1 =: g(\beta).$$

We get

$$g\left(\frac{\Omega_{pq}}{\sqrt{\Omega_{pp}\Omega_{qq}}}\right) = \frac{\Omega_{pq}^2 - \Omega_{pp}\Omega_{qq}}{\Omega_{pp}\Omega_{qq}} = \frac{\Delta_{pq}}{\Omega_{pp}\Omega_{qq}} =: \Phi_{pq}. \quad (21.2)$$

This formula now encodes in a square-root-free fashion a number that encapsulates all information about the distance between two points. If  $\Phi_{pq} = \Phi_{ab}$  in a given geometry, then  $p, q$  has the same distance as  $a, b$ .

The nice fact about this formula is that it is simply a polynomial identity in the coordinates of the points involved and the fundamental object. We obtain an equal distance if and only if

$$\Delta_{pq}\Omega_{aa}\Omega_{bb} = \Delta_{ab}\Omega_{pp}\Omega_{qq}. \quad (21.3)$$

But still this formula carries another surprise. As mentioned before, if we apply the formula in case of a degenerate length measurement, then we simply get  $0 = 0$ . The reason for this is that in this case  $X$  and  $Y$  will coincide and the discriminant

$$\Delta_{pq} = - \begin{vmatrix} p^T A p & p^T A q \\ p^T A q & q^T A q \end{vmatrix}$$

will be zero. One might attempt to compensate for a degeneration process by the following strategy. Assume that in a limit process the matrix  $A$  passes from a nondegenerate matrix to a degenerate one. During this process one could systematically replace  $A$  in the discriminant with  $k \cdot A$ , with a suitable factor  $k$  that compensates for the degeneration of  $A$ . Such a factor would cancel out in equation (21.3). Fortunately, there is a direct way to replace the discriminant  $\Delta_{pq}$  with some other quadratic form that still carries information in the degenerate case and is a multiple of  $\Delta_{pq}$  in all nondegenerate situations. To see this, let us first analyze the geometric meaning of  $\Delta_{pq}$ . The discriminant  $\Delta_{pq}$  is a polynomial that occurs when we want to calculate the points of intersection of a line spanned by  $p$  and  $q$  with the fundamental object. If we represent the points on the line by  $\lambda p + \mu q$ , then the two intersection points are given by equation (21.1). They coincide if  $\Delta_{pq} = 0$ . In other words,  $\Delta_{pq} = 0$  if the line **join**( $p, q$ ) is tangent to the fundamental conic. Furthermore,  $\Delta_{pq}$  is quadratic in both  $p$  and  $q$ . For the nondegenerate case there is another expression with exactly the same properties. The line  $l = p \times q$  being tangent to the fundamental conic can also be tested by checking  $l^T B l = 0$ , the quadratic form of the dual conic. Thus the expression

$$(p \times q)^T B (p \times q)$$

is also quadratic in both  $p$  and  $q$  and has exactly the same zero set (the pairs of points  $p$  and  $q$  that lead to a tangent situation). Hence, up to a factor  $k$  it must equal the discriminant. Thus we can rewrite equation (21.3) as

$$(p \times q)^T B (p \times q) \cdot \Omega_{aa}\Omega_{bb} = (a \times b)^T B (a \times b) \cdot \Omega_{pp}\Omega_{qq}. \quad (21.4)$$

In the nondegenerate case (where  $A$  has still at least rank 2) this equation is equivalent to (21.3). In the degenerate case the part that replaced the discriminant is in general still not zero, and we can still use the formula to compare measurements. A suitable limit argument shows that by passing



from a nondegenerate to the degenerate case with rank  $A$  being 1, everything behaves continuously and the above equation corresponds to the test for equal distance in the degenerate case.

In the case that  $p$  and  $q$  are in located such that an exceptional measurement arises, both expressions  $(p \times q)^T B (p \times q)$  and  $\Omega_{pp} \Omega_{qq}$  vanish, so that formula (21.4) is satisfied independently of the position of  $a$  and  $b$ . As already mentioned, this case does not allow for a reasonable distance measurement.

We want to explore the geometric and invariant-theoretic meaning of the above equation in the degenerate case. The interesting situation arises when the primal conic described by  $A$  degenerates to a double line. Then  $B$  might have either rank 2 or rank 1. We analyze the case in which  $B$  has rank 2. In this case  $B$  describes two distinct points on the double line. We will call these two points  $I$  and  $J$  for the moment, to remind ourselves of the special role played by the points  $I$  and  $J$  in Euclidean geometry. In the standard embedding of Euclidean geometry we have  $I = \mathbf{I}$  and  $J = \mathbf{J}$ . The dual conic is described by the matrix  $B = IJ^T + JI^T$ . The primal conic is described by the matrix  $A = (I \times J)(I \times J)^T$ . Plugging this into our condition for equal measurement, we get

$$\begin{aligned} & (p \times q)^T (IJ^T + JI^T) (p \times q) \cdot a^T (I \times J) (I \times J)^T a \cdot b^T (I \times J) (I \times J)^T b \\ &= (a \times b)^T (IJ^T + JI^T) (a \times b) \cdot p^T (I \times J) (I \times J)^T p \cdot q^T (I \times J) (I \times J)^T q. \end{aligned}$$

At first sight this looks frightening, but let us analyze the different parts of this formula. We get

$$(p \times q)^T (IJ^T) (p \times q) = ((p \times q)^T I) \cdot (J^T (p \times q)) = [pqI][pqJ]$$

and

$$(p \times q)^T (JI^T) (p \times q) = ((p \times q)^T J) \cdot (I^T (p \times q)) = [pqJ][pqI].$$

Hence

$$(p \times q)^T (IJ^T + JI^T) (p \times q) = 2[pqI][pqJ].$$

Furthermore, we get

$$a^T (I \times J) (I \times J)^T a = (a^T (I \times J)) \cdot ((I \times J)^T a) = [aIJ]^2.$$

We get similar translations for the other subexpressions involved. Thus the entire large equation translates to

$$[pqI][pqJ][aIJ]^2[bIJ]^2 = [aIb][aJb][pIJ]^2[qIJ]^2.$$

This is exactly expression (18.1), which we obtained in Section 18.8, now generalized to general Cayley-Klein geometries with a degenerate primal matrix and a rank-2 dual matrix. For reversed roles of a degenerate angle and nondegenerate distance measurement a corresponding dual formula applies.

Thus instead of considering the function  $\Phi_{pq}$  we consider the function

$$\Psi_{pq} := \frac{(p \times q)^T B(p \times q)}{p^T A p \cdot q^T A q}.$$

For the nondegenerate case these two functions differ only by a factor dependent only on  $A$  and  $B$ . However, this factor is chosen so well, that it compensates for the degeneration in a limiting process and that  $\Psi_{pq}$  is still suitable for comparing distances also in degenerate cases. Let us briefly examine what this expression becomes for concrete typical choices of degenerate  $A$  and rank-2 matrix  $B$ . First notice that in this formula  $p$  and  $q$  both occur quadratically in the numerator and in the denominator. Hence the result of the formula is independent of a specific choice of a representative of these points. The standard embedding of Euclidean geometry is given by the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Representing  $p = (p_x, p_y, 1)^T$  and  $q = (q_x, q_y, 1)^T$  by the standard Euclidean embedding with last coordinate 1, we get  $p^T A p = q^T A q = 1$ . Furthermore, we get

$$p \times q = \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} \times \begin{pmatrix} q_x \\ q_y \\ 1 \end{pmatrix} = \begin{pmatrix} p_y - q_y \\ q_x - p_x \\ p_x q_y - p_y q_x \end{pmatrix}.$$

Hence

$$\frac{(p \times q)^T B(p \times q)}{p^T A p \cdot q^T A q} = (p \times q)^T B(p \times q) = (p_x - q_x)^2 + (p_y - q_y)^2,$$

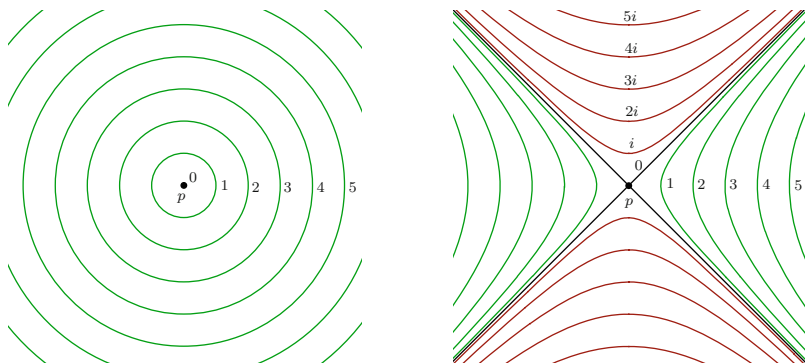
the squared Euclidean distance! Replacing  $A$  or  $B$  by a scalar multiple of these matrices only results in a global rescaling of this measurement, which is an inherent ambiguity of Euclidean distance measurement anyway. If we evaluate the same expression with the fundamental object defined by the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we get squared distance measurement in pseudo-Euclidean geometry:

$$\frac{(p \times q)^T B(p \times q)}{p^T A p \cdot q^T A q} = (p_x - q_x)^2 - (p_y - q_y)^2.$$

Compared to the Euclidean case this expression may as well be positive as negative. Thus we may have points with real or imaginary distances. Figure 21.3 (left) shows circles around a point  $p$  of radius 0 (black), 1, 2, 3, 4, 5 (green) in Euclidean geometry. On the right the corresponding circles in pseudo-Euclidean geometry and in addition circles with the imaginary radii



**Fig. 21.3** Circles in Euclidean and pseudo-Euclidean geometry.

$i, 2i, 3i, 4i, 5i$  (red) are shown. Notice that in pseudo-Euclidean geometry the circle of radius 0 is not just a point but a real pair of lines.

## 21.5 Reflections and Pole/Polar Pairs

It is a particularly interesting task to study *reflections* in the context of Cayley-Klein geometries. A reflection will be a  $\mathcal{K}$ -motion  $\tau$  with  $\tau^2 = \text{Id}$ . In planar Euclidean geometry we are used to distinguishing between line reflections (mirrors) and point reflections. We will see that at least in the nondegenerate Cayley-Klein geometries this distinction is not necessary. We start by considering involutory projective transformations on a general level. We will later on see that every reflection in a Cayley-Klein geometry turns out to be a particular such projective involution.

**Theorem 21.2.** *Let  $\tau$  be a projective transformation in  $\mathbb{RP}^2$  with  $\tau^2 = \text{Id}$ . Then*

- (i) *there is a line  $m$  with  $\tau(p) = p$  for every point  $p$  incident with  $m$ ,*
- (ii) *there is a point  $o$  with  $\tau(l) = l$  for every line  $l$  incident with  $o$ .*

*Proof.* We may assume that  $\tau \neq \text{Id}$  since otherwise the theorem holds trivially. Assume that the (primal) projective transformation is given by a matrix  $T$  with  $\tau(p) = Tp$ . The matrix  $T$  is nonsingular, and we may (possibly after rescaling) assume that  $\det(T) = 1$ . The matrix  $T$  must be similar to a Jordan matrix  $J = S^{-1}TS$ . Hence also  $J^2 = E$ . The only Jordan matrices with this property are diagonal matrices with diagonal entries  $+1$  or  $-1$ . Hence  $T$  is diagonalizable, the eigenvalues are  $\pm 1$ , and there exists a basis of eigenvectors. Since the product of the eigenvalues is  $\det(T) = 1$  and  $T$  is not the

identity, exactly two eigenvalues are  $-1$  and one eigenvalue is  $+1$ . We have  $S^{-1}TS = J = \text{diag}(-1, -1, 1)$ . Now let  $s_1, s_2, s_3$  be a basis of eigenvectors (the columns of  $S$ ). We have  $Ts_1 = -s_1$ ,  $Ts_2 = -s_2$ , and  $Ts_3 = s_3$ . The  $s_i$  correspond to fixed points of  $\tau$ , since their image differs from the preimage only by a scalar multiple. Now consider a point  $p = \lambda s_1 + \mu s_2$  on the line  $m = \text{join}(s_1, s_2)$ . We get  $Tp = T(\lambda s_1 + \mu s_2) = -\lambda s_1 - \mu s_2 = -p$ . Thus also  $p$  is a fixed point of  $\tau$ . Hence  $\tau$  leaves all points on  $m$  invariant. This proves (i). Furthermore,  $\tau$  leaves the point  $o := s_3$  invariant. Since the eigenvectors form a basis,  $p$  does not lie on  $l$ . Thus every line  $l$  through  $o$  can be expressed as the join of  $o$  and a suitable point  $p$  on  $m$ . Since both points  $o$  and  $p$  are invariant under  $\tau$ , the line  $l$  is also invariant under  $\tau$ . This proves (ii).  $\square$

We call the point  $o$  the *center* of the involution and the line  $m$  its *mirror*. If a projective involution has more fixed points than the points of  $m$  and  $o$ , it must necessarily be the identity, since in this case we find a projective basis that remains fixed. We call an involution *proper* if this is not the case. Once the line  $m$  and the point  $o$  of a proper projective involution are given, it is easy to construct concretely the involution  $\tau$ .

**Theorem 21.3.** *Let  $\tau$  be a proper involution with mirror  $m$  and center  $o$ . Then the image of a point  $p \neq o$  under  $\tau$  can be constructed as the unique point  $p'$  such that  $(p, p'; p_m, o) = -1$ , with  $p_m$  being the intersection of  $m$  and the line joining  $o$  and  $p$ .*

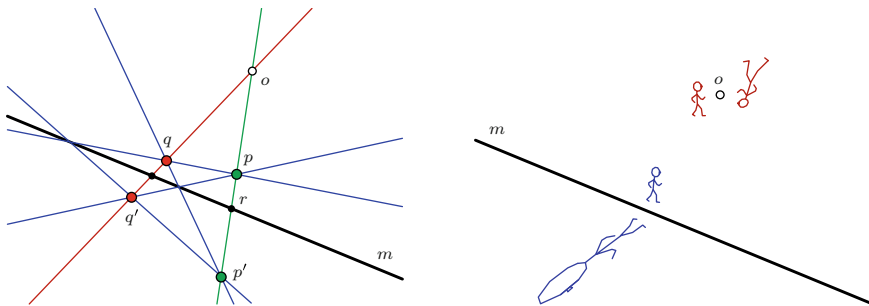
*Proof.* The image  $\tau(p)$  of  $p$  under  $\tau$  must lie on  $l := \text{join}(o, p)$ , since this line is invariant under  $\tau$ . Let  $p_m$  be the intersection of  $m$  and  $l$ . This is also a fixed point. Projective transformations leave cross-ratios invariant. Hence we have

$$\begin{aligned} (o, p_m; p, \tau(p)) &= (\tau(o), \tau(p_m); \tau(p), \tau(\tau(p))) \\ &= (o, p_m; \tau(p), p) \\ &= 1/(o, p_m; p, \tau(p)). \end{aligned}$$

This implies  $(o, p_m; p, \tau(p)) = -1$  which proves the claim.  $\square$

There is also a purely geometric way to derive the harmonic condition for the reflected point. To see this, consider Figure 21.4 (left). Assume that we have a projective transformation  $\tau$  that leaves every point on  $m$  invariant and in addition leaves the point  $o$  invariant. We consider two points  $p$  and  $q$  and their reflected images  $p' = \tau(p)$  and  $q' = \tau(q)$ . The image of the line  $\text{join}(p, q)$  is  $\text{join}(p', q')$ . The intersection of these two lines must lie on the mirror  $m$ . Similarly,  $\text{join}(p, q')$  and  $\text{join}(p', q)$  also must meet at  $m$ . Furthermore, the lines  $\text{join}(p, p')$  and  $\text{join}(q, q')$  are invariant and must pass through  $o$ . These conditions uniquely determine the positions of  $p'$  and  $q'$ . The resulting figure forms a witness configuration that  $o, r; p, p'$  are in harmonic position, where  $r$  is the intersection of  $m$  with the line joining  $o$  and  $p$ .

Figure 21.4 (right) shows the effect of an involution on two little copies (red and blue) of a drawing (for this we borrowed the character of *Dr. Stickler* from the marvelous book *Indra's Pearls* [94]). We see that close to the mirror line  $m$



**Fig. 21.4** The effect of a projective involution.

the projective involution behaves like a distorted Euclidean line reflection. Close to the center  $o$  it behaves like a distorted Euclidean point reflection. In fact, if  $o$  moves to a properly chosen point on the Euclidean line at infinity, we get an undistorted Euclidean line reflection in  $m$ . Conversely, if  $o$  is a finite point and  $m$  is the line at infinity, we get an undistorted Euclidean point reflection in  $o$ . We will now see that these are special cases of general reflections in Cayley-Klein geometries.

Before coming to this point we will give an explicit formula for the matrix of a reflection once  $m$  and  $o$  are given.

**Theorem 21.4.** *The point transformation matrix  $T$  of a projective involution  $\tau$  with center  $o$  and mirror  $m$  is given by*

$$\langle o, m \rangle E - 2om^T.$$

*Proof.* Let  $p$  be distinct from  $o$ . How can we calculate the reflected point  $p' = \tau(p)$  of  $p$ ? It must lie on the line  $l$  spanned by  $o$  and  $p$ , and thus it has coordinates of the form  $p' = \lambda p + \mu o$ . The intersection  $r$  of the lines  $l$  and  $m$  is (by the usual Plücker's  $\mu$  trick)  $r = \langle o, m \rangle p - \langle p, m \rangle o$ . To satisfy the harmonic condition  $(p, p'; o, r) = -1$  we may restrict ourselves to a local coordinate system on  $l$  given by the basis  $p, o$ . These two points correspond to the unit vectors  $(1, 0)$  and  $(0, 1)$ . The point  $p'$  has local coordinates  $(\lambda, \mu)$ , and  $r$  has local coordinates  $(\langle o, m \rangle, -\langle p, m \rangle)$ . The harmonic condition then reads

$$\frac{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \lambda & \langle o, m \rangle \\ \mu & -\langle p, m \rangle \end{vmatrix}}{\begin{vmatrix} 1 & \langle o, m \rangle \\ 0 & -\langle p, m \rangle \end{vmatrix} \cdot \begin{vmatrix} \lambda & 0 \\ \mu & 1 \end{vmatrix}} = -1.$$

Hence we have

$$-\lambda \langle p, m \rangle - \mu \langle o, m \rangle = \langle p, m \rangle \lambda.$$

Resolving for  $\lambda$  and  $\mu$  gives (up to a scalar multiple)

$$(\lambda, \mu) = (\langle o, m \rangle, -2\langle p, m \rangle).$$

Thus we can express  $p'$  as

$$p' = \langle o, m \rangle p - 2\langle p, m \rangle o = (\langle o, m \rangle E - 2om^T)p.$$

Thus the transformation matrix is  $T = \langle o, m \rangle E - 2om^T$ .  $\square$

Let us now define reflections in general Cayley-Klein geometries. They will be projective involutions that leave the fundamental conic invariant. We want to define reflections in a way that is as general as possible and also covers degenerate cases. For this we will need the concept of a pole/polar pair of point and line.

**Definition 21.2.** Let  $(A, B)$  be the primal/dual pair of matrices that defines the fundamental conic of a Cayley-Klein geometry  $\mathcal{K}$ . A pole/polar pair  $(o, m)$  of  $\mathcal{K}$  is a line  $m$  and a point  $o$  such that there exist constants  $\lambda, \mu \in \mathbb{R}$  with  $\lambda m = Ao$  and  $\mu o = Bm$ .

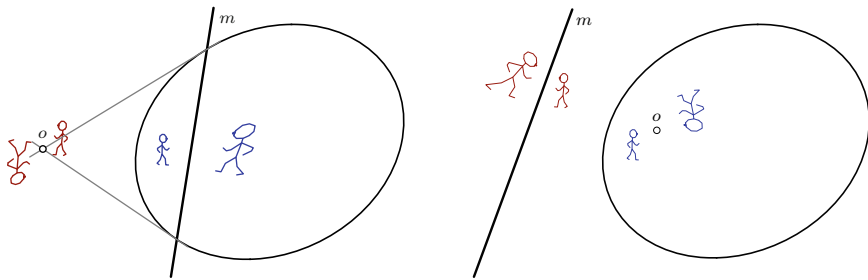
We get the usual degeneracy/nondegeneracy effects for the different geometries. In the case of a nondegenerate fundamental conic each member of a pole/polar pair  $(o, m)$  uniquely defines the other. So  $m$  is simply the polar of  $o$  with respect to the fundamental conic in the sense of Section 9.2. If  $(A, B)$  come from a singly degenerate Cayley-Klein geometry, then one of the members might uniquely define the other, but not vice versa. For instance, in Euclidean geometry an arbitrary finite line  $m$  forms a pole/polar pair with the point  $o$  on the line at infinity in the direction orthogonal to  $m$ . The line at infinity forms a pole/polar pair with every point.

We now may define a proper reflection in a given Cayley-Klein geometry

**Definition 21.3.** Let  $(o, m)$  be a pole/polar pair of a Cayley-Klein geometry  $\mathcal{K}$  such that  $o$  is not incident with  $m$ . Then the projective involution with center  $o$  and mirror  $m$  is called a  $\mathcal{K}$ -reflection.

**Theorem 21.5.** *Every  $\mathcal{K}$ -reflection is a  $\mathcal{K}$ -motion.*

*Proof.* We have to show that a  $\mathcal{K}$ -reflection leaves the fundamental conic of  $\mathcal{K}$  invariant. Let  $(o, m)$  be the corresponding pole/polar pair and let  $(A, B)$  be the primal/dual pair of the fundamental conic. Hence we have  $\lambda m = Ao$  and  $\mu o = Bm$ . By Theorem 21.4 every point is mapped according to  $p \mapsto Tp$  with  $T = \langle o, m \rangle E - 2om^T$ . We may choose the scalar factor of the elements involved such that the determinant of  $T$  becomes 1 and we have  $T^2 = E$  or equivalently  $T^{-1} = T$ . The matrix  $A$  is transformed by  $T$  according to  $A \mapsto T^T A T$ ; the matrix  $B$  is transformed according to  $B \mapsto T B T^T$  (compare Section 9.4). We get



**Fig. 21.5** Cayley-Klein reflections in the hyperbolic case.

$$\begin{aligned}
 TBT^T &= (\langle o, m \rangle E - 2om^T) \cdot B \cdot (\langle o, m \rangle E - 2om^T)^T \\
 &= \langle o, m \rangle^2 B - 2\langle o, m \rangle Bmo^T - 2\langle o, m \rangle om^T B + 4om^T Bmo^T \\
 &= \langle o, m \rangle^2 B - 2\langle o, m \rangle \mu oo^T - 2\langle o, m \rangle \mu oo^T + 4\mu o(m^T o)o^T \\
 &= \langle o, m \rangle^2 B.
 \end{aligned}$$

In this chain of equations the second equality results simply from expanding the term while transposing the matrix  $om^T$ . The third equality follows due to the fact that  $\mu o = Bm$ . Since we assumed in the theorem that  $o$  and  $m$  are not incident,  $\langle o, m \rangle^2$  is nonzero. Therefore  $B$  is mapped to a nonzero multiple of itself. The corresponding property for  $A$  follows dually.  $\square$  This proof

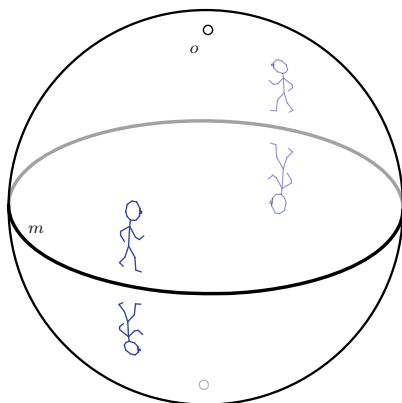
also states in a more concrete manner *how* the matrices  $A$  and  $B$  behave. We encapsulate the basic facts in a theorem that will be needed later:

**Theorem 21.6.** *Let  $T$  be the primal transformation matrix of a  $\mathcal{K}$ -reflection. Then  $T^T A T = \det(T)^2 A$  and  $T B T^T = \det(T)^2 B$ .*

*Proof.* For  $T = \langle o, m \rangle E - 2om^T$  we have  $T^2 = \langle o, m \rangle^2 E$ . Thus  $\det(T)^2 = \langle o, m \rangle^2$ . The dual part of the theorem follows from this and the equation  $T B T^T = \langle o, m \rangle^2 B$  derived in the proof of the last theorem. The primal part follows analogously.  $\square$

Figure 21.5 shows two possible situations of a reflection for a Cayley-Klein geometry of type II. Since the fundamental object remains invariant under the reflection in this case, no  $\mathcal{K}$ -motion can interchange the interior and the exterior of the fundamental conic. Thus we obtain two qualitatively different situations depending on the position of the center  $o$  with respect to the conic. If  $o$  is outside the conic, then  $m$  cuts the conic, and we get a “mirror-reflection-like” behavior in the interior of the conic. If  $o$  is inside, we get a “point-reflection-like” behavior in the interior of the conic. The situation that  $o$  is incident with the conic does not lead to a proper reflection, since in this case  $m$  and  $o$  coincide, and this was explicitly forbidden by Definition 21.3.

The situation is subtly different in the case of Cayley-Klein geometries of type I (the elliptic case). Figure 21.6 illustrates the situation. We have seen



**Fig. 21.6** Cayley-Klein reflections in the spherical case.

that (up to projective equivalence) this geometry is nicely represented by considering the sphere with antipodal pairs identified as a double cover of the projective plane. Cayley-Klein measurement then corresponds to length measurement with geodesics (shortest paths on the sphere). The fact that in elliptic geometry antipodal pairs of the sphere are identified implies that on the sphere every object is present twice. The antipodal object occurs with reversed orientation. A  $\mathcal{K}$ -reflection corresponds to a reflection in a great circle of the sphere. There is no distinction of point and line reflection in this geometry! A reflection with respect to the line  $m$  may equally well be achieved by a rotation of  $180^\circ$  with respect to the pole  $o$ .

In Euclidean geometry a  $\mathcal{K}$ -reflection leads again to two different situations. If the line  $m$  is finite, the point  $o$  lies on the fundamental object (the line at infinity) and we get a line reflection. If  $o$  is finite then  $m$  is necessarily the line at infinity and we get a point reflection. The reader is invited to work out the different situations that may arise in the other Cayley-Klein geometries.

We have seen that depending on the degree of degeneration of a Cayley-Klein geometry we obtain different degrees of freedom for the associated transformations. While the nondegenerate geometries (types I and II) admit three degrees of freedom, the singly degenerate types (types III to VI) admit four degrees of freedom. Finally, the doubly degenerate geometry (type VII) admits five degrees of freedom. In types I and II every  $\mathcal{K}$ -motion leaves distances and angles invariant. In types III and VI there exist in addition  $\mathcal{K}$ -motions that alter the angle. In types V and VI there exist  $\mathcal{K}$ -motions that alter the distance (for instance scalings in Euclidean geometry). In type VII there exist  $\mathcal{K}$ -motions that alter both angles and distances. Now, reflections



have the remarkable property that they leave also measurements invariant.<sup>1</sup> We will prove this for distance measurement. The proof for angle measurements is just the dual. We will base the distance comparison directly on the function

$$\Psi_{pq} = \frac{(p \times q)^T B (p \times q)}{p^T A p \cdot q^T A q},$$

since this function is also applicable in the case of degenerate measurements.

**Theorem 21.7.** *Let  $p$  and  $q$  be two points and let  $\tau$  be a  $\mathcal{K}$ -reflection. Let  $p' = \tau(p)$  and  $q' = \tau(q)$ . Then we have*

$$\Psi_{pq} = \Psi_{p'q'}.$$

*In other words, the distance between two points (even for degenerate measurements) is invariant under  $\tau$ .*

*Proof.* Let  $T$  be the transformation matrix that realizes  $\tau$  via  $p \mapsto Tp$ . Without loss of generality we may assume that  $T$  is chosen such that  $\det(T) = 1$ . For arbitrary points  $p$  and  $q$  we have  $(Tp \times Tq) = T^{-1T}(p \times q)$  (since  $p \times q$  represents the line through  $p$  and  $q$ ). Since  $T$  is a reflection, we have  $T^{-1} = T$ . Evaluating the term  $\Psi_{p'q'}$ , we get

$$\Psi_{p'q'} = \frac{(Tp \times Tq)^T B (Tp \times Tq)}{(Tp)^T A (Tp) \cdot (Tq)^T A (Tq)} = \frac{(p \times q)^T \overbrace{(TBT^T)}^B (p \times q)}{p^T \underbrace{(T^T AT)}_A p \cdot q^T \underbrace{(T^T AT)}_A q} = \Psi_{pq}.$$

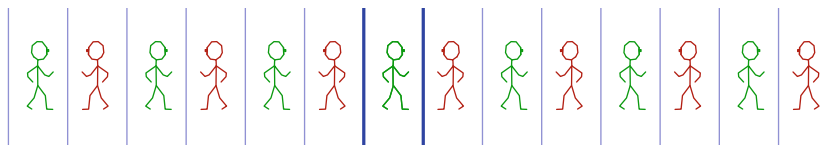
The replacements of  $T^T AT$  and  $TBT^T$  use  $\det(T) = 1$  and Theorem 21.6.  $\square$

In fact, all distance-preserving transformations can be generated by combinations of reflections. We will not prove this fact here formally, but we will (in the following section) analyze some of the different situations that may appear as the product of an even number of reflections and play the role of generalized rotations and translations.

## 21.6 From Reflections to Rotations

Perhaps you once visited a hall of mirrors, in which you could observe iterated mirror reflections of yourself. If you have two parallel mirrors and place yourself between them, you see a mirror image in each of the mirrors. But the mirror images are reflected again, and you see reflected, reflected images

<sup>1</sup> In a sense this is not too surprising, since we know this effect from Euclidean geometry, which is a degenerate Cayley-Klein geometry: A mirror image has the same size as its preimage.

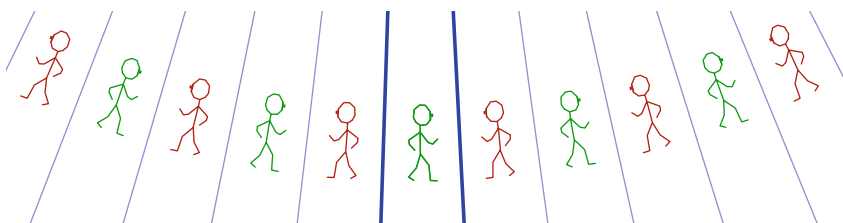


**Fig. 21.7** Iterated reflections in parallel mirrors.

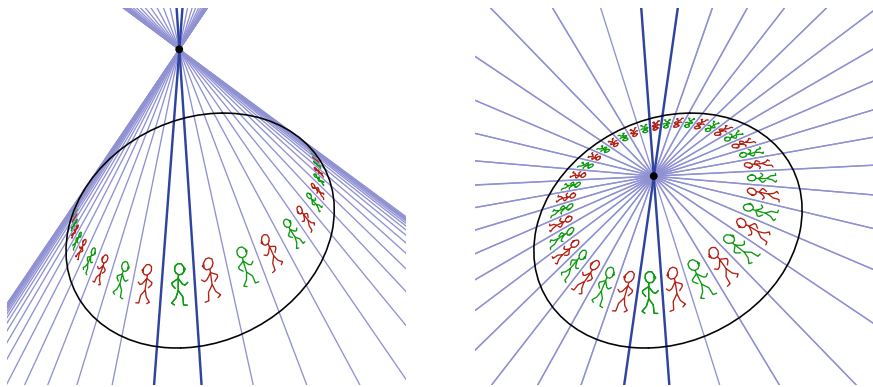
of yourself. They are reflected again, . . . . . What you see is an endlessly repeating chain of copies of yourself reaching out in both directions to infinity. Alternatingly, these mirror images are left-handed and right-handed (if you have some writing on your sweater you can properly read it in every second image). Figure 21.7 gives an impression of the geometry of this situation. You (or your alter ego Dr. Stickler) are green and stand between the two bold lines that act as mirrors. The light lines indicate the reflected mirror axes. The red/green coloring indicates the handedness of the mirror images. If for a moment you concentrate only on the green copies, you observe that they form a chain of translated copies of yourself. If the mirror operations are called  $A$  and  $B$ , then one of the green copies next to you arises as  $A(B(\text{yourself}))$ , while the other arises as  $B(A(\text{yourself}))$ . The intersection of the two mirror axes (which is infinitely far away) is a fixed point of both reflections and hence a fixed point of the translation  $A \circ B$ .

What happens if you slightly tilt one of the mirrors so that the parallelism is broken? You will observe the chain of reflected copies of yourself bending in a circular arc. Each second copy is a slightly rotated copy of yourself. The center of the rotations is again the common fixed point of the two reflections. However, this time, since the mirror axes are not parallel, it is located at a finite point. The angle of rotation is twice the angle enclosed by the mirror axes. Figure 21.8 illustrates the situation.

So why do we care about such elementary Euclidean considerations? The effects that we observe in the Euclidean case are again only very special cases of general effects occurring in the more general Cayley-Klein geometries.



**Fig. 21.8** Iterated reflections in nonparallel mirrors.



**Fig. 21.9** Dr. Stickler between two hyperbolic mirrors.

Assume that you have two different  $\mathcal{K}$ -reflections  $\tau_S$  and  $\tau_T$  in a Cayley-Klein geometry  $\mathcal{K}$ . The composition of these two reflections will again be a  $\mathcal{K}$ -motion. Since  $\tau_S$  and  $\tau_T$  are both angle- and distance-preserving, the composition  $\tau = \tau_S \circ \tau_T$  preserves distances and angles as well.

The  $\mathcal{K}$ -motion  $\tau$  has (at least) one fixed point and (at least) one fixed line. If  $m_T$  and  $p_T$  were the mirror and center of the reflection  $\tau_T$ , and  $m_S$  and  $p_S$  were the mirror and center of the reflection  $\tau_S$ , then the point  $c = \text{meet}(m_S, m_T)$  is not affected by either transformation and is hence a fixed point of  $\tau$ . Dually, the line  $\text{join}(o_T, o_S)$  is not affected by the reflection and is (as a whole) fixed under  $\tau$ . It is instructive to study the different combinations that may arise in Euclidean geometry.

- If  $\tau_T$  and  $\tau_S$  are both line reflections (i.e., the poles  $o_T$  and  $o_S$  both lie on the line at infinity), then  $\tau$  is a rotation if the intersection of the mirrors is finite. The fixed point is the center of the rotation and the fixed line is the line at infinity.
- If  $\tau_T$  is a line reflection (finite mirror  $m_T$ , infinite center  $o_T$ ) and  $\tau_S$  is a point reflection (finite center  $o_S$ , infinite mirror  $l_\infty$ ), then the fixed point of  $\tau$  is the infinite point of  $m_T$ . The fixed line is the line through  $o_S$  orthogonal to  $m_T$ . This (orientation-reversing) motion turns out to be a glide reflection.
- If both  $\tau_T$  and  $\tau_S$  are point reflections, then the fixed line is the join of the two centers. All points on the line at infinity remain fixed (since the mirror lines coincide) and we obtain a translation.

Thus by composing two reflections we may obtain rotations, translations, or glide reflections. Rotations arise only if the intersection of the mirror lines is finite.

The situation becomes more involved (or better, “less commonly known”) in the other Cayley-Klein geometries. We will briefly have a look at what happens for Cayley-Klein geometries of type II for which the fundamental conic is nondegenerate and real. Figure 21.9 illustrates two qualitatively different scenarios. Again the two mirror lines are drawn as bold blue lines. In the left picture the two mirror lines intersect outside the fundamental conic. Within the line bundle through the intersection of the mirrors we have a hyperbolic angle measurement. This results in an endless chain of reflected mirror images reaching out to both sides and never approaching the boundary. If you were Dr. Stickler, for you the situation would look much like the situation between two parallel Euclidean mirrors—endless repetition in both directions. However, projectively the situation is slightly different, since the two limiting iterations never will touch each other, as they did in the Euclidean case. In a sense, we could call such a transformation composed of the two reflections a *hyperrotation*—a rotation whose rotation center is “more than infinitely far away” (see Chapter 23). Such a hyperrotation has a single fixed line that passes through the interior of the fundamental object. This is a remarkable difference to the Euclidean case of parallel mirrors, where a translation has infinitely many fixed lines.

The right picture represents the situation in which the intersection of the mirror line is in the finite region. The angle measurement around the intersection of the mirrors is elliptic. Here the situation is much like the Euclidean case of nonparallel mirrors. If in addition the angle is a divisor of the full circle, the chain of reflections closes up and the group generated by the two reflections is finite. This is the situation shown in the picture. The composition of the two reflections is called a *rotation*. This operation also has a fixed line. However, this line is outside the horizon. It is the polar of the rotation center.

In between these two cases lies the situation in which the intersection of the mirrors is on the boundary of the fundamental object. This case is even more like the situation of Euclidean parallel mirrors, since now the limiting pictures in both directions again meet in a single point.

*Remark 21.2.* It should be finally mentioned that in many interesting situations the transformation  $\tau = \tau_T \circ \tau_S$  can be considered a member of a continuous family of distance-preserving  $\mathcal{K}$ -motions that connects  $\tau$  continuously with the identity transformation (as in Euclidean geometry, for which a rotation or translation can be considered a continuous process of moving an object to a new position). We briefly discuss this in type-II geometries. There the mirror uniquely determines the center of a reflection, which makes things a little easier. Take  $\tau_T$  and  $\tau_S$  to be reflections with axis  $m_T$  and  $m_S$ . Consider the continuous family of reflections  $\tau_\lambda$  with axis  $(1 - \lambda)m_T + \lambda m_S$ . As  $\lambda$  moves from 0 to 1, the reflection  $\tau_T \circ \tau_\lambda$  interpolates between the identity and  $\tau$ . As long as throughout this process  $\tau_\lambda$  is always admissible (i.e., the mirror is not a tangent to the fundamental conic), we get the desired continuous family of transformations.

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## Cayley-Klein Geometries at Work

*Diese Überlegung gibt uns ein schönes Beispiel dafür, wie die dualen Sätze der elliptischen Geometrie beim Übergang zur euklidischen Geometrie in Trümmer fallen.*

*Felix Klein on the angle bisector theorem,  
Vorlesungen über Nicht-Euklidische Geometrie, 1928*

Based on the measurement in a Cayley-Klein geometry, we can now define specific geometric objects and relations. For instance a *circle* may be defined as the set of all points that have a constant distance to a given point. Being *orthogonal* may be defined as a certain angle relation between two lines. In each type of a Cayley-Klein geometry the objects and relations will have very specific properties. In this chapter we will deal with aspects of elementary geometry in the context of Cayley-Klein geometries. Following the spirit of this book, we will again focus on (algebraic and geometric representations of) geometric primitive operations, on incidence theorems, and on invariance properties. Again we try to present the definitions and statements in a way that they apply as generally as possible to degenerate Cayley Klein geometries. Still some statements may break down if the geometric configurations or the underlying geometry becomes too degenerate. Since we do not want to spend most of the exposition mainly with pathological degenerate cases, we will base our definitions whenever possible on constructive approaches that allow us to explicitly calculate the objects involved. The reader should be aware that this chapter is by far more about the “how” than about the “what.” The ways theorems are interpreted and proved is more central than the theorems themselves. Nice treatments of elementary geometric theorems may also be found in [49, 86, 136].

## 22.1 Orthogonality

We start with the notion of *orthogonality*. In Euclidean geometry two distinct lines  $l$  and  $m$  are considered orthogonal if the oriented angle  $\angle(l, m)$  is identical to the oriented angle  $\angle(m, l)$ . To transfer this notion into the context of Cayley-Klein geometries, assume that  $p$  is the intersection of  $l$  and  $m$  and that  $X$  and  $Y$  are the two tangents from  $p$  to the fundamental object. Thus we are looking for a line with the property  $(l, m; X, Y) = (m, l; X, Y) \neq 1$ . Using the fact that interchanging the roles of  $l$  and  $m$  inverts the cross-ratio, we see that this can be the case only if  $(m, l; X, Y) = -1$ . As usual,  $X$  and  $Y$  are the two tangents of  $p$  to the fundamental conic. In order to cover also degenerate cases of coinciding lines in which both numerator and denominator of the cross-ratio vanish, we prefer the following characterization.

**Definition 22.1.** Two distinct lines  $l$  and  $m$  are orthogonal in the Cayley-Klein geometry  $\mathcal{K}$  if (with notions as above)

$$[h, l, X][h, m, Y] = -[h, l, Y][h, m, X] \quad (22.1)$$

for a line  $h$  not through the intersection  $p$  of  $l$  and  $m$ .

We will now derive an equivalent more concise algebraic definition for two lines  $l$  and  $m$  being orthogonal that directly relates orthogonality to the fundamental conic. For this we introduce a local coordinate system on the line bundle around  $p$  with  $l$  and  $m$  as a basis. Every line through  $p$  is expressible as a linear combination  $\lambda l + \mu m$ . In particular,  $X$  and  $Y$  have such a representation. In Section 21.3 we made a consideration dual to this in order to eliminate the points  $X$  and  $Y$  in a distance measurement. Now we want to do exactly the same with the lines  $X$  and  $Y$ . By an argument exactly dual to the content of Section 21.3 we can prove that with the abbreviations  $\Theta_{ll} = l^T B l$ ,  $\Theta_{lm} = l^T B m$ ,  $\Theta_{mm} = m^T B m$ , and  $\Delta = \Theta_{lm}^2 - \Theta_{ll}\Theta_{mm}$ , the condition (22.1) is equivalent to

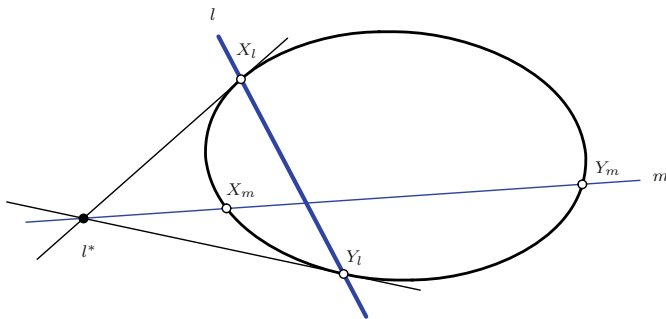
$$\Theta_{lm} + \sqrt{\Delta} = -\Theta_{lm} + \sqrt{\Delta}.$$

The expressions on the left and the negative of the expression on the right of this equation represent the numerator and the denominator of the cross-ratio. From this we easily get the following theorem.

**Theorem 22.1.** *Let  $\mathcal{K}$  be a Cayley-Klein geometry and let  $B$  be the matrix representing the dual fundamental conic. Then a pair of distinct lines  $l, m$  is orthogonal in  $\mathcal{K}$  if  $l^T B m = 0$ .*

*Proof.* In the expression  $\Theta_{lm} + \sqrt{\Delta} = -\Theta_{lm} + \sqrt{\Delta}$  the term  $\sqrt{\Delta}$  cancels. So it is equivalent to  $0 = \Theta_{lm} = l^T B m$ , as claimed.  $\square$

We can use this condition also in the case of coinciding lines  $l$  and  $m$ . In this case  $l^T B m = 0$  means that  $l = m$  is a tangent to the fundamental



**Fig. 22.1** Constructing a perpendicular.

conic. In other words, a tangent to the fundamental conic may be considered orthogonal to itself.

If the term  $l^T B$  is nonzero, it represents the pole of  $l$  with respect to the fundamental conic  $\mathcal{F}$ . Thus in this situation  $m$  is orthogonal to  $l$  if it passes through the pole of  $l$  with respect to  $\mathcal{F}$ . Similarly, if  $Bm$  is nonzero, then  $l$  and  $m$  are orthogonal if  $l$  passes through the pole of  $m$ . In most cases the situation will be suitably nondegenerate so that each line passes through the pole of the other.

It is instructive to analyze what the condition  $l^T B m = 0$  means for various situations in various geometries.

**Nondegenerate fundamental conic:** The situation is easiest in nondegenerate geometries (type I and type II). There  $B$  is regular and the poles always exist. Thus  $l$  passes through the pole of  $m$  and vice versa. Figure 22.1 shows the situation for the case of a nondegenerate real fundamental conic. If the line  $l$  intersects the conic in two real points, then the usual tangent construction provides the polar  $l^*$ . Comparing this image with Figure 10.13, we may also derive another characterization of perpendicularity in the case of a nondegenerate fundamental conic.

**Theorem 22.2.** *In a Cayley-Klein geometry with nondegenerate fundamental conic  $\mathcal{F}$ , two lines  $l$  and  $m$  are perpendicular if their intersections  $X_l, Y_l$  and  $X_m, Y_m$  are in harmonic position considered as points on a conic, i.e.,  $(X_l, Y_l; X_m, Y_m)_{\mathcal{F}} = -1$ .*

*Proof.* The fact that  $m$  has to pass through the pole of  $l$  leads to the construction of Figure 22.1. Comparing this with the Theorem 10.8 immediately proves the claim.

**Euclidean and pseudo-Euclidean geometry:** If  $B$  has rank 2 (i.e., we are either in type V or in type VI), the fundamental conic consists of a double line  $\ell_{\infty}$  with two points  $I$  and  $J$  on it. The orthogonality condition says that the intersections of the two lines with  $\ell_{\infty}$  and these two points form a

harmonic quadruple. As expected, this agrees nicely with the characterization of Euclidean orthogonality we derived in Theorem 18.6. In pseudo-Euclidean geometry the two points  $I$  and  $J$  are real. A line passing through these points is self-orthogonal.

There is one interesting case covered by  $l^T B m = 0$  that is still missing: The double line  $\ell_\infty$  is orthogonal to every other line  $l$ . This is the case since  $B\ell_\infty = (IJ^T + JI^T)\ell_\infty = 0$ . In this situation the pole of  $l$  is no longer uniquely specified.

**The remaining cases:** In all other cases the matrix  $B$  has rank 1 and is of the form  $B = qq^T$ . The point  $q$  is the double point representing the dual fundamental object. The situation  $l^T B m = 0$  can arise only if either  $l$  or  $m$  is incident with  $q$ . Thus if one line does not pass through  $q$ , then in case of orthogonality the other must pass through  $q$ . The point  $q$  is the pole of every line that does not already pass through it.

We now turn to simple theorems involving orthogonality. In the case that the expression  $Bl$  is not zero, it represents the pole of  $l$  with respect to the fundamental conic. Then we get, for instance, an easy explicit method to construct a perpendicular to  $l$  through a given point  $p$ . Expressed in algebraic terms this reads as:

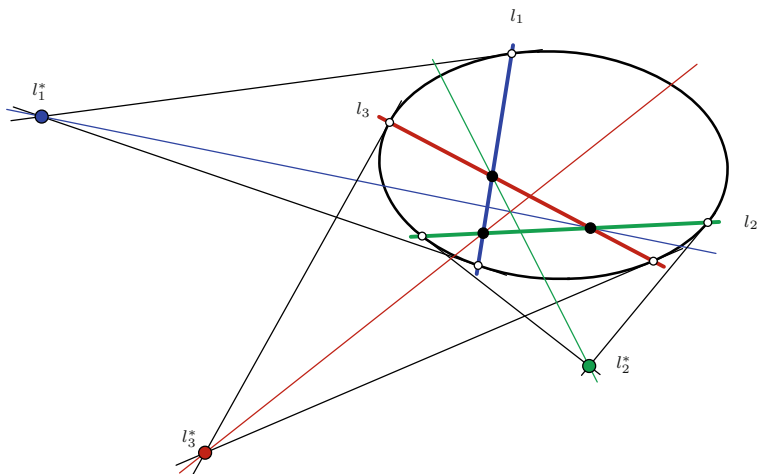
- 1: Construct the pole  $l^* = B \cdot l$  of  $l$ .
- 2: Join  $l^*$  with  $p$ .

As an application of expressing perpendicularity in a Cayley-Klein geometry we will show that also in a Cayley-Klein geometry the altitudes of a triangle meet in a point. A corresponding picture for a nondegenerate fundamental conic  $\mathcal{F}$  is shown in Figure 22.2. The three black points in the interior are the vertices of the triangle. Its sides are the lines  $l_1, l_2, l_3$ . The polars of these lines are shown in corresponding colors. Joining the polars with the corresponding opposite vertices results in the three altitudes that meet in a point. Essentially this theorem of metric character is again translated into an entirely projective statement. We could provide geometric proofs for the non-degenerate and the degenerate cases. However, we prefer to give an algebraic proof that simultaneously covers both cases.

**Theorem 22.3.** *Let  $\mathcal{K}$  be a Cayley-Klein geometry with dual fundamental conic  $B$  and let  $l_1, l_2, l_3$  be the sides of a triangle. If  $Bl_1$ ,  $Bl_2$ , and  $Bl_3$  are all nonzero, then the altitudes of the triangle are uniquely defined and meet in one point.*

*Proof.* Let  $l_1, l_2, l_3$  be the lines supporting the sides of the triangle and let  $B$  be the matrix of the dual fundamental conic. The polars of the lines are given by  $Bl_1, Bl_2, Bl_3$ . The altitude  $h_1$  to line  $l_1$  thus can be uniquely (!) written as  $h_1 = \text{join}(\text{meet}(l_2, l_3), Bl_1)$ . In terms of cross products this can be expressed





**Fig. 22.2** The altitudes meet in a point.

as  $(l_2 \times l_3) \times Bl_1$ . Applying the identity  $(v_1 \times v_2) \times v_3 = \langle v_1, v_3 \rangle v_2 - \langle v_2, v_3 \rangle v_1$ , we get

$$h_1 = \langle l_2, Bl_1 \rangle l_3 - \langle l_3, Bl_1 \rangle l_2.$$

Similarly, we get

$$\begin{aligned} h_2 &= \langle l_3, Bl_2 \rangle l_1 - \langle l_1, Bl_2 \rangle l_3, \\ h_3 &= \langle l_1, Bl_3 \rangle l_2 - \langle l_2, Bl_3 \rangle l_1. \end{aligned}$$

Now we want to prove that  $h_1, h_2, h_3$  are concurrent. Thus we want to expand the determinant  $[h_1, h_2, h_3]$ . Since each row in this expression has two summands, this expansion has eight summands. However, it is easy to check that all terms involving a left term and a right term of the above equations must vanish identically, since two rows become linearly dependent. Thus we get

$$\begin{aligned} [h_1, h_2, h_3] &= +[\langle l_2, Bl_1 \rangle l_3, \langle l_3, Bl_2 \rangle l_1, \langle l_1, Bl_3 \rangle l_2] \\ &\quad - [\langle l_3, Bl_1 \rangle l_2, \langle l_1, Bl_2 \rangle l_3, \langle l_2, Bl_3 \rangle l_1] = 0. \end{aligned}$$

This proves the claim.  $\square$

## 22.2 Constructive versus Implicit Representations

One might wonder how important the nondegeneracy assumptions “ $Bl_1, Bl_2$ , and  $Bl_3$  are all nonzero” in Theorem 22.3 were. In fact, we formulated the theorem in a way that kept us on the safe side. Let us now explore to what extent a statement like the following is true:

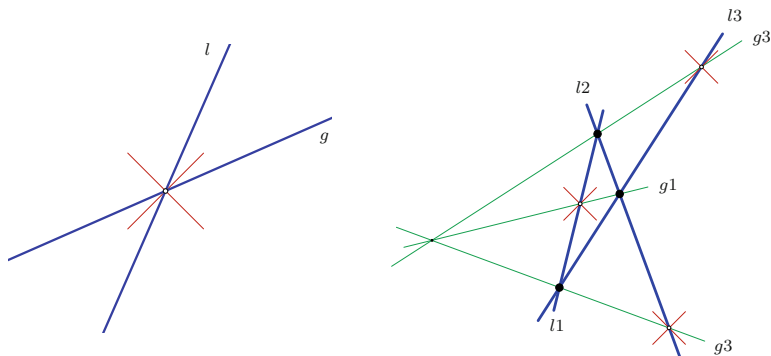
*Let  $l_1, l_2, l_3$  be lines that support the three sides of a triangle and let  $g_1, g_2, g_3$  be three lines such that  $l_i$  and  $g_i$  for  $i = 1, \dots, 3$  are orthogonal and  $g_i$  passes through the vertex  $\mathbf{meet}(l_j, l_k)$  with  $\{i, j, k\} = \{1, 2, 3\}$  for  $i = 1, \dots, 3$  (i.e., the  $g_i$  are altitudes). Then  $g_1, g_2, g_3$  have a point in common.*

Compared to Theorem 22.3 this statement characterizes orthogonality only implicitly via the equations  $l_i^T B g_i = 0$ . This statement is not true in general, but still many cases not covered by Theorem 22.3 turn out to be true. Let us explore the different situations. If all poles exist, the statement is true as a consequence of Theorem 22.3. Let us first consider the case of Euclidean geometry. The only situation not already covered is the case that one of the lines (say  $l_1$ ) is the line at infinity (every finite line has a unique pole). In this case two of the triangles vertices (those incident with  $l_1$ ) are the infinite points of  $l_2$  and  $l_3$ . The altitude  $g_2$  of  $l_2$  passes through the pole of  $l_2$  (this is an infinite point) and through the infinite point of  $l_3$ . Hence this altitude is the line at infinity. Similarly,  $g_3$  is also the line at infinity. This implies that no matter what  $g_1$  is, the three altitudes have a point in common. Thus the above statement also holds in general in the Euclidean plane. The same argument applies to pseudo-Euclidean geometry.

In the remaining cases the dual conic is a double point  $q$ . As long as the three lines  $l_1, l_2, l_3$  do not pass through this point, they have a unique polar (the point  $q$  itself), and Theorem 22.3 applies and proves that the altitudes meet. Actually, they will meet in the double point itself. What happens if one or more lines go through the double point? In the case of only one line (say  $l_1$ ) the two altitudes  $g_2$  and  $g_3$  must still go through  $q$ . Since also  $l_1$  goes through  $q$ , the two altitudes  $g_2$  and  $g_3$  must coincide with  $l_1$ , and again the statement is true.

There are not many cases left, but now the statement finally breaks apart if at least two of the lines  $l_i$  (say  $l_1$  and  $l_2$ ) pass through the double point  $q$ . In this case every line  $g_1$  is orthogonal to  $l_1$  and every line  $g_2$  is orthogonal to  $l_2$ . Thus no matter where  $g_3$  is, these lines can be chosen in a way that the three altitudes *do not meet*.

This is what Felix Klein meant in the quotation used as the epigraph to this chapter when he said that “theorems fall to pieces” if the underlying geometry is too degenerate. What makes the theorem remain true in so many cases and finally collapse in the last mentioned case? The deep reason for this is that each the “good” cases could be considered a limiting case of a nondegenerate situation. However, in the last case we forced the poles of  $l_1$  and  $l_2$  to move to different places and still represent a double-point degeneracy. This can never happen as a limiting case of a sequence of nondegenerate situations.

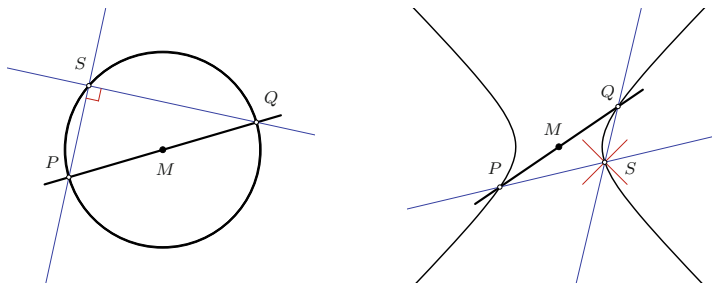


**Fig. 22.3** The altitudes theorem in pseudo-Euclidean geometry.

## 22.3 Commonalities and Differences

We have seen that a theorem like “the altitudes meet in a point” in essence remains true throughout all Cayley-Klein geometries. However, concerning *orthogonality* there are many fine points of commonalities and differences between the different Cayley-Klein geometries that are worth mentioning. We only will sketch a few of them in the hope that the reader works out many others on his/her own. Let us start with comparing the notion of orthogonality in Euclidean geometry with that in the closely related pseudo-Euclidean geometry. For both geometries we assume that they are given by the standard embedding. Thus if again  $I$  and  $J$  are the two points of the (dual) fundamental conic, we choose them as  $I = I$  and  $J = J$  in Euclidean geometry and as  $I = (1, 1, 0)^T =: I_P$  and  $J = (1, -1, 0)^T =: J_P$  in pseudo-Euclidean geometry. We now will look at the pictures of pseudo-Euclidean geometry with “Euclidean eyes.” Two finite lines  $l$  and  $g$  that intersect in  $p$  in pseudo-Euclidean geometry are orthogonal if their infinite points are in harmonic position with  $I_P$  and  $J_P$ . Since in the standard embedding the lines  $\text{join}(p, I_P)$  and  $\text{join}(p, J_P)$  form lines with slopes 1 and  $-1$ , this means that  $l$  and  $m$  have inverse slopes. In other words, with respect to the Euclidean (!) notion of reflection they are symmetric with respect to a slope-1 line through  $p$ . Figure 22.3 (left) illustrates the situation. From this we can get a Euclidean interpretation of the pseudo-Euclidean altitude theorem. *Draw a triangle with sides  $l_1, l_2, l_3$ . For each line  $l_i$  construct a line  $g_i$  with inverse slope through the vertex not on  $l_i$ . These three lines  $g_1, g_2, g_3$  meet in a point.* Figure 22.3 (right) illustrates the situation. The observant reader may recognize that we met this theorem before in a different context. It is the “mirroring slopes theorem” shown in Figure 8.12.

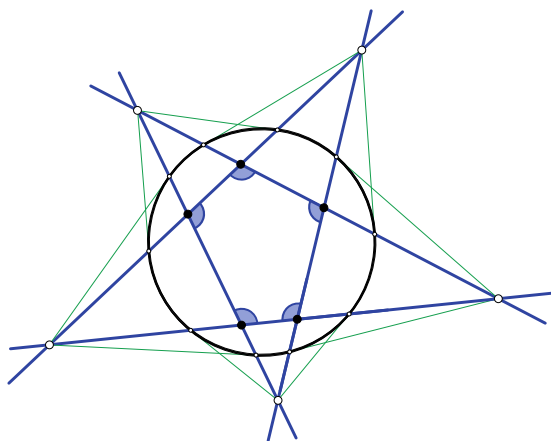
One remarkable fact has to be mentioned. The angle measurement in pseudo-Euclidean geometry is hyperbolic. Thus usually the constant  $c_{\text{dist}}$  is chosen to be  $-1/2$ . This implies that the angle between two orthogonal



**Fig. 22.4** Thales' theorem in pseudo-Euclidean geometry.

lines is the *complex* number  $-i\pi/2$ . In fact, in pseudo-Euclidean geometry it is not possible to move a line into one of its orthogonals by any sequence of finite rotations (i.e., those with a real rotation angle). Figure 22.4 shows another interesting instance of a theorem that remains true in pseudo-Euclidean geometry: Thales' theorem. In this geometry circles are (Euclidean) hyperbolas whose asymptotes have slopes 1 and  $-1$ . Thales' theorem states that each point  $S$  on a circle "sees" the endpoints  $P$  and  $Q$  of any diameter of the circle in a *right angle*.

Let us now turn to hyperbolic geometry (i.e., we have a real nondegenerate fundamental object). Perhaps one of the most striking and important differences to Euclidean geometry is the one indicated in Figure 22.5. There a pentagon is shown. Every side of the pentagon is incident with the poles of the two adjacent sides. Hence the edges of the pentagon meet orthogonally at the corner. We get a pentagon with five right vertex angles. Indeed, it is possible for every  $n \geq 5$  to have  $n$ -gons with only right angles. We will have



**Fig. 22.5** A hyperbolic pentagon with only right angles.

a closer look at this important phenomenon later when we discuss hyperbolic geometry.

## 22.4 Midpoints and Angle Bisectors

Two other interesting geometric primitive operations are the construction of midpoints and of angle bisectors. In Cayley-Klein geometries these two concepts are intimately related, since one is the dual of the other. The reason that these concepts are not obviously dual in Euclidean geometry (there are two angular bisectors of a pair of lines but only one midpoint of a pair of points) is that there the fundamental conic itself is not self-dual and induces qualitatively different behavior for angles and for distances.

Let us begin with the consideration of midpoints of two points  $p$  and  $q$  in a general Cayley-Klein geometry. We first treat the case of a nondegenerate distance measurement given by a primal matrix  $A$  of rank 2 at least. Again we abbreviate  $\Omega_{pq} = p^T A q$ . Let  $p$  and  $q$  be arbitrary points. A midpoint<sup>1</sup> of  $p$  and  $q$  is a point on the line  $l = \mathbf{join}(p, q)$  with  $\mathbf{dist}_K(p, m) = \mathbf{dist}_K(m, q)$ . Here the distance measurement is considered an oriented measurement with respect to the line  $l$ . Thus if  $X$  and  $Y$  are the two points where  $l$  meets the fundamental object, we are looking for a point  $m$  such that

$$(p, m; X, Y) = (m, q; X, Y).$$

We will now explicitly calculate the solutions of this equation. While performing the calculation we will have to make a few nondegeneracy assumptions—we will remove them in later considerations wherever possible. The cross-ratio equation above is equivalent to

$$(p, X; m, Y) = (m, X; q, Y). \quad (22.2)$$

This equation is obtained by permuting the two middle letters on each side of the equation. For the moment we will assume that  $p$  and  $q$  are distinct points. So we can introduce a local coordinate system on  $l$  by representing every point  $\lambda p + \mu q$  by the coordinates  $(\lambda, \mu)$ . By multiplying by the denominators of the cross-ratios and canceling  $[X, Y]$  we can now write condition (22.2) as the bracket expression

$$[p, m][m, Y][X, q] = [m, q][p, Y][X, m]. \quad (22.3)$$

We set  $m = \lambda p + \mu q$  and try to find the  $(\lambda, \mu)$  that satisfy the equation. We get the following condition:

$$[p, \lambda p + \mu q][\lambda p + \mu q, Y][X, q] = [\lambda p + \mu q, q][p, Y][X, \lambda p + \mu q].$$

---

<sup>1</sup> Note that we deliberately do not speak of *the* midpoint.

Expanding, eliminating brackets with double letters, collecting terms, and canceling the nonzero factor  $[p, q]$  yields

$$\mu^2[X, q][q, Y] = \lambda^2[p, Y][X, p].$$

Now we use the fact that in Section 21.3 we found that  $X, Y$  are represented in the form  $\lambda p + \mu q$  by the two points

$$\left( \Omega_{pq} \pm \sqrt{\Omega_{pq}^2 - \Omega_{pp}\Omega_{qq}} \right).$$

Furthermore,  $p$  corresponds to  $(1, 0)$  and  $q$  to  $(0, 1)$ . Plugging in these values, the above condition yields

$$\mu^2\Omega_{qq}^2 = \lambda^2(\Omega_{pq}^2 - (\Omega_{pq}^2 - \Omega_{pp}\Omega_{qq})).$$

Canceling  $\Omega_{qq}$  once on each side, we get

$$\mu^2\Omega_{qq} = \lambda^2\Omega_{pp}.$$

Thus we get the amazingly simple formula

$$m = \sqrt{\Omega_{qq}} \cdot p \pm \sqrt{\Omega_{pp}} \cdot q. \quad (22.4)$$

Let us for a moment analyze the nondegeneracy assumptions that entered this calculation. There were three of them. In the very last step we divided by  $\Omega_{qq}$ . In fact, this is not much of a restriction, since there is a completely analogous way of deriving the result by dividing by  $\Omega_{pp}$ . However, it is essential that at least one of these expressions is nonzero. Otherwise, expression (22.4) would be identically zero. The second nondegeneracy assumption was that  $p$  and  $q$  are distinct points. So let us see what happens to (22.4) if  $p$  and  $q$  are identical. Evaluating the expression with the “+” sign results in the point  $p$  again (which is a reasonable choice for the midpoint of  $p$  and  $p$ ). Evaluating the expression with the “−” sign results in the zero vector, a sign of degeneracy. Finally, we assumed that the distance measurement is nondegenerate. This means that  $X$  and  $Y$  are distinct and we can divide by  $[X, Y]$ . We made use of this when we derived the bracket expression (22.3). However, this expression is still meaningful for coinciding points  $X$  and  $Y$ . If  $X$  and  $Y$  coincide, then this equation becomes

$$[p, m][m, X][X, q] = [m, q][p, X][X, m].$$

This condition is satisfied either if  $X$  and  $m$  coincide or if  $(m, X; p, q)$  are a harmonic quadruple. This agrees with our usual interpretation of midpoints in a degenerate distance measurement. For instance, in Euclidean geometry the midpoint of  $p$  and  $q$  is the harmonic conjugate of the infinite point  $X$

on  $\mathbf{join}(p, q)$  with respect to the pair  $(p, q)$ ; and there is another point that has the same distance to  $p$  and  $q$ , namely  $X$  itself. Thus we can without any problem use our derivations (and in particular the nice formula (22.4)) also for a degenerate distance measurement.

It is an amazing fact that by this point of view we also obtain *two* points that deserve the name *midpoint of  $p$  and  $q$*  in Euclidean geometry. The ordinary midpoint and the corresponding infinite point on  $\mathbf{join}(p, q)$ . We will briefly inspect how formula (22.4) behaves for degenerate distance measurements. In this case the matrix  $A$  has rank 1. Thus it is of the form  $A = \ell\ell^T$ , where  $\ell$  represents the line that plays the role of the line at infinity. Expression (22.4) becomes

$$m = \sqrt{q^T \ell \ell^T q} \cdot p \pm \sqrt{p^T \ell \ell^T p} \cdot q = \langle q, \ell \rangle \cdot p \pm \langle q, \ell \rangle \cdot q.$$

The formula  $\langle q, \ell \rangle \cdot p + \langle q, \ell \rangle \cdot q$  gives the usual midpoint. The expression  $\langle q, \ell \rangle \cdot p - \langle q, \ell \rangle \cdot q$  can be interpreted as Plücker's  $\mu$  trick for calculating a point on  $\ell$  and on  $\mathbf{join}(p, q)$ . Thus the second midpoint is, as expected, the intersection of  $\mathbf{join}(p, q)$  with the line at infinity. This point has the same distance to  $p$  and  $q$ : it is *infinitely far away*.

We also want to mention the following important relation between the two midpoints of  $p$  and  $q$ .

**Theorem 22.4.** *Let  $m^+$  and  $m^-$  be the two midpoints of  $p$  and  $q$ . Then the pairs  $(p, q)$  and  $(m^+, m^-)$  are in harmonic relation.*

*Proof.* Expanding the condition for harmonicity

$$[p, m^+][q, m^-] + [p, m^-][q, m^+] = 0$$

yields under the use of  $m^+ = \lambda p + \mu q$  and  $m^- = \lambda p - \mu q$

$$\begin{aligned} & [p, m^+][q, m^-] + [p, m^-][q, m^+] \\ &= [p, \lambda p + \mu q][q, \lambda p - \mu q] + [p, \lambda p - \mu q][q, \lambda p + \mu q] \\ &= -\mu\lambda[p, q][q, p] + \mu\lambda[p, q][q, p] \\ &= 0, \end{aligned}$$

which is obviously true. □

*Remark 22.1.* The observant reader should notice that we did essentially the same calculation earlier, namely when we proved in Section 19.3 that the two Euclidean angle bisectors of a line pair are perpendicular.

Let us turn to a generalization of the Euclidean “the medians meet in a point” theorem. We will see that from the six possible midpoints of the points of a

triangle, we get even more collinearities and concurrences than in the usual Euclidean statement. Nevertheless, if we include infinite objects in Euclidean geometry these extra collinearities and concurrences will be present there as well.

**Theorem 22.5.** *Let  $p, q, r$  be three points such that none of them lies on the fundamental conic. Let*

$$\begin{aligned} m_{pq}^{\sigma_1} &= \sqrt{\Omega_{qq}} \cdot p + \sigma_1 \sqrt{\Omega_{pp}} \cdot q, \\ m_{qr}^{\sigma_2} &= \sqrt{\Omega_{rr}} \cdot q + \sigma_2 \sqrt{\Omega_{qq}} \cdot r, \\ m_{pr}^{\sigma_3} &= \sqrt{\Omega_{pp}} \cdot r + \sigma_3 \sqrt{\Omega_{rr}} \cdot p, \end{aligned}$$

*be the three midpoints for  $\sigma_i \in \{+1, -1\}$ . Then  $m_{pq}^{\sigma_1}, m_{qr}^{\sigma_2}, m_{pr}^{\sigma_3}$  are collinear if  $\sigma_1 \sigma_2 \sigma_3 = -1$ . Furthermore, the lines  $\mathbf{join}(m_{pq}^{\sigma_1}, r)$ ,  $\mathbf{join}(m_{qr}^{\sigma_2}, p)$ ,  $\mathbf{join}(m_{pr}^{\sigma_3}, q)$  are concurrent if  $\sigma_1 \sigma_2 \sigma_3 = 1$ .*

*Proof.* Calculating the determinant of  $m_{pq}^{\sigma_1}, m_{qr}^{\sigma_2}, m_{pr}^{\sigma_3}$  yields

$$[m_{pq}^{\sigma_1}, m_{qr}^{\sigma_2}, m_{pr}^{\sigma_3}] = \sqrt{\Omega_{pp}\Omega_{qq}\Omega_{rr}}[p, q, r] + \sigma_1 \sigma_2 \sigma_3 \cdot \sqrt{\Omega_{pp}\Omega_{qq}\Omega_{rr}}[p, q, r].$$

This equation becomes zero if either all or exactly one of the  $\sigma_i$  is zero. This corresponds to all four combinations of the claimed collinearities.

The statement about the concurrence of certain line triples is a direct consequence of the harmonicity conditions of Theorem 22.4, the collinearity just proved, and Theorem 19.4, which relates harmonicity, collinearity, and concurrence of points on triangle sides.  $\square$

Figure 22.6 illustrates the full theorem. The original triangle vertices are bold and black. The six midpoints are white. The claimed collinearities correspond to the blue lines. The six medians are the green lines. The little black dots indicate the four places where the medians meet. Considering only the blue and the green lines and the points at which triples of them meet, we get exactly Desargues's configuration (see Section 15.1).

The theorem is true in all Cayley-Klein geometries, in particular in Euclidean geometry. There the points  $m_{pq}^-, m_{qr}^-, m_{pr}^-$  are the infinite points on the lines supporting the triangle sides.

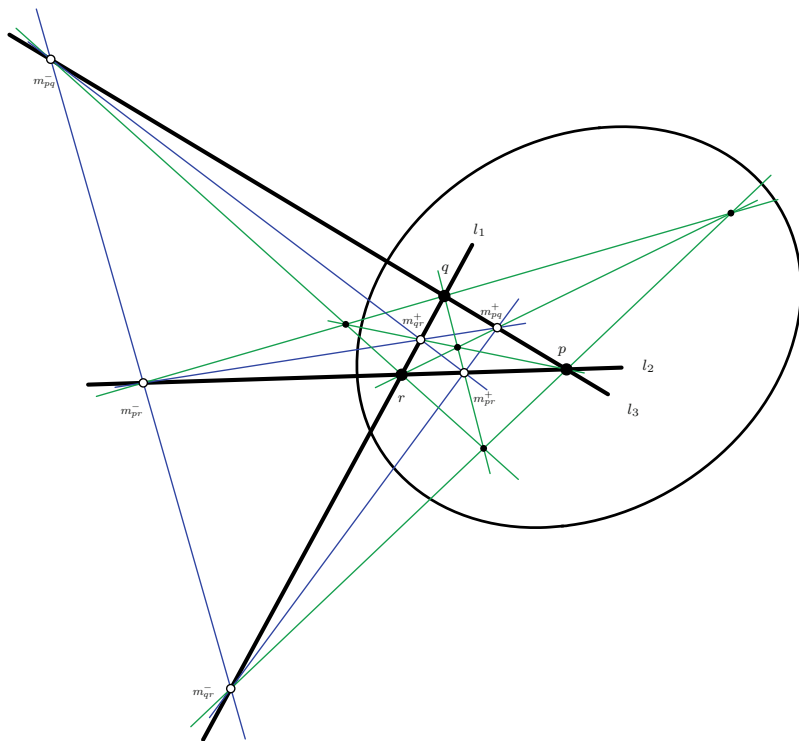
Dual to the concept of midpoints is the concept of *angle bisectors*. Given two lines  $l$  and  $g$  meeting at  $p$  we are looking for lines  $a$  through  $p$  with

$$(l, a; X, Y) = (a, g; X, Y).$$

The calculations we just did apply in direct duality, and we the following equation for the two angle bisectors:

$$a = \sqrt{\Theta_{ll}} \cdot g \pm \sqrt{\Theta_{gg}} \cdot l.$$





**Fig. 22.6** The complete midpoint theorem.

In particular, the dual of Theorem 22.5 corresponds to the angle bisector theorem in its full generality. (As an exercise try to interpret this theorem for the special case of Euclidean geometry.)

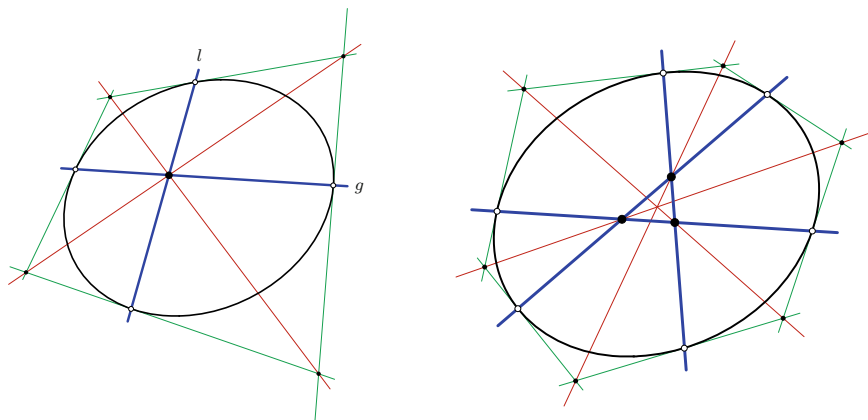
We want to mention one more amazing connection of the angle bisector theorem to another theorem of projective geometry. For this we have to consider the special case of this theorem in hyperbolic geometry (nondegenerate real fundamental object). We first consider two lines  $g$  and  $l$  passing through the interior of the fundamental object. In what follows we assume that  $m$  does not lie on the fundamental conic. Thus we have  $m^T A m \neq 0$ . The following construction generates their angle bisectors.

1. For each line construct the two tangents  $(t_l^1, t_l^2)$  and  $(t_g^1, t_g^2)$  of its intersection points with the fundamental conic to that conic.
2. The lines

$$\text{join}(\text{meet}(t_l^1, t_g^1), \text{meet}(t_l^2, t_g^2)),$$

$$\text{join}(\text{meet}(t_l^1, t_g^2), \text{meet}(t_l^2, t_g^1))$$

are the two angle bisectors.

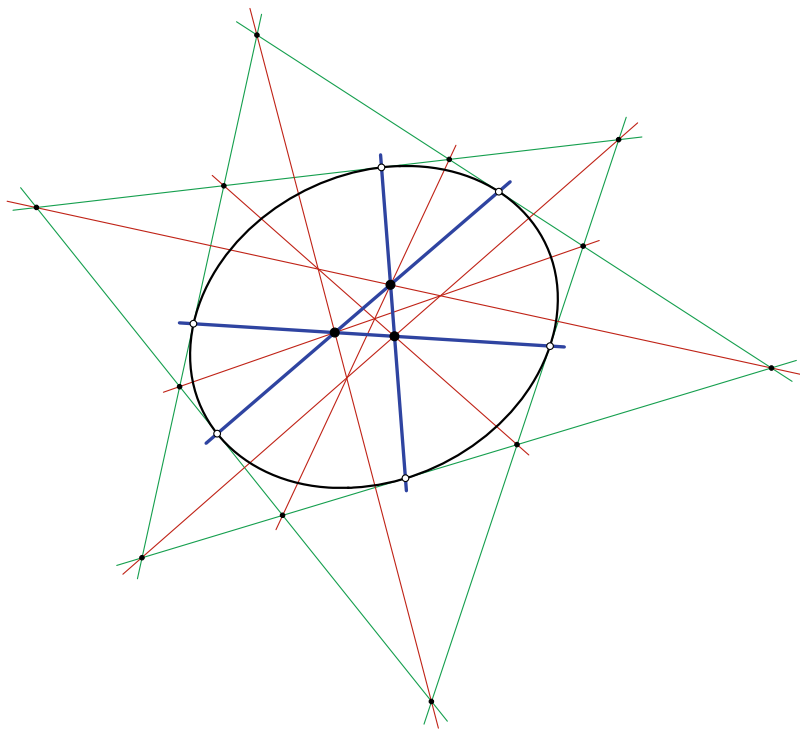


**Fig. 22.7** Constructing angle bisectors.

The construction is illustrated in Figure 22.7 (left). The lines  $l$  and  $g$  are blue, the tangents are green, and the resulting angle bisectors are red. One way to see the correctness of this construction is to apply a projective transformation that maps the fundamental conic to a circle and the intersection of the lines to the center of the circle. For this situation the correctness of the construction is an obvious Euclidean fact that follows by symmetry. The reader is invited to create a purely projective proof based on the notion of hyperbolic reflections.

We now want to construct the angle bisectors of a triangle by this construction. Figure 22.7 (right) shows the construction of the three interior angle bisectors. We already know that they must meet in a point, but this figure indicates another projective reason. If we focus on the green lines, the white points, and the conic, they form the hypotheses of Brianchon's theorem (see Theorem 10.7). This theorem then states that the three red lines must meet in a point. Figure 22.8 shows a configuration for which all six angle bisectors were constructed in this way.

*Remark 22.2.* The same construction also works in elliptic geometry. However, there the tangents turn out to have complex coordinates. The angle bisectors are real objects again.



**Fig. 22.8** The complete angle bisector theorem.

## 22.5 Trigonometry

Large parts of classical Euclidean, spherical (elliptic in our setup), and hyperbolic geometry are dedicated to the classical topic of *trigonometry*. This topic deals with interrelations of angles and distances in triangles. Angle functions such  $\sin(\dots)$ ,  $\cos(\dots)$ ,  $\tan(\dots)$  and in the hyperbolic case  $\sinh(\dots)$ ,  $\cosh(\dots)$ ,  $\tanh(\dots)$  play a prominent role in this context. In fact, almost all trigonometric formulas can be considered shadows of projective relations. These projective relations very often cover a broader context and specialize in different flavors to the well-known trigonometric formulas in the different geometries. On the level of projective geometry the theorems very often have almost trivial proofs (at least from a projective perspective). We want to exemplify this effect with just one theorem—the *law of sines*. In Euclidean geometry it says that in a triangle with sides  $a, b, c$  and corresponding opposite interior angles  $\alpha, \beta, \gamma$  the relations

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

hold (with interior angles always measured with positive sign). There is a corresponding theorem in elliptic geometry (which we for a moment consider as geometry on the sphere, ignoring the identification of antipodal points). We consider a sphere of radius 1, so that the total circumference is  $2\pi$ . If we have a spherical triangle with (geodesic) side lengths  $a, b, c$  and corresponding opposite interior angles  $\alpha, \beta, \gamma$ , the relation

$$\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\beta)}{\sin(b)} = \frac{\sin(\gamma)}{\sin(c)}$$

holds. In hyperbolic geometry (we will learn more about hyperbolic geometry in Chapter 24) the corresponding formula assumes the shape

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}.$$

The occurrence of the  $\sinh(\dots)$  function reflects the fact that the length measurement on a hyperbolic line is hyperbolic.

In what follows we will first connect the trigonometric functions to their projective counterparts. We will see that all three formulas are just incarnations of the same projective theorem. Then we will provide a conceptually simple proof for this underlying projective theorem. To avoid the sign ambiguity that arises from measuring angles and distances in a certain direction we will consider squares of the fractions in the above expressions.

Already in Section 21.4 we connected trigonometric functions to our projective ways of measuring. We will briefly redo this here in a self-contained way in order to cover as well the case of the  $\sinh(\dots)$  function. For this recall that

$$\sin(x) := \frac{e^{ix} - e^{-ix}}{2i}, \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

In Section 20.3 we introduced the standard constants  $1/2i$  and  $-1/2$  as scaling factors for elliptic, resp. hyperbolic, measurements. Using these constants, we obtain for an elliptically measured size  $\alpha = (1/2i) \cdot \ln(\Xi)$  (with  $\Xi$  playing the role of the cross-ratio)

$$\begin{aligned} \sin(\alpha) &= \sin\left(\frac{1}{2i} \cdot \ln(\Xi)\right) \\ &= \frac{e^{i \frac{1}{2i} \cdot \ln(\Xi)} - e^{-i \frac{1}{2i} \cdot \ln(\Xi)}}{2i} \\ &= \frac{e^{\frac{1}{2} \cdot \ln(\Xi)} - e^{-\frac{1}{2} \cdot \ln(\Xi)}}{2i} \\ &= \frac{\sqrt{\Xi} - 1/\sqrt{\Xi}}{2i}. \end{aligned}$$

Analogously, for  $\sinh(\alpha)$  in a hyperbolic measurement  $\alpha = (-1/2) \cdot \ln(\Xi)$  we get

$$\sinh(\alpha) = \frac{1/\sqrt{\Xi} - \sqrt{\Xi}}{2}.$$

This is up to a factor the same expression. So, how can we interpret the term  $\sqrt{\Xi} - 1/\sqrt{\Xi}$ ? In order to be able to neglect the sign right away we instead consider  $(\sqrt{\Xi} - 1/\sqrt{\Xi})^2 = \Xi + 1/\Xi - 2$ . In the following considerations we refer to distance measurements; angle measurements are completely analogous. In Section 21.3 we proved that for a (distance) measurement between  $p$  and  $q$  the involved cross-ratio is

$$\Xi = (p, q; X, Y) = \frac{\Omega_{pq} + \sqrt{\Delta_{pq}}}{\Omega_{pq} - \sqrt{\Delta_{pq}}}.$$

Inserting this in the expression  $\Xi + 1/\Xi - 2$  and abbreviating  $\Omega := \Omega_{pq}$  and  $\Delta = \Delta_{pq}$ , we get

$$\begin{aligned} \Xi + 1/\Xi - 2 &= \frac{\Omega + \sqrt{\Delta}}{\Omega - \sqrt{\Delta}} + \frac{\Omega - \sqrt{\Delta}}{\Omega + \sqrt{\Delta}} - 2 \\ &= \frac{2\Omega^2 + 2\Delta}{\Omega^2 - \Delta} - 2 \\ &= \frac{4\Delta}{\Omega^2 - \Delta} \\ &= \frac{4\Delta_{pq}}{\Omega_{pp}\Omega_{qq}}. \end{aligned}$$

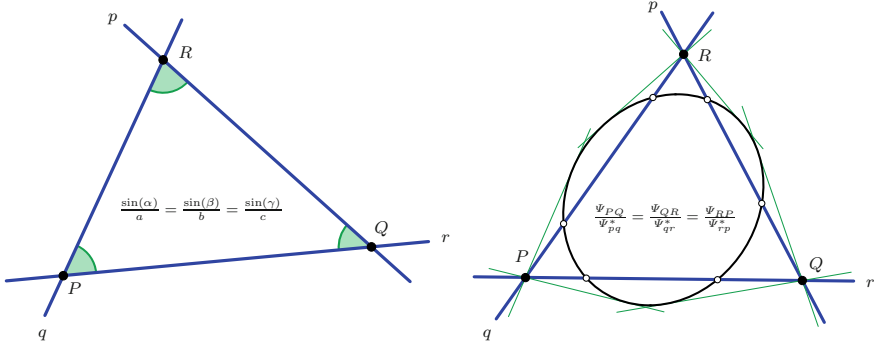
This is (up to the factor of 4) exactly the function  $\Phi_{pq}$  that we defined in Section 21.4 in equation (21.2). There we also showed that this function (again up to a constant factor depending only on  $A$  and  $B$ ) equals the expression

$$\Psi_{pq} = \frac{(p \times q)^T B (p \times q)}{p^T A p \cdot q^T A q}.$$

This expression was also used as a squared distance measurement for degenerate Cayley-Klein geometries. Thus the expression  $\Psi_{pq}$  can be interpreted in three different ways. Up to a constant scalar factor it is ...

- ...  $(\sin(\mathbf{dist}_{\mathcal{K}}(p, q)))^2$  in an elliptic distance measurement,
- ...  $(\sinh(\mathbf{dist}_{\mathcal{K}}(p, q)))^2$  in a hyperbolic distance measurement,
- ... the squared length in a degenerate length measurement.

Corresponding dual statements for angle measurements hold for the expression



**Fig. 22.9** The law of sines in its projective version relates the cross-ratios (or equivalently the measurements) at the points and sides of a triangle, with respect to the fundamental object.

$$\Psi_{lg}^* := \frac{(l \times g)^T A (l \times g)}{l^T B l \cdot g^T B g}.$$

Now we are in a position to base all three versions of the law of sines on a unified projective basis. All three statements essentially express the same theorem (which extends even to the other types of Cayley-Klein geometries). In the following theorem the role of  $A$  and  $B$  is usually played by the primal and dual matrices of the fundamental conic of a Cayley-Klein geometry, although this is not essential for the theorem to hold.

**Theorem 22.6.** *Let  $P, Q, R$  be three distinct points in the projective plane and let  $p, q, r$  coordinates of lines connecting these points:  $p = \mathbf{join}(Q, R)$ ,  $q = \mathbf{join}(R, P)$ ,  $r = \mathbf{join}(P, Q)$ . Let  $A$  and  $B$  be two arbitrary  $3 \times 3$  matrices. Assume that none of the points is on the primal conic described by  $A$  and none of the lines is tangent to the dual conic described by  $B$ . Then we have*

$$\frac{\Psi_{PQ}}{\Psi_{pq}^*} = \frac{\Psi_{QR}}{\Psi_{qr}^*} = \frac{\Psi_{RP}}{\Psi_{rp}^*}.$$

*Proof.* For symmetry reasons it suffices to prove only one equality between the involved fractions. We consider  $\frac{\Psi_{PQ}}{\Psi_{pq}^*} = \frac{\Psi_{QR}}{\Psi_{qr}^*}$ . The nondegeneracy assumption in the theorem ensures that none of the quadratic forms in the following expressions becomes zero. Expanding the functions  $\Psi$  and  $\Psi^*$ , we get

$$\begin{aligned} & \frac{(P \times Q)^T B (P \times Q)}{P^T A P \cdot Q^T A Q} \bigg/ \frac{(p \times q)^T A (p \times q)}{p^T B p \cdot q^T B q} = \\ & \frac{(Q \times R)^T B (Q \times R)}{Q^T A Q \cdot R^T A R} \bigg/ \frac{(q \times r)^T A (q \times r)}{q^T B q \cdot r^T B r}. \end{aligned}$$

Rewriting this into multiplicative form and canceling terms that occur on the left and on the right, we are left with

$$(P \times Q)^T B(P \times Q) \cdot p^T Bp \cdot R^T AR \cdot (q \times r)^T A(q \times r) = \\ P^T AP \cdot (p \times q)^T A(p \times q) \cdot (Q \times R)^T B(Q \times R) \cdot r^T Br.$$

Notice that each letter involved occurs equally often on the left and on the right of the equation. So multiplying any of them by a nonzero scalar factor does not change the truth of the equation. Now we can use the fact that there are certain incidence relations between the points and lines. We have

$$r = \mathbf{join}(P, Q); \quad R = \mathbf{meet}(p, q); \quad p = \mathbf{join}(Q, R); \quad P = \mathbf{join}(q, r).$$

In fact, it is possible to choose the scaling of the homogeneous coordinates such that we have

$$r = P \times Q; \quad R = p \times q; \quad p = Q \times R; \quad P = q \times r.$$

We will encapsulate the proof of this nice little fact in a lemma after this proof. Inserting these relations into our above expression, we get the replacements

$$\overbrace{(P \times Q)^T}^{r^T} B \overbrace{(P \times Q)}^r \cdot p^T Bp \cdot R^T AR \cdot \overbrace{(q \times r)^T}^{P^T} A \overbrace{(q \times r)}^P = \\ P^T AP \cdot \underbrace{(p \times q)^T}_{R^T} A \underbrace{(p \times q)}_R \cdot \underbrace{(Q \times R)^T}_{p^T} B \underbrace{(Q \times R)}_p \cdot r^T Br,$$

which leaves us with

$$r^T Br \cdot p^T Bp \cdot R^T AR \cdot P^T AP = P^T AP \cdot R^T AR \cdot p^T Bp \cdot r^T Br.$$

Obviously every quadratic form on the left occurs also on the right, which proves the theorem.  $\square$

Although the proof contains quite a number of letters and symbols, the reader should be aware of the fact that it is structurally extremely simple. After the replacement of the crossproducts we have only a simple cancellation argument. In essence, it boils down to a strategy of the form, *Formulate the theorem in the appropriate form—see immediately that it is true*. Furthermore, the proved fact is by far more general than the usual law of sines. It even applies to matrices  $A$  and  $B$  that have nothing to do with each other.

Still we have to prove the little lemma required for the theorem.

**Lemma 22.1.** *For three distinct points  $P, Q, R$  of a triangle and the three lines  $p, q, r$  supporting the sides, let*

$$r = \mathbf{join}(P, Q); \quad R = \mathbf{meet}(p, q); \quad p = \mathbf{join}(Q, R); \quad P = \mathbf{join}(q, r).$$

Then we can choose the concrete coordinates such that

$$r = P \times Q; \quad R = p \times q; \quad p = Q \times R; \quad P = q \times r.$$

*Proof.* We choose  $P, Q, p, q$  as coordinates representing the corresponding points that in addition satisfy  $\langle Q, p \rangle = 0$ . This can be done by suitable scaling of  $Q$ . We set  $r = P \times Q$  and  $R = p \times q$ . Inserting  $r = P \times Q$  into  $P = q \times r$ , we get

$$q \times r = q \times (P \times Q) = \langle q, P \rangle Q - \langle q, Q \rangle P = \langle q, Q \rangle P.$$

Similarly, we get

$$Q \times R = Q \times (p \times q) = \langle Q, p \rangle q - \langle Q, q \rangle p = \langle Q, p \rangle p.$$

This proves the claim.  $\square$

Let us at the end of this section summarize several projective algebraic terms and provide a table of useful relations and what they mean with respect to the various measurements. We assume that the constants are as usual chosen to be  $1/2i$  for elliptic measurements and  $-1/2$  for hyperbolic measurements. We assume that we measure the distance between two points  $p$  and  $q$ . The cross-ratio  $(p, q; X, Y)$  will be abbreviated by  $\Xi$ . The distance in the various measurements is  $a := \mathbf{dist}_K(p, q)$ . The expressions in each row are identical.

$\Xi$ -expression	$\Omega$ -expression	elliptic	hyperbolic	Euclidean
$\frac{1/\Xi + \Xi - 2}{4}$	$\frac{\Delta_{pq}}{\Omega_{pp}\Omega_{qq}}$	$-\sin^2(a)$	$\sinh^2(a)$	$k \cdot  a ^2$
$\frac{1/\Xi + \Xi + 2}{4}$	$\frac{\Omega_{pq}^2}{\Omega_{pp}\Omega_{qq}}$	$\cos^2(a)$	$\cosh^2(a)$	$k \cdot  a ^2 + 1$

The expressions in the second row have been discussed at length in this section. The factor  $k$  depends on the unit scale in the corresponding Euclidean measurement. The expressions in the third row can be easily derived from the second row using the identities  $\Delta_{pq} = \Omega_{pq}^2 - \Omega_{pp}\Omega_{qq}$ ,  $\cos^2(a) + \sin^2(a) = 1$  and  $\cosh^2(a) - \sinh^2(a) = 1$ . As usual, there are also analogous statements for angle measurements.



## Circles and Cycles

*Do not disturb my circles.*

*Archimedes of Syracuse*  
(287–212 BCE)

Let us come to another interesting class of geometric objects: *circles*. Speaking about circles in general Cayley-Klein geometries first raises a conceptual issue. Euclidean circles have several geometric properties on which one could base a more general definition. They are...

- ... objects of constant distance to a given point  $m$ —the circle’s center,
- ... objects that have the property that every incident point “sees” a given segment with endpoints on the circle under the same (directed) angle,
- ... objects that admit a continuous group of isometries under which any point on the object can be mapped to any other.

At first sight it is not at all clear that these properties are all equivalent for different Cayley-Klein geometries (in fact, they are not). So, what are reasonable properties on which one should base a general definition of a circle? Closely related to this question is, again, the problem of degenerate situations. Questions like “Should one admit centers on the line at infinity?” “How should one deal with degenerate measurements?” arise naturally in this context. In fact, it turns out that (with a proper treatment of infinite radii) the above concepts are equivalent only for Euclidean and pseudo-Euclidean geometry. For different reasons these properties begin to diverge in all other geometries. We here will again take a road that is on the pragmatic side and forms a compromise between *conceptual understanding* and *easily accessible concrete calculational formulas*. We first will stick mainly to the first property

(constant distance to the center). At the end of this chapter we will discuss alternative definitions and dedicate a short section to the subtleties of circles in type-VII (Galilean) geometries.

## 23.1 Circles via Distances

So for now let us take the obvious definition and define circles to be *objects of constant distance to a given point  $m$ , the circle's center*. We start our investigations by expressing this property in terms of the quadratic forms  $A$  and  $B$  of the primal/dual pair of the fundamental conic  $\mathcal{F}$  of the geometry. In Section 21.4 we showed that (also for degenerate measurements) the function

$$\Psi_{pq} := \frac{(p \times q)^T B(p \times q)}{p^T A p \cdot q^T A q} \quad (23.1)$$

is bijectively related to the distance between  $p$  and  $q$ : different values of  $\Psi_{pq}$  mean different distances and vice versa. Applying this to the case of circles with center  $m$ , we see that a circle is the set of points  $p$  for which

$$(m \times p)^T B(m \times p) = k \cdot m^T A m \cdot p^T A p \quad (23.2)$$

for some constant  $k$ . There is one important difference between the two formulas (23.1) and (23.2). By rewriting (23.1) in the form (23.2) we have silently dealt with the case of exceptional measurements between  $m$  and  $p$ . In (23.1) they lead simply to an undefined expression  $0/0$ . In (23.2) they lead to equations of the form  $0 = 0$ , independently of the choice of  $k$ . Thus if  $m$ , is a proposed center and  $e$  is an exceptional point with respect to  $m$  then  $e$  automatically lies on *every* circle around  $m$ . We know this effect from Euclidean geometry. The points I and J are exceptional with respect to every center  $m$ , their distance to any  $m$  is undetermined, and every circle passes through them. Depending on the fundamental conic  $\mathcal{F}$  and on the position of  $m$  relative to it, the exceptional points with respect to  $m$  may be real or complex.

Equation (23.2) imposes a quadratic condition on  $p$ . Thus in particular a circle (including the exceptional points with respect to  $m$ ) is a special conic. We can make the quadratic condition more explicit if we rewrite the  $p \times m$  operator using matrix multiplication. For  $m = (x, y, z)^T$  we set

$$M := \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

and get  $m \times p = M \cdot p$  (we used this trick already in Section 11.1). Thereby and with  $k' = k \cdot m^T A m$ , equation (23.2) becomes

$$p^T M^T B M p = k' \cdot p^T A p.$$

Equivalently for suitable  $\lambda$  and  $\mu$  we can rewrite this equation as

$$p^T \underbrace{(\lambda \cdot M^T B M + \mu \cdot A)}_{C_{\lambda, \mu}} p = 0.$$

The  $\lambda, \mu$  notion has the advantage that (as commonly in projective geometry) we can also deal with the case  $k' = \infty$  in a clean way. The matrix  $C_{\lambda, \mu}$  represents the quadratic form of the circle. The circles with center  $m$  are parameterized by the homogeneous coordinates  $(\lambda, \mu)$ . We get two extreme situations  $C_{1,0} = M^T B M$  and  $C_{0,1} = A$ . The first quadratic form  $p^T C_{1,0} p = p^T M^T B M p = 0$  encodes the geometric condition that the line  $p \times m$  is tangent to the fundamental conic. For the other extreme situation the equation  $p^T C_{0,1} p = p^T A p = 0$  simply says that  $p$  is on the fundamental conic. It is illustrative to analyze the corresponding radii for these extreme situations. We have to do this for degenerate and non-degenerate distance measurement separately.

If the measurement is degenerate, we have seen at the end of Section 21.4 that  $\Psi_{mp}$  itself is (up to a scalar normalization factor) the squared distance measure. For a point with  $p^T M^T B M p = 0$  the numerator of  $\Psi_{mp}$  becomes zero. Thus the circle represented by  $C_{1,0}$  is a circle with radius 0. For a point with  $p^T A p = 0$  the denominator of  $\Psi_{mp}$  becomes zero. Thus the circle represented by  $C_{0,1}$  is a circle with radius  $1/0 = \infty$ .

The situation is the same in the nondegenerate case, although there the reasoning is slightly different. For a point  $p$  on  $p^T M^T B M p = 0$  the line  $\text{join}(m, p)$  is tangent to the fundamental conic. Thus the distance measurement is degenerate, since  $X$  and  $Y$  will coincide and we get a distance of zero. A point  $p$  with  $p^T A p = 0$  is on the fundamental conic. Thus it coincides with  $X$  or  $Y$  and the distance measurement will become infinite.

In both cases the different circles  $C_{\lambda, \mu}$  with center  $m$  can be considered a linear interpolation between a circle with zero radius and a circle with infinite radius around  $m$ . If, for instance, we want to create a circle with center  $m$  and through a given point  $p$ , then we can calculate its coordinates by the usual Plücker's  $\mu$  trick and obtain immediately the following result:

**Theorem 23.1.** *Let  $\mathcal{K}$  be a Cayley-Klein geometry and let  $m$  be a point not on the fundamental conic of  $\mathcal{K}$ . Let  $p$  be an arbitrary point not on the fundamental conic. Then the set of all points  $q$  having the same distance to  $m$  as  $p$  united with the points exceptional with respect to  $m$  is a conic given by the quadratic equation  $q^T X q = 0$  with*

$$X = (p^T A p) \cdot M^T B M - (p^T M^T B M p) \cdot A.$$

*Proof.* Let  $k$  be the constant such that

$$p^T M^T B M p = k \cdot p^T A p.$$

The quadratic equation  $q^T X q = 0$  expands to

$$0 = (p^T A p) \cdot (q^T M^T B M q) - (p^T M^T B M p) \cdot (q^T A q).$$

Inserting the above expression gives

$$0 = (p^T A p) \cdot (q^T M^T B M q) - (k \cdot p^T A p) \cdot (q^T A q).$$

Canceling  $p^T A p$  proves that  $q$  lies on the same circle around  $m$  as  $p$ , which proves the claim.  $\square$

## 23.2 Relation to the Fundamental Conic

We will now study how a general circle (with the constant-distance-from-center definition and center not on the fundamental conic) is related to the fundamental conic. We start with the situation in nondegenerate Cayley-Klein geometries. In these geometries there is one more circle around  $m$  that deserves special attention. In Section 21.4 we have seen that in the nondegenerate case the discriminant  $\Delta_{mp} = \Omega_{mp}^2 - \Omega_{mm}\Omega_{pp}$  up to a scalar multiple equals the expression  $(m \times p)^T B(m \times p)$ . We can rewrite the discriminant as a quadratic form in  $p$  by

$$\Delta_{mp} = p^T A m m^T A p - m^T A m p^T A p = p^T \underbrace{(A m m^T A - m^T A m \cdot A)}_{\alpha \cdot C_{1,0}} p.$$

Thus up to a scalar multiple the matrices  $A m m^T A - m^T A m \cdot A$  and  $M^T B M = C_{1,0}$  are identical. Rewriting the above expression with  $A m m^T A$  on one side of the equality sign gives

$$A m m^T A = \alpha \cdot C_{1,0} + \beta \cdot C_{0,1} = C_{\alpha,\beta}.$$

This implies that in the nondegenerate case the matrix  $A m m^T A$  represents a special circle around  $m$ . The line  $m^* = A m$  is the polar line of  $m$  with respect to the fundamental conic. So the circle represented by the matrix  $A m m^T A = (m^*)^T (m^*)$  is the doubly counted polar line of the center  $m$ . This means that in particular this doubly covered polar of  $m$  must be considered a circle with center  $m$ . This tells us something about the relation of general circles to the fundamental conic. In Section 11.4 we studied the problem of intersecting two conics given by matrices  $C$  and  $D$ . The strategy there was to generate a degenerate conic in the bundle  $\lambda C + \mu D$  that

consisted only of two lines. Intersecting this degenerate conic with the conic represented by  $C$ , we get the intersections of the two original conics. Now, in case of intersecting a circle  $C_{\mu,\lambda}$  different from  $\mathcal{F}$  having center  $m$  with the fundamental conic  $\mathcal{F}$ , our considerations imply that we always get the following degenerate situation. The degenerate conic  $Amm^T A$  is in the bundle spanned by the two conics. Consequently, the intersections of  $C_{\mu,\lambda}$  and  $\mathcal{F}$  are the same as the intersections of  $Amm^T A$  with  $\mathcal{F}$ . Since  $Amm^T A$  represents a double line, we get two pairs of coinciding points of intersection. These points that are at the same time on the polar of  $m$  and on  $\mathcal{F}$  are exactly the two points that have an exceptional measurement with respect to  $m$ . All circles around  $m$  have these exceptional points with respect to  $m$  in common. This is where they touch the fundamental conic. The situation is illustrated in Figure 23.1. All in all we obtain the following:

**Theorem 23.2.** *In a nondegenerate Cayley-Klein geometry every circle of finite positive radius meets the fundamental object in two double points.*

In other words, in nondegenerate Cayley-Klein geometries a circle is *touching* the fundamental conic in two (generally different) points. The reader should at this point look back at Figures 20.7, 20.8, 20.9, and 20.10. The curves shown there are circles. The double contact is obvious for the last three figures, where the center lies outside the fundamental object. In Figure 20.7 the center lies in the interior of the fundamental conic. One might wonder where the points of double contact are in these situations. They are still present, only they become complex. So far, we have not treated the situation that the center lies *on* the fundamental conic. This is a limiting case of the situation described above in which the two touching points become coincident and the circles have even fourfold contact with the fundamental conic.

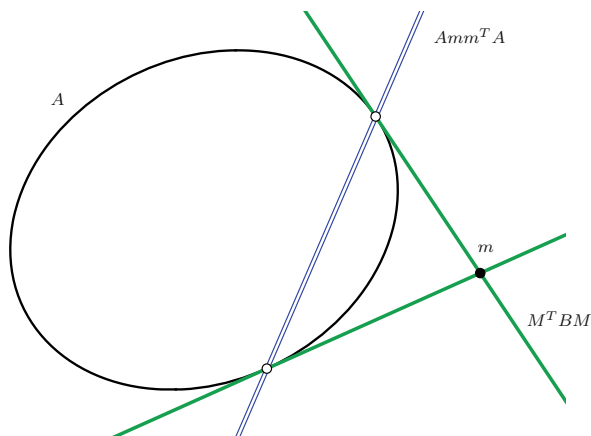
What happens in other geometries? We will not go encyclopedically through all different situations. We mention only Euclidean and pseudo-Euclidean geometry. There the dual conic is of the form  $B = \mathbf{l}\mathbf{j}^T + \mathbf{j}\mathbf{l}^T$ . The primal conic is of the form  $A = (\mathbf{l} \times \mathbf{j})(\mathbf{l} \times \mathbf{j})^T$ . A circle around  $m$  must satisfy the equation

$$(p \times m)^T (\mathbf{l}\mathbf{j}^T + \mathbf{j}\mathbf{l}^T) (p \times m) = k \cdot p^T (\mathbf{l} \times \mathbf{j})(\mathbf{l} \times \mathbf{j}) p.$$

This equation can be rewritten as

$$2[p, m, \mathbf{l}][p, m, \mathbf{j}] = k \cdot [p, \mathbf{l}, \mathbf{j}]^2.$$

In particular, this equation is satisfied by  $p = \mathbf{l}$  and  $p = \mathbf{j}$ . As one might have guessed, circles are conics through  $\mathbf{l}$  and  $\mathbf{j}$ .



**Fig. 23.1** Three interesting circles around  $m$ : The fundamental conic with radius  $\infty$  (black), the pair of tangents through  $m$  with radius 0 (green) and the doubly counted polar of  $m$  (blue). The white points have exceptional measure with respect to  $m$ .

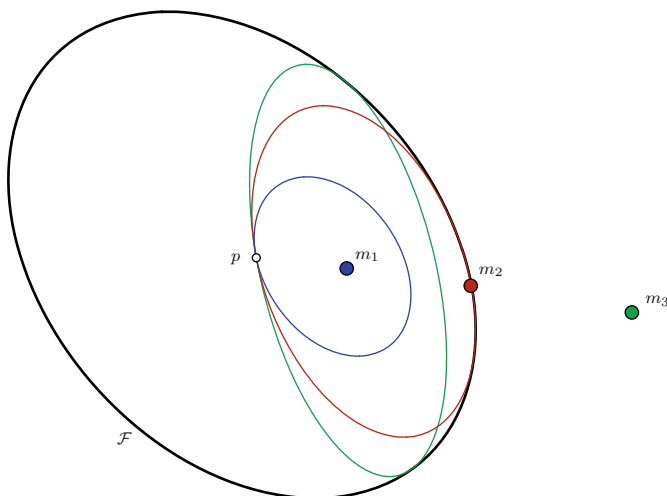
### 23.3 Centers at Infinity

So far, we have always assumed that the center  $m$  of the circle does not lie on the fundamental conic. If the center  $m$  is on the fundamental conic then all points have an infinite or exceptional distance to  $m$  (independently of the distance measurement being degenerate or not). Again limit considerations are necessary to derive a reasonable concept of a circle. However, we will only briefly touch this subject.

For nondegenerate Cayley-Klein geometries the situation is comparably simple. We have seen that for a fixed center  $m$  all circles lie in the span

$$\lambda \cdot M^T BM + \mu \cdot A. \quad (23.3)$$

The matrix  $M^T BM$  describes the degenerate conic consisting of the two tangents from  $m$  to  $\mathcal{F}$ . In the limit case that  $m$  is on the fundamental conic given by  $A$ , these two tangents coincide, and  $M^T BM$  becomes a rank-1 matrix and is up to a scalar factor identical to  $Amm^T A$ . The conic described by this matrix is still different from the fundamental object. Accordingly,  $\lambda \cdot M^T BM + \mu \cdot A$  still describes a one-dimensional bundle of conics. Each conic in this bundle may be considered a conic with “center”  $m$ . The two exceptional points with respect to  $m$  coincide, and these circles touch the fundamental object in one point. “Circles” of this type are called *horocycles*. They can occur as real objects only in the case of a nondegenerate and real fundamental conic. For good reasons they are called “cycles” instead of circles, since the usual *points-of-same-distance* definition no longer applies to them. Figure 23.2 shows three different types of circle-like objects



**Fig. 23.2** Three types of hyperbolic cycles. A circle (blue), a horocycle (red), and a hypercycle (green)

in hyperbolic geometry (the general term is *cycle*). The blue object is a usual circle. The corresponding center  $m_1$  lies inside the fundamental conic. All points on the circle have real distance to the center. The circle has no real intersections with the fundamental conic. The red cycle is a horocycle. There the two contact points with the fundamental conic coincide. This means that it touches the fundamental conic of order 4. The center  $m_2$  lies on (!) the cycle. All points of the cycle have an infinite distance to the center. However, this is no longer a characterizing property for them. Finally, the green cycle is a so-called *hypercycle*. Its center  $m_3$  is outside the fundamental conic. It touches the fundamental conic in the two points that have exceptional measure with respect to  $m_3$ . All points on the hypercycle have complex distance to the center.

The situation is slightly more complicated in the case of Euclidean or pseudo-Euclidean geometry. We study the situation in the standard embedding of Euclidean; geometry, the situation for pseudo-Euclidean geometry is analogous. The fundamental conic and a point  $m = (a, b, c)^T$  are given by

$$m = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We get

$$M^T BM = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} c^2 & 0 & -ac \\ 0 & c^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{pmatrix}.$$

If  $m$  is at infinity (this means  $c = 0$ ), then the matrix  $M^T BM$  is a multiple of the matrix  $A$ , and  $\lambda \cdot M^T BM + \mu \cdot A$  no longer generate a bundle of circles. Let us study the limit case in which  $c$  approaches 0 and we keep a finite point  $p = (x, y, 1)$  on the circle fixed. The corresponding circle is given by

$$(p^T AP) \cdot M^T BM - (p^T M^T BM p) \cdot A.$$

Plugging in the value of  $p$ , we get

$$1^2 \cdot M^T BM - (a^2 + b^2 - 2axc - 2byc + c^2(x^2 + y^2)) \cdot A,$$

which leads to the matrix

$$\begin{pmatrix} c^2 & 0 & -ac \\ 0 & c^2 & -bc \\ -ac & -bc & 2axc + 2byc - c^2(x^2 + y^2) \end{pmatrix}.$$

From this matrix we now can extract a factor  $c$ . Extracting this factor first, canceling it by homogenization, and setting  $c = 0$  in the remaining expression leads to the matrix

$$\begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & -b \\ -a & -b & 2ax + 2by \end{pmatrix}.$$

This matrix describes a degenerate conic consisting of the line at infinity and a line through  $p$  in the direction orthogonal to  $m$ . This can be easily seen by checking that each infinite point and each point of the form  $p + \alpha(-b, a, 0)^T$  lies on the corresponding conic.

It is interesting that in the limiting case it is reasonable to consider also the line at infinity as part of the circle. (Depending on the geometric setup of a certain problem, this may make sense or not.) Considering the line at infinity as part of a circle with “infinite radius” and center at infinity is consistent with our observation that circles are conics through **I** and **J** (resp. **I** and **J**, in general). The line at infinity passes through these two points, and those conics that have this line as a component are exactly the degenerate conics that consist of the line at infinity and an arbitrary other line.

## 23.4 Organizing Principles

So what are general properties that characterize a general circle in an arbitrary Cayley-Klein geometry? This question is unexpectedly difficult to



answer, since it depends a bit on the point of view and the properties that one wants to generalize. We will discuss here only some approaches and collect a few general properties that all cycles (this is what we will call the generalizations of circles) should have. Some of the issues will be touched only briefly, since they are closely related to differential geometry and we do not want to develop here the necessary differential-geometric machinery. In what follows, we will use the more general name *cycle* for circle-like objects and reserve the word circle strictly for the *points-with-constant-distance-to-center* definition.

**Cycles as limiting cases:** We have seen that in the nondegenerate cases (geometries of type I and type II and finite center) it is clear what a circle should look like. It is the set of all points having a fixed distance from the center  $m$  together with the points that have exceptional measure with respect to  $m$ , or in algebraic terms the points  $p$  that satisfy the equation

$$p^T M^T B M p = k \cdot p^T A p.$$

Perhaps the most reasonable definition for a general cycle is to say that these are all objects that arise as a continuous limit case of a sequence of nondegenerate circles. During the limiting process the matrices  $A$  and  $B$  may approach the matrices of a degenerate Cayley-Klein geometry and/or the point  $m$  may approach a position on the fundamental conic.

One might wonder why there is any difficulty at all, since we already characterized circles with infinitely distant centers in Section 23.3 for different geometries and from there derived general definitions of a cycle in these cases. We obtained there:

- In hyperbolic and elliptic geometry circles are conics that have tangential contact with the fundamental conic in two points (that may be complex). If the center is on the fundamental conic, these points of contact may coincide and generate an osculation of order four.
- In Euclidean and pseudo-Euclidean geometry cycles are conics through  $I$  and  $J$ , the two special points on the (degenerate) fundamental conic. This definition also covers the case of an infinitely distant center. Then the cycle degenerates into two lines one of which is the (primal) fundamental conic itself.

One can prove (we will not do this here) that by these characterizations all limiting cases in these geometries are properly covered. So all that is left is the case of dual Euclidean and dual pseudo-Euclidean geometry and the case of doublydegenerate (Galilean) geometry. In fact, the definition for dual Euclidean and dual pseudo-Euclidean geometry can be easily dealt with by dualizing the second of the above characterizations (we will see this in a minute). However, in the case of Galilean geometry the concepts of circles and cycles break apart seriously. We will come back to this issue in Section 23.5 in some detail. But we will now at least roughly state what happens.

The reason why we can deal in a reasonably tame way with limit cases of circles in the geometries of types I to VI is that there the limiting considerations can be made intrinsically within a single specific geometry:

*In Cayley-Klein geometries of types I to VI the limits of circles that we obtain by moving the center  $m$  to the fundamental conic are exactly the same as the limit cases we get when moving at the same time the fundamental conic  $(A, B)$  and  $m$  to a degenerate situation.*

The proof of this fact is quite technical, and we will not carry it out here. The statement allows us to do the limit considerations of circles with infinite center without changing the fundamental conic. This fact breaks down for the geometries of type VII for which as well the distance as the angle measurement is degenerate. Let us exemplify this for a concrete choice of matrices  $(A, B)$  that represent a Galilean geometry. We set

$$m = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the fundamental conic consists of the (doubly covered) line with coordinates  $l_\infty = (0, 0, 1)^T$  (this is the usual line at infinity in our standard embedding) and on it the double point  $p_\infty = (0, 1, 0)^T$  (this is the infinite point of the usual  $y$ -axis). We get

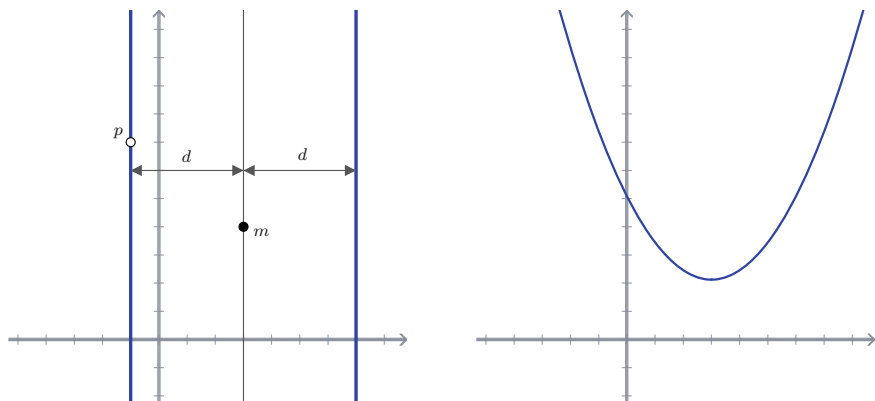
$$M^T B M = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} c^2 & 0 & -ac \\ 0 & 0 & 0 \\ -ac & 0 & a^2 \end{pmatrix}.$$

The squared distance from  $m$  to a point  $p = (x, y, z)^T$  is given by

$$\frac{p^T M^T B M p}{p^T A p \cdot m^T A m} = \frac{(cx - az)^2}{z^2 c^2}. \quad (23.4)$$

For finite points  $p$  and  $m$  we can normalize this equation by setting  $c = z = 1$  and simply obtain  $(x - a)^2$ . Thus the distance of  $p = (x, y, 1)^T$  and  $m = (a, b, 1)^T$  in the standard embedding is (up to sign) simply the difference of the  $x$ -coordinates of the points. The set of all points equidistant to  $p$  with respect to the center  $m$  is a pair of vertical lines, one of which passes through  $p$ , and the other lies symmetrically with respect to  $m$  (see Figure 23.3, left). If  $m$  is moved to an infinite point (for instance by moving it vertically up), then this qualitative property of being a conic that decomposes into a pair of lines does not change. Performing a calculation similar to the one we did for Euclidean geometry, we obtain that in the limit situation the points equidistant to  $p$  still form a pair of parallel vertical lines.

If we consider the more general situation in which we *simultaneously* deform the primal dual pair  $(A, B)$  and the position of  $m$ , we get more general

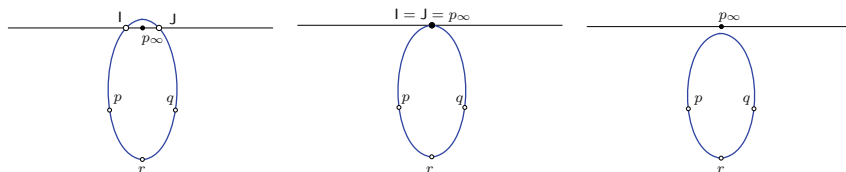


**Fig. 23.3** A Galilean circle vs. a Galilean cycle.

conics. For this we consider a primal/dual pair of matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If the parameter  $k$  traverses the interval  $[-1, 1]$ , we move continuously from the situation of pseudo-Euclidean geometry ( $k < 0$ ) via the degenerate case of Galilean geometry ( $k = 0$ ) to the Euclidean case ( $k > 0$ ). The points  $I$  and  $J$  are the solutions of  $k \cdot x^2 + y^2 = 0$  and  $z = 0$ . In the first situation  $I$  and  $J$  are real. In the process of deformation they approach each other. They coincide for  $k = 0$  with  $I = J = (0, 1, 0)^T = p_\infty$ , and branch off as complex conjugates when  $k$  enters the positive region. We now consider the situation of a circle through three finite points  $p, q, r$ . We could make the effort to perform the rather tedious explicit calculation, but we may also argue directly in geometric terms. For each choice of  $k \neq 0$  the corresponding circle is the unique conic through the five points  $p, q, r, I, J$ . Throughout the deformation process the line  $\text{join}(I, J)$  stably remains  $l_\infty = (0, 0, 1)^T$ , the primal fundamental conic. In the limiting case  $k = 0$  the points  $I$  and  $J$  coincide, and the conic through the five points becomes a conic through  $p, q, r$  and  $p_\infty = I = J$  tangent to  $l_\infty$ . Figure 23.4 illustrates this passage for the situation in which  $l_\infty$  and  $p_\infty$  were moved to finite positions. The middle picture corresponds to the case of Galilean geometry. If, as in our original choice of the fundamental conic,  $l_\infty = (0, 0, 1)^T$  and  $p_\infty = (0, 1, 0)^T$ , then in the pseudo-Euclidean case cycles look like (usual) hyperbolas, since they intersect  $l_\infty$  twice. In the Euclidean case they look like (usual) ellipses, since they do not intersect  $l_\infty$ , and in the Galilean case they look like parabolas of the form  $y = \beta(x - \alpha)^2 + \gamma$ , since they touch  $l_\infty$  at the point  $p_\infty$ . Thus



**Fig. 23.4** Limiting case for a Galilean cycle.

each parabola with vertical symmetry axis corresponds to a cycle of Galilean geometry. As special limiting cases we here again get degenerate parabolas that either consist of two vertical lines (these are the circles) or consist of  $l_\infty$  and an arbitrary other line. Figure 23.3 (right) shows one possible cycle.

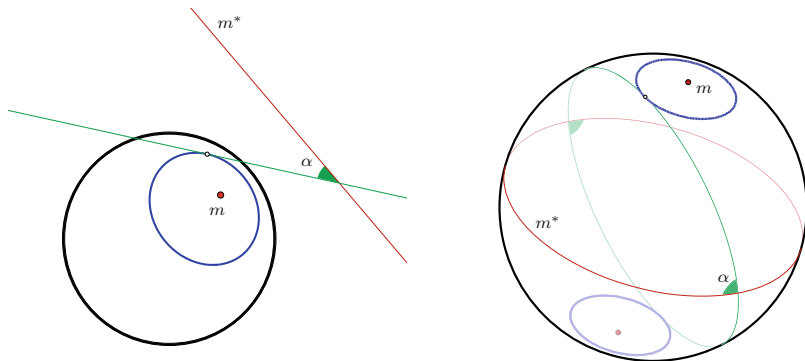
One might wonder what these general parabola-shaped cycles have to do with ordinary circles in Galilean geometry. For instance, one might be interested in the question, “where is the center?” In fact, a careful analysis shows what happens to the center: it moved to the position  $p_\infty$ , and indeed this point has the same distance to all points of a parabola. The distance is *infinite*. However, this is no longer a characterizing property. Later, in Section 23.5, we will learn about other surprising (and more illuminating) properties that are shared by a circle and a parabola.

**Duality of circles:** We now want to highlight another property of general cycles. The attribute of being a circle carries some nice duality properties. Knowing only about Euclidean geometry, these duality properties are usually obscured by the fact that the fundamental conics of this geometry is not self-dual. The duality properties are best observed in self-dual geometries where the primal and dual fundamental conic have the same degree of degeneracy (these are hyperbolic, elliptic, and Galilean geometry). So let us start our investigations with the definition of a usual (nondegenerate) circle and dualize it word by word.

**Primal:** *A circle is a curve consisting of all points that have a fixed constant distance to a given center.*

**Dual:** *A dual circle is a curve all of whose tangents have a fixed constant angle to a given line.*

At first sight it is not evident *whether* and if so *how* these primal and dual circles are related. It turns out that for nondegenerate geometries both definitions describe exactly the same class of objects. For every circle we find a suitable line with respect to which it is a dual circle. We can even be more specific. If  $m$  is the center of the circle, then its dual  $m^* = Am$  with respect to the fundamental conic of the geometry is the line that appears in the dual circle definition. By switching to algebraic terms we may right away include the case of circles with infinite radius. The following theorem formalizes this



**Fig. 23.5** The (oriented) angle of tangents to a circle with the polar of the center is constant.

duality. We recall that for nondegenerate geometries the matrix of a circle with center  $m$  can be written as a linear combination  $\alpha \cdot Amm^T A + \beta \cdot A$ . A circle with infinite radius corresponds to  $(\alpha, \beta) = (0, 1)$ . Analogously, a dual circle can be described by a linear combination  $\alpha' \cdot B(m^*)(m^*)^T B + \beta' \cdot B$ .

**Theorem 23.3.** *Let  $(A, B)$  be the primal/dual pair of a nondegenerate Cayley-Klein geometry. Let  $m$  be a point and let  $X = \alpha \cdot Amm^T A + \beta \cdot A$  be the matrix representing a circle with center  $m$ . Then the line  $m^* = Am$  has the property that for suitable  $\alpha', \beta'$  the matrix  $X' := \alpha' \cdot Bm(m^*)^T B + \beta' \cdot B$  satisfies  $X \cdot X' = \mu E$  for some parameter  $\mu$  (and hence  $X'$  describes a conic dual to the circle  $X$ ).*

*Proof.* We prove this theorem by directly multiplying  $X$  and  $X'$ , thereby deriving conditions for  $\alpha'$  and  $\beta'$ . Since we are in the nondegenerate case, we may without loss of generality assume that  $B = A^{-1}$ . We thus have  $m^* = Am$  and  $m = Bm^*$ . The matrices  $X$  and  $X'$  may be rewritten as

$$X = \alpha(m^*)(m^*)^T + \beta A \quad \text{and} \quad X' = \alpha' m m^T + \beta' A^{-1}.$$

The case  $(\alpha, \beta) = (0, 1)$  immediately leads to the condition  $(\alpha', \beta') = (0, 1)$ . Since only the ratio of the parameter pair  $(\alpha, \beta)$  is relevant, we can (in order to deal with the remaining cases) without loss of generality set  $\alpha = 1$ . The following calculation shows that we find a suitable pair  $(\alpha', \beta')$  also for  $\alpha' = 1$ . In this case we get

$$\begin{aligned}
X \cdot X' &= ((m^*)(m^*)^T + \beta A) \cdot (mm^T + \beta' A^{-1}) \\
&= (m^*)(m^*)^T mm^T + \beta Amm^T + \beta'(m^*)(m^*)^T A^{-1} + \beta\beta' AA^{-1} \\
&= ((m^*)^T m)(m^*)m^T + \beta(m^*)m^T + \beta'(m^*)m^T + \beta\beta' E \\
&= \underbrace{((m^*)^T m + \beta + \beta')}(=:0)(m^*)m^T + \beta\beta' E.
\end{aligned}$$

The condition  $(m^*)^T m + \beta + \beta' = 0$  can be easily satisfied by setting  $\beta' = -\beta - m^T Am$ . Thus  $(\alpha', \beta') = (1, -\beta - m^T Am)$  is the choice whose existence is claimed by the theorem.  $\square$

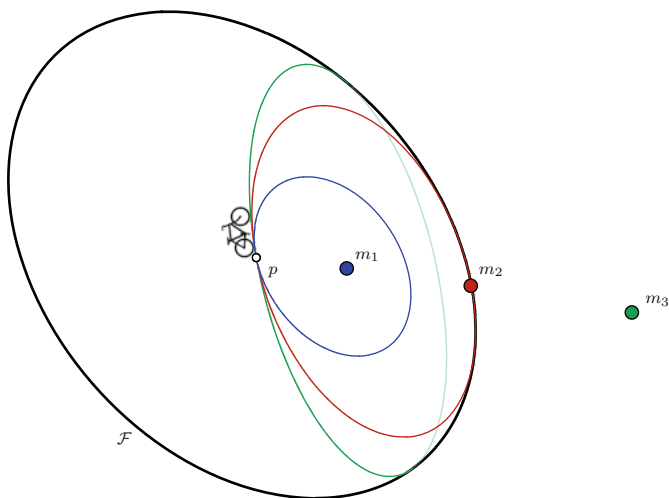
This theorem also implies that the tangents to a circle form a constant angle with the polar of the center of the circle. Figure 23.5 illustrates this theorem for a hyperbolic circle (left) and for an elliptic circle (right). The elliptic picture is as usual represented on the sphere that is a double cover of the projective plane. There the theorem becomes an almost obvious geometric fact. The figure consisting of the center, the circle, and the polar of the center forms an object that has a rotational symmetry whose axis is the line joining the two antipodal representations of the center. Any tangents to the circle can be moved into each other by a simple rotation around this axis. Thus the angle under which they meet the polar must be constant, since it remains invariant under the continuous rotation.

What happens to this duality relation in Euclidean geometry? The polar of any finite point is the line at infinity. Every finite line cuts the line at infinity under the constant angle  $\pi/2$ . Thus in particular, all tangents to a circle cut the polar of the center under this angle. Unfortunately, the situation is very degenerate. This has the consequence that the definition of a dual cycle by all tangents having a fixed angle with respect to a line describes only pathological situations. If the line is finite, then all lines cutting this line at a constant angle form a bundle of parallels. The conic that has those tangents consists of a single point at infinity. If it is the line at infinity, the only angle that may occur is  $\pi/2$ . Thus the bundle of lines meeting at this angle consists of all finite lines and does no longer describe the tangents of a conic. The same reasoning applies to pseudo-Euclidean geometry.

Our last consideration also tells us something important about cycles in dual Euclidean or dual pseudo-Euclidean geometry. There the primal definition via centers contains only pathological examples. In these geometries it is more reasonable to base the definition of a cycle on the dual situation and to prefer “midlines” to centers.

**Curves of constant curvature:** Here comes another definition of a cycle that one often finds in the literature.

*In any geometry a cycle is a curve of constant curvature.*



**Fig. 23.6** Bicycling with constant curvature.

Curvature is a notion related to differential geometry. Roughly speaking, for a smooth curve it describes the amount of *turn* at each point while the curve is traversed. If you have a constant turn while you traverse, you will travel on a cycle. A good way of imagining local curvature is to assume that the curve is a road that you travel on a bicycle. The amount of local curvature corresponds to the angle of your handlebar at each moment. If you fix the handlebar of your bicycle at a certain angle, you will travel along a circular path. This includes the limiting case of fixing the handlebar at a straight position (frontwheel aligned to rearwheel) that takes you on a straight linear road.

In contrast to our previous characterizations, the cycle definition via curvature does not a priori state the existence of a center (or a midline), nor does it state that the resulting curve has the shape of a conic. It uses only local differential properties, and the concrete shape and the existence of centers can be derived as secondary properties. However, one has to be very careful, since this local definition introduces some subtle (but extremely important) differences to the concepts we introduced in our previous sections. Very often in the literature these differences are overlooked or ignored leading to partial misunderstandings, at least for the novice in the field. Since we here do not want to develop all the necessary tools from differential geometry, we explain these differences on a more phenomenological level.

Compared to our more algebraic language, this more local differential geometric approach has some remarkable differences. They already become visible in Euclidean geometry. For the special case of a handlebar fixed in the straight position this definition gives us the line as a cycle, but does not contain the line at infinity as a component. The situation becomes more drastic

in hyperbolic geometry. For this we refer to Figure 23.6 (this figure is very closely related to Figure 23.2, in which we described the qualitatively different types of circles). Imagine you start your bicycling trip at point  $p$ . You may fix your handlebar at a position that gives you just the right amount of turn to traverse the blue circle. What will happen? With this fixed curvature you will traverse the blue circle round and round. You may start again this time fixing the handlebar to traverse the red curve (the horocycle). What will happen? While you traverse the curve you will get closer and closer to the point where the horocycle touches the fundamental conic. However, you will never reach it, since in the metric of the corresponding Cayley-Klein geometry the path to this point has infinite length. If we look at the situation “from the outside,” your bike seems to shrink as it approaches the boundary (like the fishes in an Escher circle limit picture). You yourself will never recognize this shrinking process, since during your trip all parts of the environment (trees, houses, your eye, your brain) will shrink as well in the same proportional amount. By starting from point  $p$  in the opposite direction with the same turning angle you may still traverse the other part of the curve that is below point  $p$ . Although the red curve is still connected in the inner topology of your habitat, it is topologically different from the blue circle. It seems like a line infinite in both directions. The situation becomes even more drastic when you travel along the green curve. Again you will never be able to tour beyond the points where the green curve touches the fundamental conic. However, this time this prevents you from reaching a whole part of the green curve. The segment (dimmed green) between the two touching points that does not contain  $p$  is not reachable by traveling on a curve with constant curvature. Similarly, if you started on this dimmed green part you would never be able to reach the other part by traveling with the corresponding constant curvature. If we define a cycle via the local curvature property, we must consider the green curve as *two* different unconnected cycles. In our algebraic setup we treated it as *one* cycle, since both branches lie on the same conic. We will return to this issue later when we speak about hyperbolic geometry.

We will briefly mention that the effect we just explained is also very much related to generating circular shapes by a sequence of iterated reflections. If you look back at Figure 21.9, you observe that the feet of Dr. Stickler in the hall of mirrors stand on a circular path. If the two mirror axes meet inside the fundamental conic, this path will be a full circle. If they meet outside, then the mirror images will appear on only one branch of the cycle. We may consider the transformations of the green copies of Dr. Stickler as a kind of discrete version of the bicycle metaphor. To get from one green copy to the next, Dr. Stickler must take a step of constant width and then turn around by a constant angle.<sup>1</sup>

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<sup>1</sup> In a sense, this is a kind of Cayley-Klein-turtle-graphics: move forward—turn—move forward—turn—move forward—turn—...



## 23.5 Cycles in Galilean Geometry

We will finish our bestiary of circles by considering the situation in type-VII Cayley-Klein geometries (Galilean geometry) a bit more closely. We already mentioned that in this geometry the definitions of circles and cycles differ most significantly. While circles (as curves of constant distance to a finite point) are a pair of parallel vertical lines, cycles are general parabola shaped curves with vertical symmetry axis. There are significantly more cycles than circles in this geometry. The general equation of such a parabola has three free parameters; the circles have only two free parameters. Furthermore (if we include limit cases of parabolas), every circle is a cycle. In this chapter we want to show how several theorems that are well known for Euclidean geometry transfer to Galilean geometry. Before we go into the details of the theorems, we will give two more indications why the parabola shaped cycles are the right object to consider in Galilean geometry. Firstly, if we take the *path of constant curvature definition* of a cycle, then it turns out that in Galilean geometry these curves are exactly the parabolas with vertical symmetry axis. Secondly, if we consider the iterated reflections of two mirrors, we also get parabola shaped traces of the mirror images of an object. To see this, we have to reconsider our precise definition of a reflection in a Cayley-Klein geometry. This was defined by a line and a point that form a pole/polar pair in the geometry. Now, in Galilean geometry every pair of a line incident with  $p_\infty$  and a point on  $l_\infty$  form such a pole/polar pair. Figure 23.7 shows a computer experiment of placing Dr. Stickler inside a hall of mirrors consisting of two vertical mirrors (they pass through  $p_\infty$ ) with corresponding mirror points on the line at infinity (not visible in the picture). The iterated reflection generates a parabola shaped trace. The exact position of the mirror points of the reflections determine the position and steepness of the observed parabola.

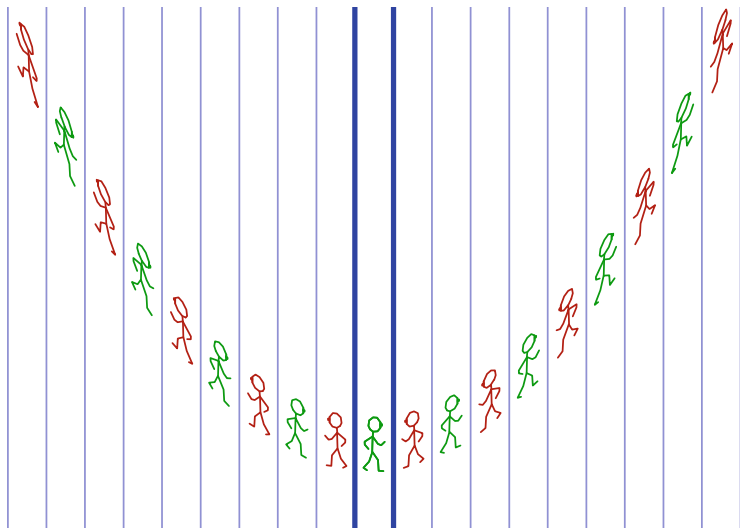
Before studying elementary geometric theorems in Galilean geometry we must discuss the precise notion of distances and angles. Both types of measurements (distances and angles) are degenerate, and concerning distances we have already done this. In equation (23.4) we have shown that in the standard embedding with  $l_\infty = (0, 0, 1)^T$  and  $p_\infty = (0, 1, 0)^T$ , the squared distance of two points with homogeneous coordinates  $p = (x_p, y_p, z_p)^T$ ,  $q = (x_q, y_q, z_q)^T$  is

$$\frac{(x_p z_q - x_q z_p)^2}{z_p^2 z_q^2} = \left( \frac{x_p}{z_p} - \frac{x_q}{z_q} \right)^2.$$

We may directly use the term

$$\mathbf{dist}(p, m) = \frac{x_p}{z_p} - \frac{x_q}{z_q}$$

as oriented distance in this geometry, since along the line  $\mathbf{join}(p, m)$  the measurements always refer to the same objects on the fundamental conic.



**Fig. 23.7** Dr. Stickler in a Galilean mirror cabinet.

Since the fundamental conic is self-dual (point  $p_\infty$  incident to line  $l_\infty$ ) we get a corresponding formula for angles between lines. If  $l = (a_l, b_l, c_l)$  and  $g = (a_g, b_g, c_g)$  are given, then a similar calculation shows that the (oriented) angle is

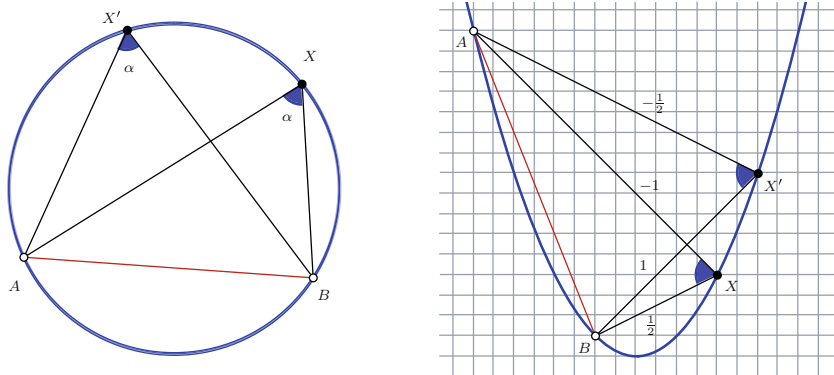
$$\text{ang}(l, g) = \frac{a_l}{b_l} - \frac{a_g}{b_g}.$$

Bearing in mind that in the standard embedding the line  $l$  has the equation  $x \cdot a_l + y \cdot b_l + c_l = 0$ , which (if  $l$  is nonvertical) is equivalent to

$$y = -\frac{a_l}{b_l} \cdot x - \frac{c_l}{b_l},$$

we see that (up to a sign change) the Galilean angle of two lines is just the difference of their slopes. Since Galilean geometry is a rather degenerate environment, the distance and angle measurement turn out to be performable without much calculation. For the moment we may make our life a little easier and strip off our projective framework and directly express Galilean measurements in unusual  $\mathbb{R}^2$  coordinates. For this we will simply ignore points at infinity (i.e., points on  $l_\infty$ ), and ignore vertical lines (i.e., lines through  $p_\infty$ ) as well.<sup>2</sup> In this simplified framework the points  $p$  and  $q$  are given by the two-dimensional coordinates  $p = (x_p, y_p)$  and  $q = (x_q, y_q)$ . The lines  $l$  and  $g$  are represented by equations  $y = k_l \cdot x + r_l$  and  $y = k_g \cdot x + r_g$ . The formulas for distance and included angle then simply become

<sup>2</sup> In a sense, we are dealing for a moment with Galilean geometry in the same way we dealt with Euclidean geometry before projective geometry was invented.



**Fig. 23.8** The peripheral angle theorem in Euclidean and Galilean geometry. In the Euclidean case it states that the Euclidean angles at  $X$  and  $X'$  are identical. In the Galilean case it states that the Galilean angles at  $X$  and  $X'$  are identical. This means that the differences of corresponding line slopes are equal.

$$\text{dist}(p, m) = x_p - x_q \quad \text{and} \quad \text{ang}(l, g) = k_l - k_g.$$

We will soon consider theorems that involve exactly one Galilean cycle. By applying a suitable Galilean transformation this cycle can without loss of generality always be chosen to be the unit parabola  $y = x^2$ . This can be seen as follows. Galilean transformations are those projective transformations that leave  $l_\infty$  and  $p_\infty$  invariant. In other words, they preserve (Euclidean) parallelism *and* the property of being a vertical line. Expressed in  $\mathbb{R}^2$ , two types of such transformations are translations  $(x, y) \mapsto (x + t_x, y + t_y)$  and scaling of the  $y$ -axis  $(x, y) \mapsto (x, y \cdot s)$  with  $s \neq 0$ . Using a combination of these two operations every parabola can be transformed into the form  $y = x^2$ .

Equipped with these preconsiderations that allow for reasonably simple calculations, we now study a few theorems about cycles in Galilean geometry. The first problem is to identify theorems in other Cayley-Klein geometries that may have a chance to be still not too degenerate to be meaningful in Galilean geometry. The problem is that theorems involving, for instance, *centers* of circles (or cycles) automatically become too degenerate. If a cycle is a circle (two vertical lines), then it has infinitely many possible centers, namely all points on the middle axes of the two lines that constitute the circle. If the cycle is a parabola, then the center is automatically  $p_\infty$ , so all such cycles share this center. As a consequence, also the notion of a “diameter” is quite degenerate. It is a line through *the* center of a cycle. So for a parabola shaped cycle this is just any vertical line. The following are interesting exercises that I want to leave to the reader. What are the Galilean analogues for Thales’

theorem, the angle bisector theorem for a triangle, and the medians-meet-in-a-point theorem?

So, what are possible interesting theorems that may be interpreted in Galilean geometry? Such theorems should directly make statements about measurements and their relations without making reference to objects that become degenerate. One such theorem is the peripheral angle theorem in Euclidean geometry:

*Let  $A$  and  $B$  be two points on a circle. Then every other point  $X$  on the circle “sees”  $A$  and  $B$  under the same oriented angle (see also Theorem 17.1).*

A corresponding theorem in Galilean geometry translates to this:

*Let  $A$  and  $B$  be two points on a parabola then for every other point  $x$  on the parabola the difference of the slopes of the lines  $\overline{AX}$  and  $\overline{BX}$  is a constant depending only on  $A$  and  $B$ .*

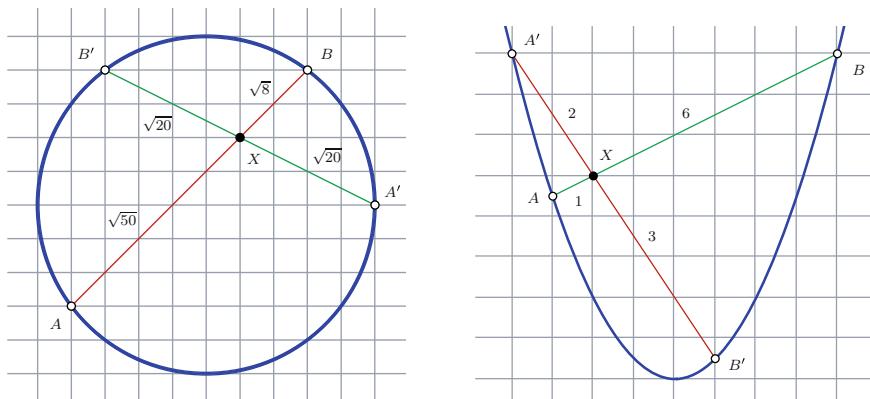
The difference of the slopes corresponds to the angle under which the points are seen in the Euclidean version.

In principle, this Galilean version follows instantly by a limit argument that deforms Euclidean geometry into Galilean geometry and the fact that the peripheral angle theorem does hold in Euclidean geometry. We here provide a sketch of the limit argument, which goes as follows. We know that the peripheral angle theorem holds in Euclidean geometry. We deform the fundamental object by introducing a parameter  $k$  that traverses the interval  $[0, 1]$  from 1 to 0 moving  $I = (-ik, 1, 0)$  and  $J = (ik, 1, 0)$  to finally become coincident at  $p_\infty$ . As long as  $k > 0$ , the geometry thereby defined is still Euclidean (although not identical to our standard embedding with  $I = J$  and  $J = J$ ). Hence with the corresponding measurement of the geometry the peripheral angle theorem still holds. In the limit case (at  $k = 0$ ) the geometry becomes Galilean. The circle becomes a cycle in this Galilean geometry, lines remain lines, and the comparison of angle measurement in Euclidean geometry asymptotically becomes the comparison of angle measurement in Galilean geometry (this requires a little work analogous to our considerations of Section 20.4). As a consequence, the corresponding Galilean theorem holds as well.

Nevertheless, since the Galilean statement is of a very elementary nature, we also want to provide an elementary proof for it.

*Proof.* Without loss of generality we may limit ourselves to the case of a standard parabola  $y = x^2$ . The points involved in the theorem have coordinates  $A = (a, a^2)$ ,  $B = (b, b^2)$ , and  $X = (x, x^2)$ . The slopes of the lines  $\overline{AX}$  and  $\overline{BX}$  calculate to

$$\frac{a^2 - x^2}{a - x} \quad \text{and} \quad \frac{b^2 - x^2}{b - x}.$$



**Fig. 23.9** The product of the lengths of the two sections of a secant depends only on the circle and the cutting point  $X$ . This holds as well in Euclidean as in Galilean geometry. In the latter the lengths are the differences of the ordinates.

Since  $X$  was assumed to be distinct from  $A$  and  $B$ , we may factor out the denominators and obtain the slopes

$$a + x \quad \text{and} \quad b + x.$$

Thus the difference of the slopes is simply  $a - b$  and does not depend on the position of  $X$ .  $\square$

Let us close this section with one more example of a Euclidean theorem that nicely translates to Galilean geometry.

*Let  $P$  be an arbitrary point and let  $\mathcal{C}$  be a circle. Let  $l$  be a line through  $p$  that intersects  $\mathcal{C}$  in two distinct points  $A$  and  $B$ . Then the product of lengths the of the segments  $(P, A)$  and  $(P, B)$  depends only on  $P$  and  $\mathcal{C}$ .*

The Galilean analogue of this theorem becomes the following:

*Let  $P = (x_p, y_p)$  be an arbitrary point and let  $\mathcal{P}$  be a parabola. Let  $l$  be a line through  $p$  that intersects  $\mathcal{P}$  in two distinct points  $A = (x_a, y_a)$  and  $B = (x_b, y_b)$ . Then the product  $(x_a - x_p)(x_b - x_p)$  depends on  $P$  and  $\mathcal{P}$ .*

Again, the Galilean theorem can be proved by limit considerations from the Euclidean one. But again we will provide an elementary proof.

*Proof.* We may assume that  $\mathcal{P}$  is the unit parabola  $y = x^2$ . Then  $A$  and  $B$  may be represented by coordinates  $A = (a, a^2)$  and  $B = (b, b^2)$ . Let  $P$  have coordinates  $P = (x, y)$ . The collinearity of  $P, A, B$  may be expressed as

$$0 = \det \begin{pmatrix} a & b & x \\ a^2 & b^2 & y \\ 1 & 1 & 1 \end{pmatrix} = ab^2 + by + a^2x - ay - ba^2 - b^2x = (ab + y - (b+a)x)(b-a).$$

Dividing by  $(b-a)$  (the points  $A$  and  $B$  are distinct) gives

$$(ab + y - (b+a)x) = 0. \quad (23.5)$$

Now we consider the product  $(a-x)(b-x)$ , which is claimed to be constant by the theorem. Expanding gives

$$(a-x)(b-x) = ab - x(a+b) + x^2.$$

Subtracting (23.5) from this expression, which means subtracting zero, gives

$$(a-x)(b-x) = -y + x^2. \quad (23.6)$$

Thus the product depends only on  $x$  and  $y$ , as claimed by the theorem.  $\square$

There is a remarkable connection of this theorem to a theorem we already proved earlier. Lemma 10.1 provided a kind of mechanical multiplication device based on a parabola (see also Figure 10.9). The theorem we just proved can be directly used to demonstrate that this multiplication device works properly. Lemma 10.1 just corresponds to the special case  $x = 0$  in which equation (23.6) becomes  $a \cdot b = -y$ .

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## Non-Euclidean Geometry: A Historical Interlude

*Some of the men stood talking in this room, and at the right of the door a little knot had formed round a small table, the center of which was the mathematics student, who was eagerly talking. He had made the assertion that one could draw through a given point more than one parallel to a straight line; Frau Hagenström had cried out that this was impossible, and he had gone on to prove it so conclusively that his hearers were constrained to behave as though they understood.*

*Thomas Mann, Little Herr Friedemann*

*History will teach us nothing.*

*Sting*

Imagine you are a two-dimensional being living in the interior of a real non-degenerate fundamental conic of a Cayley-Klein geometry. All your measurements (distances and angles) are done with respect to this Cayley-Klein geometry, and you have no knowledge of the fact that your world is embedded in some larger space (the projective plane in which the Cayley-Klein geometry is defined). One day your dog, your ruler, and you decide to take a long, long walk always following the same direction. How would that feel? In a sense it would not feel very exciting, and this is indeed an exciting thing. The three of you simply go on without anything remarkable happening. A person from the outside observing you will see you all getting smaller and smaller as you approach the boundary of the fundamental object. With your legs shrinking, your step size (observed from the outside) is getting smaller and smaller, too. Every single step will be very small compared to your current distance to the

boundary. You yourself will recognize nothing of this change of size. Together with you your dog, your shoes, your ruler, everything shrinks by the same amount. So when in regular intervals of time you measure the size of your dog (this is an old habit of yours), it neither shrinks nor grows. What does measuring mean? Well, you take your ruler and compare it to the length of your dog. Your dog was three times as long as your ruler when you started your trip and it is so every time you redo the measurement. So who could daresay that something has changed?

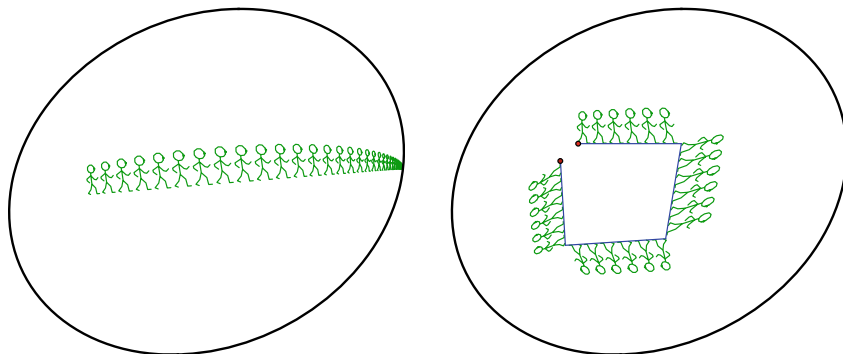
## 24.1 The Inner Geometry of a Space

In this little story there are two observers: you inside the conic and a person looking from the outside. Both describe the same situation in different terms. You from the inside observe an infinite space. The person from the outside observes you constantly shrinking within a finite space. Both descriptions are perfectly legitimate. Yours describes the *inner geometry* of the space you live in. The person outside describes the scenario he sees in terms of the geometry he lives in (and perhaps as an irony of fate his world is also embedded in some strange way into a larger space he does not know about).

Do you as an inhabitant of the interior of the conic have any chance to find out that the world you live in is indeed the finite interior of a conic? Does this question even make sense? In a way yes, in a way no, and a good answer to these questions becomes philosophical sooner or later. You as a being in your world have very good reasons to call it *infinite in every direction*. However, your intellect (after quite a lot of mathematical thought) may tell you that there are ways to realize your world (to find a model for it consistent with your experiences) in a way that it fits into the finite region of the interior of a conic. However, in order to make some predictions about the geometric behavior of your habitat it does not matter which way you describe it. So both standpoints (the inside/infinite view and the outside/finite view) may be equally appropriate as long as they make the same predictions about what will happen.

We have seen in the previous chapters that compared to the Euclidean geometry we learn in school (and which is perhaps—at least locally—the geometry of the space we are living in), the geometry in a general Cayley-Klein geometry may be significantly different (one may have, for instance, hyperbolic angle measurement, so that it is not possible to turn around one's own axis and face the same direction again). But what is it like inside the nondegenerate and real conic? There angle measurement is elliptic as we are used to from Euclidean geometry: turning around by  $360^\circ$  brings you back to your old position. Distance measurement is in a good sense infinite in each direction, as in Euclidean geometry. However, the differences to Euclidean geometry come into play whenever distance and angle measurements interact.





**Fig. 24.1** Dr. Stickler making an infinite walk (left) and performing the square-walk experiment (right).

So here is a little experiment one might perform that might be illuminating about the inner geometry of the space one is living in:

- First construct a device with which you can measure a right angle (perhaps take a piece of paper, fold it and fold it again so that your first fold comes to lie on itself. After unfolding the paper you see four right angles).<sup>1</sup>
- Put a pin at your start position,
- Then choose any direction and walk 1000 steps straight in this direction. Make a right turn by exactly  $90^\circ$ , walk 1000 steps straight, make a right turn again, walk 1000 steps, make a final right turn again, and finally walk once more 1000 steps.

After this procedure, where are you? In Euclidean geometry you would exactly end up at the position you started. In the Cayley-Klein geometry inside the nondegenerate fundamental conic you would end up at some different place. (The prediction where you end up does not depend on the inner or outer description. Both are equally legitimate and describe the same situation.)

The geometric situation within the real nondegenerate fundamental conic has historically played an eminently important role in mathematics, since it differs only so subtly from the commonly used Euclidean geometry. Its discovery was pivoted for several branches of mathematics and is closely interwoven with the way we nowadays treat mathematics. Several equivalent models of this space were discovered—one of them is in terms of Cayley-Klein geometries. Others use only intrinsic properties of local curvatures, still others embed this space into regions of the complex number plane. All these

<sup>1</sup> Let us neglect the fact that paper-folding requires three dimensions to be performed.

approaches are equally legitimate, since they all lead to the same qualitative results. The Cayley-Klein approach has the advantage to embed this special geometry into a larger system of other possible geometries.

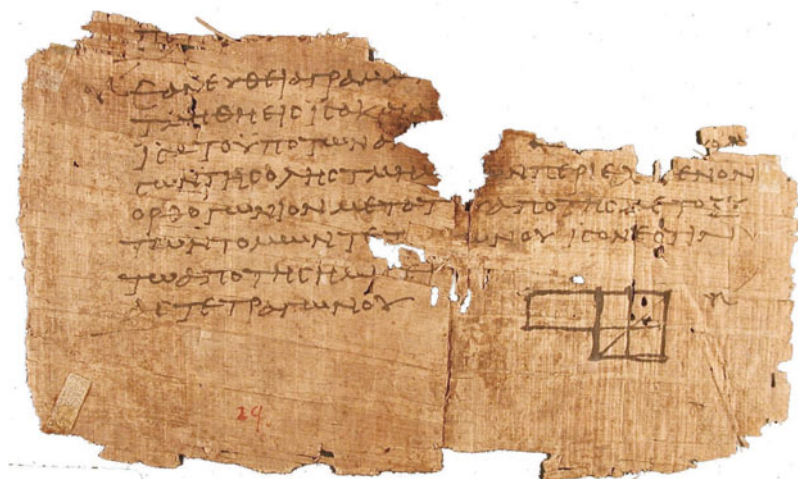
This chapter is perhaps a bit different from the others in this book and offers more philosophical and historical background dealing with the *nature of space* and the *nature of geometry*. Still we here can only scratch the surface and advise the reader to consult some of the really brilliant books and papers on this topic. We here want to recommend four books that provide interesting insights in the history of development of these ideas. The very wellwritten book of Marvin Jay Greenberg on hyperbolic geometry [49] dedicates entire chapters to the historical and philosophical aspects of hyperbolic geometry. The books by Jeremy Gray *Worlds out of nothing* [48] and by Isaak Moiseevich Yaglom *Felix Klein and Sophus Lie—Evolution of the Idea of Symmetry in the Nineteenth Century* [137] tell essentially the entire story of the development of hyperbolic geometry, including the origins of modern algebra and projective geometry. Finally, F. Klein's *Development of mathematics in the 19th century* [69, 70] gives a kind of first-hand treatment of the subject. Although it sometimes lacks the distance of a historian, it gives very deep insights into important streams of thought of these times.

## 24.2 Euclid's Postulates

In the books (the *Elements*) that Euclid wrote around 300 BCE about mathematics and geometry, he tried to lay a rigorous foundation of the matters he was writing about. Compared to modern standards there are still a few gaps in his reasoning, but he came pretty close. In a sense, his treatment of mathematics served as a blueprint for the way we do mathematics nowadays: Base your formal treatment on axioms and definitions; then do admissible formal reasoning to derive theorems based only on these axioms, definitions, and things you have already proved. Thus ultimately the proof of every single theorem can be traced back to your axioms, definitions, and logically admissible reasoning steps.

In Euclid's treatment of geometry, after specifying the objects (points, segments, circles), which he treats in very general terms by sentences like "A point is that which has no part" etc.), he describes the relations between these objects by a set of five axioms, often called his *postulates*:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.



**Fig. 24.2** One of the oldest surviving fragments of Euclid's *Elements*, found at Oxyrhynchus and dated to circa 100 CE. The diagram accompanies Book II, Proposition 5.

5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

All further conclusions are based on these five axioms and logical reasoning (at least they are intended to be). Already a superficial look at these axioms shows a significant difference between the first four and the fifth postulate. The first four are brief and concise, while the fifth postulate looks more like the fine print in a license agreement of a cell phone contract. The reasons for this remarkable difference are somehow buried in history. One could speculate that Euclid himself was of the opinion that the geometric property expressed by the fifth postulate is of crucial importance and he wanted to be on the watertight side. At least Euclid was somehow reluctant to make use of the fifth postulate, and his first 28 theorems are proved without referring to it.

No matter what was the actual reason for Euclid's reluctance, at some time mathematicians were beginning to speculate whether the fifth postulate was indeed necessary. It might well be a consequence of the first four postulates. Attempts to prove the fifth postulate from the remaining four can be traced back to a time more than 1000 years ago. Ibn al-Haytham (Alhazen) (965–1039), an Iraqi mathematician, is the first person known to have worked on this problem. Over the years, along with the many attempts to prove the fifth postulate from the other four (or to show its independence), mathematicians (too many to be named all here) came up with several concise formulations

equivalent to the fifth postulate in the presence of the other four. Here we list a few of them:

- a. At most one line can be drawn through any point not on a given line parallel to the given line in a plane.
- b. The sum of the angles in every triangle is  $180^\circ$ .
- c. There exists a triangle whose angles add up to  $180^\circ$ .
- d. There exists a pair of similar, but not congruent, triangles.
- e. If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
- f. There exists a quadrilateral of which all angles are right angles.
- g. In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagorean theorem).
- h. There is no upper limit to the area of a triangle.

The first of these formulations is often called the *parallel postulate*, and it traces back to John Playfair (1748–1819). Formulations b. to d. demonstrate that the problem is intimately related to the angular sum of a triangle. Formulations e. and f. show that our *walk in a square* experiment from the last chapter was ultimately a test for the truth of the fifth postulate. Formulation g. shows that the Pythagorean theorem is essentially equivalent to the fifth postulate, and finally formulation h. shows that the area of a triangle will be bounded if the contrary of the fifth postulate holds.

### 24.3 Gauss, Bolyai, and Lobachevsky

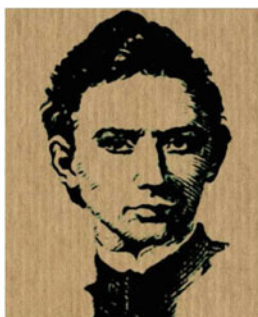
Surprisingly, the problem of the independence of the fifth postulate was not resolved to full satisfaction until approximately between 1820 and 1882, and the Cayley-Klein geometries we discuss in in this book play a crucial role in this development. The story is long, surprisingly emotional, involves personal fate and has several surprising twists. Here we will give only a brief outline of what happened in the final and crucial years.

The main actors in this part of mathematical history were Carl Friedrich Gauss (1777–1855), János Bolyai (1802–1860), Nikolai Ivanovich Lobachevsky (1792–1856), Eugenio Beltrami (1835–1900), Felix Christian Klein (1849–1925) and Jules Henri Poincaré (1854–1912). One should also mention Farkas Bolyai (1775–1856), the father of János, because of his personal involvement and the fact that he somehow formed a bridge between Gauss and his own son.

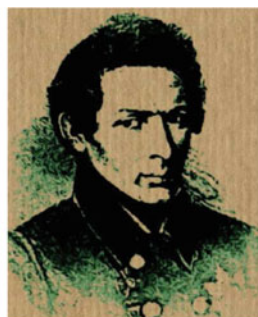
In short, the story can be told as follows. In the first half of the nineteenth century several mathematicians were attempting to show that the parallel



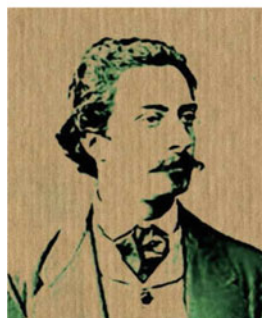
C.F. Gauss



E. Beltrami



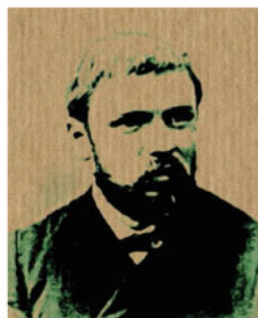
N.I. Lobachevsky



E. Beltrami



F.C. Klein



J.H. Poincaré

**Fig. 24.3** The non-Euclidean geometry hall of fame.

postulate (or some of the other equivalent formulations) was indeed dependent on the other four of Euclid's postulates. Many of these attempts were based on a "reductio ad absurdum" argument: *Assume the fifth postulate is not true and try to derive a contradiction from this assumption.* Thereby several mathematicians drew conclusions about a system consisting of the first four postulates and the negation of the fifth postulate. Some of them proceeded without finding a contradiction and at some point gave up with more or less desperation. Others stopped at a point when they arrived at a conclusion that in their opinion contradicted common sense (like "the area of a triangle is bounded"), claiming that this proves the dependence of the fifth postulate. In fact, they were wrong, as we will soon see.

There were three persons who went in a sense further than others without ending up in desperation: Gauss, Bolyai (junior), and Lobachevsky. A point on which historians agree nowadays is that their closely related results were indeed individual and independent developments. All three of them developed a geometric system based on the postulates one to four and the negation of the fifth postulate. The system derived was comparably rich as Euclidean geometry, and seemingly free of contradictions. In this new *non-Euclidean*

geometry there were perfectly reasonable notions of elementary geometry, differential geometry, mechanics, etc. Many Euclidean theorems had a non-Euclidean analogue that was often related to the corresponding Euclidean one by just a small but essential twist.

The way the three mathematicians dealt with their results was quite different. Gauss was perhaps the first person who was convinced that there is an equally justified non-Euclidean geometry. However, he was reluctant to publish anything on this topic. He was afraid of the at that time dominant philosophical school of Immanuel Kant. One of Kant's fundamental statements in his epistemology was that the properties of a straight line are a priori clear and cannot be further discussed. Starting an intellectual dispute about non-Euclidean geometry would exactly mean to discuss the very meaning of a straight line. Nevertheless, to the outside world there was indeed some evidence that Gauss was well aware of all the subtleties around the parallel postulate. As the leading mathematician of his day he often received letters with supposed proofs of the dependence of Euclid's fifth postulate. On these occasions he often responded very soon, pointing the finger to the place of the logical flaw in the supposed argumentation. Still, during his lifetime he did not publish anything on this subject.

Farkas Bolyai, the father of János Bolyai, was a school friend of Gauss. Farkas himself was deeply committed to the problem of the parallel postulate and worked on it for several decades (without real success). When his son told him that he was as well planning to spend significant time working on the parallel postulate, the father beseeched him not to do so. To get an impression how emotionally tense the subject was, here is an excerpt from a letter Farkas Bolyai wrote to his son János trying to prevent him from doing research on the parallel postulate. For those readers capable of reading German I include the full original passage, which may be found, for instance, in [104]. After this I include an abridged translation that can be found in [49].

Du darfst die Parallelen auf jenem Wege nicht versuchen; ich kenne diesen Weg bis an sein Ende. Auch ich habe diese bodenlose Nacht durchmessen, jedes Licht, jede Freude meines Lebens sind in ihr ausgelöscht worden; ich beschwöre Dich bei Gott! Lass die Lehre von den Parallelen in Frieden. Du sollst davor denselben Abscheu haben, wie vor einem liederlichen Umgang, sie kann Dich um all Deine Muße, um die Gesundheit, um Deine Ruhe und um Dein ganzes Lebensglück bringen. Diese grundlose Finsternis würde vielleicht tausend Newtonsche Riesentürme verschlingen, es wird nie auf Erden hell werden, und das armselige Menschengeschlecht wird nie etwas vollkommen Reines haben, selbst die Geometrie nicht; es ist in meiner Seele eine tiefe und ewige Wunde. Behüt Dich Gott, dass diese sich (bei Dir) je so tief hineinnagen möchte. Diese raubt einem die Lust zur Geometrie, zum irdischen Leben. Ich hatte mir vorgenommen, mich für die Wahrheit aufzuopfern; ich wäre bereit gewesen, zum Märtyrer zu werden, damit ich nur die Geometrie von diesem Makel gereinigt dem menschlichen Geschlecht übergeben könnte. Schauderhafte, riesige Arbeiten habe ich vollbracht, habe bei weitem Besseres geleistet als bisher (geleistet wurde), aber keine vollkommene Befriedigung habe ich je gefunden. Hier aber gilt es: Si paullum a summo discessit, vergit ad imum. Ich bin zurückgekehrt, als ich durchschaut habe, dass man den Boden dieser Nacht von der Erde aus nicht

erreichen kann, ohne Trost, mich selbst und das ganze Geschlecht bedauernd. Lerne an meinem Beispiel; indem ich die Parallelen kennen wollte, blieb ich unwissend, diese haben mir all die Blumen meines Lebens und meiner Zeit weggenommen. Hier steckt sogar die Wurzel aller meiner späteren Fehler, und es hat darauf aus den häuslichen Gewölken geregnet. Wenn ich die Parallelen hätte entdecken können, so wäre ich ein Engel geworden, wenn es auch niemand gewusst hätte, dass ich sie gefunden habe. . . . Versuche es nicht, Du wirst es nie zeigen, dass je mit den unaufhörlichen Einbiegungen desselben Maßes die untere Gerade geschnitten werde, es steckt in dieser materia ein ewig in sich zurückdrehender circulus - ein Labyrinth, das einen immer hineinlockt -, wer sich hineinbegibt, verarmt, wie ein Schatzgräber, und bleibt unwissend. Solltest Du auf was für immer ein absurdum geraten, alles ist umsonst. Du kannst es nicht als ein Axiom hinstellen. . . . Die Säulen des Herkules stehen in diesen Gegenden, gehe nicht um einen einzigen Schritt weiter, sonst bist Du verloren.

And here the abridged translation:

You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone . . . I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction. For here it is true that si paullum a summo discessit, vergit ad imum. I turned back when I saw that no man can reach the bottom of this night. I turned back unconsolated, pitying myself and all mankind. . . do not go any step further; otherwise, you are lost.

János Bolyai did not stop. He continued and developed the above-mentioned rich system of non-Euclidean geometry. After he convinced his father that he had indeed gone further in this subject than other contemporary mathematicians, his father sent his son's notes to his youth friend Gauss. He wanted to let Gauss approve (or disapprove) the mathematical thoughts of his son and to get advice how to proceed. Gauss answered in a letter, of which here is an extract (again in German):

Jetzt über die Arbeiten deines Sohnes.—Wenn ich damit anfangen, das ich solche nicht loben darf, so wirst Du wohl einen Augenblick stutzen, aber ich kann nicht anders; sie loben hieße mich selbst zu loben, denn der ganze Inhalt der Schrift, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen er geführt ist, kommen fast durchgehends mit meinen eigenen, zum Teile schon seit 20–25 Jahren angestellten Meditationen berein. In der Tat bin ich dadurch auf das Äußerste überrascht. – Mein Vorsatz war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das was ich ihnen mitteilte mit besonderem Interesse aufnahmen. Um das zu können, muß man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht mit der Zeit alles zu Papier zu bringen, daß es mit mir dereinst nicht unterginge. – Sehr bin ich also überrascht, daß gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvor gekommen ist.

Abridged translation:

If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. . . . I have found very few people who could regard with any special interest what I communicated to them on this subject. . . . it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner.

Finally János Bolyai could publish his explorations as an appendix in a book written by his father.

The third person who arrived at essentially the same system was N.I. Lobachevsky, a mathematics professor at the university of Kazan, in Russia. He as well started to derive conclusions from the negated parallel postulate and made perhaps the furthest-reaching investigations of differential-geometric structures in his non-Euclidean geometry. In a letter Gauss also expressed great appreciation of the work of Lobachevsky. Not seldom is the term Lobachevskian Geometry used synonymously with the term non-Euclidean geometry.

The story continues. The two Bolyais falsely believed that Lobachevsky was a pseudonym of Gauss under which he incognito had published his thoughts on non-Euclidean geometry<sup>2</sup>. And unfortunately, neither Bolyai nor Lobachevsky received recognition of their work until Gauss died in 1855. With Gauss's death his private letters became public, and with them also a wealth of private thoughts of the world's leading mathematician on the subject of non-Euclidean geometry, including references to Bolyai and Lobachevsky. During their lifetimes only few mathematicians knew about the groundbreaking work of these two mathematical revolutionaries.

## 24.4 Beltrami and Klein

So what was the achievement of the three protagonists of our last section? They went further than others in deriving consequences of the negated parallel postulate. By this they developed a system of geometric terms at least

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<sup>2</sup> This almost curious fact is mentioned in Klein's historical book [69] and can be traced back to a text *Bemerkungen ueber Nicolaus Lobatschefskijs Untersuchungen zur Theorie der Parallellinien* Bolyai wrote around 1851. There he writes: "Für noch wahrscheinlicher aber halte ich, daß der ohnedem an Schätzen so reiche Koloß Gauss es nicht ertragen konnte, daß ihm jemand auch in dieser Sache zuvorgekommen sei, und, da er dies durchaus nicht mehr verhindern konnte, das Werk selbst bearbeitet hat und unter Lobatschefskijs Namen hat herausgeben lassen." Further details on this fascinating part of the story may be found in [122].



as rich as Euclidean geometry in which Euclid's fifth postulate does not hold. However, by this they did *not* prove the independence of the fifth postulate from the other axioms. Still there was a slight chance that if one went further and further making logically correct conclusions one could arrive at a point where a statement and at the same time its negation could be proved. If this happened, then the system of non-Euclidean geometry would turn out to be self-contradictory and *inconsistent*.

In fact, this state of matters is not as dramatic as it may sound at first. In a sense, the new non-Euclidean geometry was in no worse state than the traditional Euclidean geometry. Also, under the assumption of Euclid's original five axioms it was not impossible that by drawing further and further conclusions one might end up with a self-contradiction of the system. A certain element of belief is necessary for both types of geometry.

From a more modern standpoint of formal logic the situation looks as follows. When doing mathematics one has first to agree on several conventional facts (the axioms) and rules of logical inference (as Euclid did). From this starting point a system of conclusions and derivations can take off and form the rich platform of an interesting mathematical subject. One cannot a priori rule out the possibility that such a system of axioms and inference rules may not be self-contradictory. The situation is even worse: In 1931 the famous mathematician Kurt Gödel proved that any such system if it is sufficiently complex<sup>3</sup> is incapable of proving its own consistency. Nevertheless, and for good reasons, there are systems in mathematics in whose consistency mathematicians "trust." The system of natural numbers is one of them; Euclid's geometry is another. The trust comes from two sources: the experience that even centuries of mathematical research did not reveal an inconsistency and the fact that these systems have counterparts (physical models) in the real world. However, also in this respect, for traditional Euclidean geometry the state of affairs is not that clear. Some of Euclid's postulates make statements about the behavior of lines of infinite extent. In practice, one does not have access to such lines. So compared to that, what is the state of a new development like non-Euclidean geometry à la Gauss, Bolyai, and Lobachevsky? First of all, it comes neither with experience nor with a directly accessible physical model. So what the triumvirate of non-Euclidean geometry achieved was in a sense developing a dense network of concepts that made it more and more likely that it would obtain the trust of the mathematical community.

The next fundamental step in the development was taken 13 years after Gauss's death and is intimately related to the names of Beltrami and Klein. In 1868, E. Beltrami was able to provide a *model* for non-Euclidean geometry, not a physical model but a formal model in terms of traditional Euclidean geometry. In other words, equipped with the language of Euclidean geometry he was able to define objects and relations between them that behave in such a way that the first four postulates and the negated fifth postulate were

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<sup>3</sup> This means sufficiently complex that one can express the natural numbers in it.

satisfied. This put non-Euclidean geometry on a new footing. If you trust in the consistency of Euclidean geometry, then you must also trust in the consistency of non-Euclidean geometry, since the latter can be consistently described in terms of the first. Also conversely it is possible to describe Euclidean geometry in terms of hyperbolic geometry. Thus either one trusts in the consistency of both geometries or in neither of them. This finally provided a satisfactory answer to the problem of the independence of the parallel postulate. No matter whether one takes Euclid's first four postulates together with the parallel postulate or with its negation to define a geometric system, it is possible to obtain a model of the other possibility inside this system.

Beltrami's work (a reprint of a translation can be found in [125]) was based essentially on differential-geometric arguments. He embedded hyperbolic geometry in a surface of constant negative curvature (the pseudosphere). This surface was itself embedded in (three-dimensional) Euclidean space. A few years later, Felix Klein was able to streamline these thoughts and prove the interrelated consistency of the two geometries without referring to differential geometry or higher-dimensional spaces. He used a method published by the brilliant British mathematician Arthur Cayley [21] in 1859 to express measurements in terms of projective relations with respect to a conic. Klein greatly generalized Cayley's methods and arrived at the Cayley-Klein geometries we have been dealing with here for four chapters. Along the way he could show that the Cayley-Klein geometry with a nondegenerate real fundamental conic restricted to the interior of this conic formed a perfect model for non-Euclidean geometry à la Gauss, Bolyai, and Lobachevsky. In two groundbreaking articles published in 1871 to 1873 entitled "On the so-called non-Euclidean geometry" (see [66, 67, 125]) Klein exposed this whole chain of ideas describing a comparably simple model of non-Euclidean geometry within Euclidean (or better projective) geometry. The same model can as well be extracted from Beltrami's works. This is why this model is often referred to as the *Beltrami-Klein model* of non-Euclidean geometry.

## 24.5 The Beltrami-Klein Model

This section is dedicated to the problem of explaining how the Beltrami-Klein model relates to the Euclidean postulates. There is one technical (or better historical) problem that we cannot completely resolve on the following two pages. Compared to modern standards, Euclid's definitions and postulates are not completely rigorous. Several implicit assumptions are made by Euclid, and it would be much better to base the following consideration on how Euclid *used* his postulates than on how he *formulated* them. For instance, when in the second postulate Euclid says that *a line segment can be extended indefinitely in a straight line* he implicitly assumes that this straight line is being infinitely extended in both directions and that it is not the case that it

topologically may close up to form a circle. He uses this implicit assumption, for instance, when he proves that the exterior angle in a triangle is always larger than each of its opposite interior angles. There are several more implicit assumptions on the notions of betweenness, the character of congruence, and so forth.

A strict (in modern standards) axiomatic treatment of what we call Euclidean geometry was done as late as 1899 in David Hilbert's book *Grundlagen der Geometrie* [58]. Hilbert there presented a set of 20 axioms forming a watertight basis for Euclidean geometry. He later on also formulated a similar set of axioms for projective geometry and for non-Euclidean geometry.<sup>4</sup> So when we here want to see how the Beltrami-Klein model is indeed a model of Euclidean geometry it would be by far more appropriate to relate it to Hilbert's axioms than to Euclid's postulates. We will not do this here and refer to other books for this topic [48]. We will content ourselves with a brief and commented version of Euclid's postulates and their relationship to the Beltrami-Klein model.

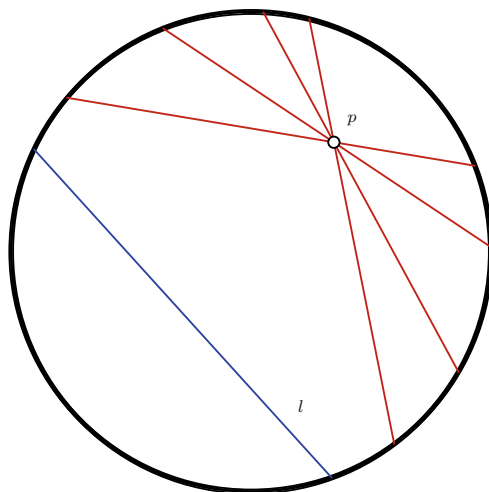
In principle, within the preceding chapters we have done all preparations to explain the Beltrami-Klein model in comparatively simple terms. So, what do we want? We need a geometric system whose objects are lines, planes, and circles, equipped with a notion of measurement for distances and angles, that satisfies the first four of Euclid's postulates and the negation of the fifth. So, here it is: We start with a nondegenerate real conic in the projective plane. For reasons of simplicity and without loss of generality we choose the unit circle  $\mathcal{F}$  defined by  $x^2 + y^2 = 1$ . This has the advantage that all points we will consider now are part of the usual Euclidean plane, and by this it is not explicitly necessary to refer to projective geometry and elements at the line at infinity. The *points* of our geometry are *all interior points of the unit circle*. The *lines* of the geometry are the (nonempty) *intersections of an ordinary line with the interior of the unit circle*. Distance measurement and angle measurement are those of the Cayley Klein geometry  $\mathcal{K} = (\mathcal{F}, -1/2, -1/2i)$ . With respect to this measurement we can define, in agreement with Euclid's definitions,<sup>5</sup> that a *circle* is the set of *all points having constant distance from another point*. Furthermore, we define that an angle between two lines is *right* if the two angles to the left and to the right of one of the lines and on the same side of the other have equal size.

Equipped with these definitions, we will now analyze how this system of geometric objects relates to Euclid's postulates. For this we take the intrinsic viewpoint. We argue as a being that lives inside the reality of the interior of the unit circle equipped with the Cayley-Klein measurement.

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<sup>4</sup> It is an interesting fact that in Hilbert's axioms the term *circle* is not at all used or defined. Circles in this setup are secondary objects whose definition is based on the axioms of measurement and incidence.

<sup>5</sup> We have not reproduced all these 23 definitions here.



**Fig. 24.4** The violation of the parallel postulate in hyperbolic geometry.

**1<sup>st</sup> Postulate:** A straight line segment can be drawn joining any two points. From the line that connects two points inside  $\mathcal{F}$  we take the segment that connects the two points inside the unit disk. This is the segment.

**2<sup>nd</sup> Postulate:** Any straight line segment can be extended indefinitely in a straight line. This is the infinite walk of you and your dog in one fixed direction described at the beginning of this chapter. Here Euclid's implicit assumption on the infinite extent of lines comes into play. This implicit assumption implies that the distance measurement is of hyperbolic type. If it were elliptic, we would eventually return to our starting point. If we dropped this implicit assumption, then also elliptic geometry would be a candidate for a non-Euclidean geometry.

**3<sup>rd</sup> Postulate:** Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center. The set of points having a fixed real distance to a point inside the unit disk constitutes the circle. In fact, the elliptic angle measurement ensures that these circles are always closed curves.

**4<sup>th</sup> Postulate:** All right angles are congruent. We have a right angle if a line  $g$  passes through the pole of the other line  $l$ . If we have right angles between two different pairs of lines, then there is always a corresponding  $\mathcal{K}$  motion of one pair that makes it congruent to the other. Here several implicit assumptions on continuous rigid motions and congruence are necessary. The  $\mathcal{K}$ -motions are in full agreement with all of them.

**5<sup>th</sup> Postulate:** Now comes the important point (and in a surprising sense it is somehow the easiest). In the Cayley-Klein model, Euclid's fifth postulate

does not hold. We here take the slightly simpler approach of considering the parallel postulate instead of the fifth postulate. The parallel postulate states that through any point  $p$  not on  $l$  there *is exactly one* line not meeting  $l$ . In the Cayley-Klein model this statement is violated in the following way: *For every line  $l$  and point  $p$  not on  $l$  there are infinitely many lines passing through  $p$  that do not cut  $l$ .* The reason for this becomes obvious if one looks at Figure 24.4. In principle, there is also another possibility to violate the parallel postulate. We could try to define a geometry in which there is *no* line through  $p$  that does not intersect  $l$ . Indeed, if we drop our implicit assumption that lines have infinite extent, then this is also possible and leads to the Cayley-Klein geometry of type I: the elliptic geometry.

To summarize, if we take the first four of Euclid's postulates (without the implicit assumption on the infinite extent of lines), they in particular imply that the angle measurement is elliptic. There are three Cayley-Klein geometries in agreement with that: type II (restricted to the interior of the fundamental conic)—*hyperbolic geometry*; type I—*spherical geometry*; type V—usual *Euclidean geometry*. They relate to the parallel postulate and the infinite length assumption as follows (let  $p$  and  $l$  be an arbitrary noncoincident point/line pair):

**Euclidean geometry:** Lines have infinite extent. There is *exactly one* line through  $p$  not cutting  $l$ .

**Hyperbolic geometry:** Lines have infinite extent. There are *infinitely many* lines through  $p$  not cutting  $l$ .

**Elliptic geometry:** Lines have only finite extent. There is *no* line through  $p$  not cutting  $l$ .

Thus in the presence of the infinite-extent assumption, Euclidean and hyperbolic geometry are the only two Cayley-Klein geometries in accordance with the first four postulates. In fact, it can be proved that every formal system that satisfies the first four postulates and the implicit -extent assumption is automatically isomorphic to one of these two geometries.

## 24.6 Poincaré

Still we left out one of the main protagonists of the early years of non-Euclidean geometry: Henri Poincaré. His contribution was to provide still another model for hyperbolic geometry. This model turned out to be a crucial link to many other branches of mathematics, including complex function theory, the theory of automorphic functions, differential equations and many more. A reprint of his essential articles can again be found in [125].

Poincaré interpreted hyperbolic geometry directly in the complex number plane  $\mathbb{C}$ . For us his model is also very interesting from a projective point

of view. While the Beltrami-Klein model is best understood embedded in the *two-dimensional real* projective space  $\mathbb{RP}^2$ , the Poincaré model is best understood embedded in the *one-dimensional complex* projective space  $\mathbb{CP}^1$ . In the *Poincaré disk model* of the hyperbolic plane the region of relevant points is as in the Beltrami-Klein model restricted to the interior points of a circle (one may again choose the unit circle for convenience). The hyperbolic lines are circular arcs that meet the unit circle orthogonally. We will give a detailed description of the Poincaré disk model later in the next chapters. Here we just collect a few of the essential features.<sup>6</sup>

At first sight one might wonder why it would be advantageous to represent hyperbolic lines by circular arcs. In fact, losing straightness of lines is only a small drawback compared to the advantages of the Poincaré disk model. The important point is that the Poincaré disk model is *conformal*. This means that it represents intersection angles in an unperturbed way. In this model the hyperbolic angle under which two hyperbolic lines meet corresponds exactly to the angle under which the associated two circular arcs meet. As a consequence of conformality, circles are represented by proper circles in the Poincaré disk model. The feature of conformality is also the reason that this model and with it hyperbolic geometry plays a fundamental role in complex function theory.

There is a very simple relationship between the Poincaré disk model and the Beltrami-Klein model. For both geometries let the points be those in the interior of the unit circle. We may identify the interior of the unit disk in  $\mathbb{CP}^1$  with the interior of the unit disk in  $\mathbb{RP}^2$  as usual by the map  $x + iy \mapsto (x, y, 1)^T$ . Now, if  $z$  with  $\|z\| < 1$  is a point in the Poincaré disk and  $w$  is the corresponding point in the Beltrami-Klein model, then these points are related by

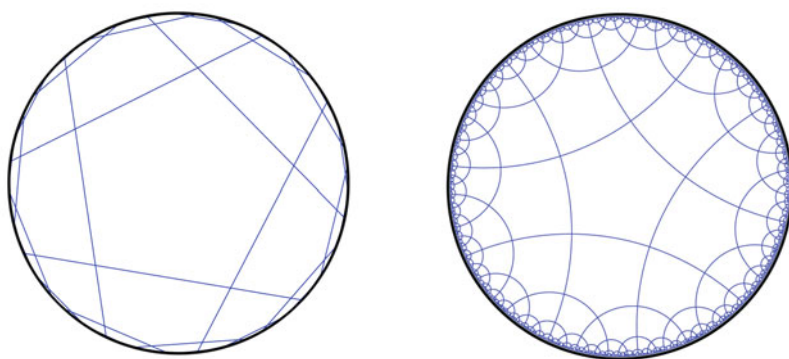
$$z \mapsto \frac{2z}{1 + \|z\|^2} = w \quad \text{and} \quad w \mapsto \frac{w}{1 + \sqrt{1 - \|w\|^2}} = z.$$

Thus the Poincaré disk is only a deformed copy of the Beltrami-Klein disk. The deformation function is centrally symmetric. In moving from the Beltrami-Klein model to the Poincaré model the interior of the disk is shrunk and the boundary is blown up. Visually, this is a very good feature, since we get a higher “resolution” at the boundary, which thereby contains most of the hyperbolic plane. Visually it is also very pleasant that intersection angles of objects appear at their proper geometric size.

As an example of the visual advantages of the Poincaré model consider Figure 24.5. It shows an identical arrangement of infinitely many lines in

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<sup>6</sup> In the literature one frequently also finds the *Poincaré half-plane model*, where the active points are all points above the real axis of  $\mathbb{C}$  and the lines are circular arcs that meet the real axis orthogonally. From a projective viewpoint this model is exactly equivalent to the disk model, since any open half-plane can be mapped to any interior of a circle by a  $\mathbb{CP}^1$  transformation. This map preserves the set of lines and circles and the intersection angle of these objects.



**Fig. 24.5** Comparing the Beltrami-Klein representation and the Poincaré representation of an arrangement of lines.

both models (left Beltrami-Klein, right Poincaré). The arrangement of lines cuts out a tessellation of the hyperbolic plane constituted of infinitely many pentagons with only right angles at the corners (yes, such weird things exist in the hyperbolic plane). Notice that in the Poincaré model one can at the same time perceive much more detail of the tessellation and directly see that the vertices of the pentagons are rectangular.

Which of the two models is preferable under algebraic aspects depends a lot on the concrete circumstances. Later on, we will give a dictionary that helps to translate concepts from one model into the other.





# Hyperbolic Geometry

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*To infinity ... and beyond!*

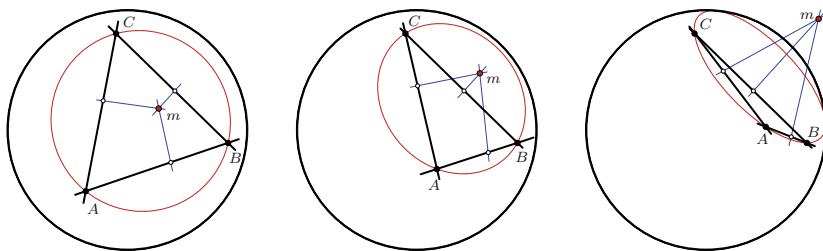
*Buzz Lightyear (Toy Story, Disney)*

Back to mathematics! This chapter is dedicated to several interesting topics in hyperbolic geometry. With our previous knowledge on the real projective plane  $\mathbb{RP}^2$ , on the complex projective line  $\mathbb{CP}^1$ , and of Cayley-Klein geometries we have an ideal departure point to explain several hyperbolic effects from an elegant and advanced standpoint. In particular, we will work out the relations between the Beltrami-Klein model and the Poincaré disk model. Compared to the general considerations in Chapters 20–23 we are now in a somewhat better situation. When we dealt with general Cayley-Klein geometries we spent a lot of our efforts on the treatment of case distinctions that arose from the various degrees of degeneracy of the fundamental conic. The algebraic structure became easier the less degenerate the fundamental conic was. Now we will deal only with one particular Cayley-Klein geometry, which in addition is nondegenerate.

## 25.1 The Staging Ground

In everything that follows we refer to the Cayley-Klein geometry defined by the following parameters. The fundamental conic  $\mathcal{F} = (A, B)$  is given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$



**Fig. 25.1** The intersection of hyperbolic medians may be a hyperinfinite point.

The scaling parameters are

$$c_{\text{dist}} = -\frac{1}{2} \quad \text{and} \quad c_{\text{ang}} = \frac{1}{2i}.$$

The fundamental conic is the unit disk in the standard embedding of  $\mathbb{RP}^2$ . We will later give reasons why this specific choice of parameters is most appropriate. Furthermore, we will (at least mostly) restrict our considerations to the points in the *interior* of the unit disk.

At this time we should once more clarify the difference between the intrinsic viewpoint of “living in hyperbolic space” and the extrinsic/algebraic viewpoint of having an appropriate algebraic description, like our Cayley-Klein geometry. We will exemplify this by the concept of a line. From the intrinsic viewpoint a line consists only of those points that lie in the interior of the unit circle. A line in  $\mathbb{RP}^2$  that does not intersect the unit circle at all is materially nonexistent from the intrinsic viewpoint. Points outside the unit circle are as well nonexistent. Now, as an example, consider the problem of intersecting the medians of a triangle  $A, B, C$  (compare Figure 25.1). If the triangle is near to being equilateral, then the medians intersect in a point  $m$  that lies in the interior of the triangle (left picture). This point is in addition the center of the circumcircle of the triangle. Now we may move point  $A$  toward the segment  $B, C$ . At a certain point the circumcenter moves out of the triangle (middle picture). This is not very spectacular because we know this effect from Euclidean geometry (there an obtuse triangle has its circumcenter outside the triangle). If point  $A$  approaches the segment  $B, C$  closer and closer the circumcenter  $m$  moves even out of the unit circle. How should we interpret this case? The answer is not that easy. From the intrinsic viewpoint the medians no longer intersect. Strictly speaking, the triangle does not even have a circumcircle, since there is no material point that has the same distance to all three triangle points. So from the intrinsic viewpoint we might say that the triangle has moved to a new qualitative state (a special case, so to speak) in which the medians simply no longer intersect and the

circumcircle disappears. Were we to do so, hyperbolic geometry would suffer from the treatment of many many case distinctions.

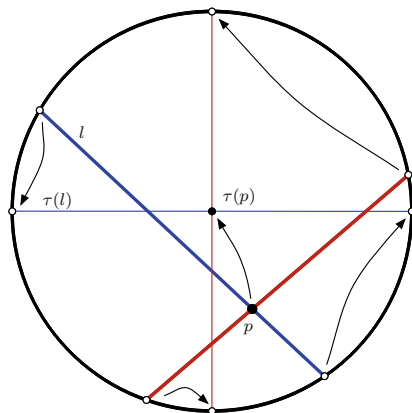
Now it may happen that a clever mathematician in the hyperbolic world comes along and says, Well, the intersection of the medians may disappear in our world, but if we introduce a suitable algebraic structure to coordinatize our points and lines (namely homogeneous coordinates) and if we further describe our angle measurement by a certain formula (Cayley-Klein geometry), then we may think of our world as realized within a certain object (the unit disk). If we do all that, then I can say that it is reasonable to extend our lines even further than their infinity points to, let's call it hyperinfinite points. In this representation two lines always intersect, and it makes total sense always to speak of the intersection of the medians. They may intersect at finite, infinite, or hyperinfinite points. There is always also a circumcenter. If it is a hyperinfinite point, its distance to the triangle vertices becomes complex. But who cares, I am a mathematician and I am allowed to do abstractions.

In a sense, the situation is similar to the introduction of infinite points in Euclidean geometry. One may either accept that there are many case distinctions or accept that there is a nicer algebraic system that gets rid of the special cases for the price of being more abstract and by adding elements that have no material counterpart. So also from an intrinsic viewpoint it is totally legitimate to represent hyperbolic geometry in terms of Cayley-Klein geometries and their measurements. This means simply to allow for infinite and hyperinfinite elements and using the corresponding algebraic representations.

In what follows we will denote the interior of the unit circle by  $\mathcal{H}$  and the unit circle itself by  $\overline{\mathcal{H}}$ . The interior  $\mathcal{H}$  is the staging ground to which we relate all our geometric considerations. However, we will freely make use of its nice algebraic representation in  $\mathbb{RP}^2$ .

## 25.2 Hyperbolic Transformations

After these quite philosophical considerations we will return to concrete calculations. We will start with a topic we already touched in Section 21.2: hyperbolic transformation. In our setup with the unit circle as fundamental object these are exactly those projective transformations in  $\mathbb{RP}^2$  that leave the unit circle invariant. Fixing the images of three points  $A, B, C \in \overline{\mathcal{H}}$  on the unit circle determines such a *hyperbolic transformation*  $\tau$  uniquely. The image points  $A', B', C' \in \overline{\mathcal{H}}$  must also lie on the unit circle. The matrix of the hyperbolic transformation can be calculated in the following way. The three points  $A, B, C$  determine a projective scale on  $\overline{\mathcal{H}}$ . Every other point  $D$  on  $\overline{\mathcal{H}}$  has a unique and well-defined cross-ratio  $(A, B; C, D)_P$  under which it is seen from a generic point  $P$  on  $\overline{\mathcal{H}}$ . So we can determine the unique position  $D$  on  $\overline{\mathcal{H}}$  such that  $(A, B; C, D)_P = -1$ . Similarly, we get a unique point  $D'$  in harmonic



**Fig. 25.2** Moving a point to the center of  $\mathcal{H}$ .

position to  $A', B', C'$ . The hyperbolic transformation is the unique projective transformation that maps  $A, B, C, D$  to the corresponding primed points (see also Figure 21.2). We recall that our Theorems 10.3 and 10.4 imply that this procedure indeed generates a map that leaves  $\overline{\mathcal{H}}$  invariant and that the group of all such transformations is isomorphic to the group of projective transformations of the real projective line  $\mathbb{RP}^1$ . Since we will quite frequently refer to the cross-ratio of four points  $A, B, C, D$  on  $\overline{\mathcal{H}}$  seen from a generic other point  $P$ , we abbreviate this cross-ratio by  $(A, B; C, D)_{\overline{\mathcal{H}}} := (A, B; C, D)_P$ . Theorem 10.1 ensures that this value is independent of the specific choice of  $P$  and hence well-defined. It is easy to show that a hyperbolic transformation automatically also maps  $\mathcal{H}$  to  $\mathcal{H}$ .

**Theorem 25.1.** *Let  $\tau$  be a projective transformation that maps  $\overline{\mathcal{H}}$  to  $\overline{\mathcal{H}}$ . Then it also maps  $\mathcal{H}$  to itself bijectively.*

*Proof.* Since projective transformations are automatically bijections, the only thing we have to prove is that under  $\tau$  the image of every point  $p \in \mathcal{H}$  is again in  $\mathcal{H}$ . For this we choose two arbitrary lines  $l$  and  $g$  through  $p$ . Since  $p$  is in the interior of the unit disk, the lines  $l$  and  $g$  intersect  $\overline{\mathcal{H}}$  in two pairs of points  $A_l, B_l$  and  $A_g, B_g$ , respectively. Since the two lines intersect in the interior of  $\overline{\mathcal{H}}$ , the intersections with  $g$  and  $l$  alternate when traversing  $\overline{\mathcal{H}}$  cyclically. This in turn implies that the cross-ratio  $(A_l, B_l; A_g, B_g)$  is negative. Hence also the cross-ratio  $(\tau(A_l), \tau(B_l); \tau(A_g), \tau(B_g))$  is negative. Thus the image points alternate as well. Accordingly also  $\tau(l)$  and  $\tau(p)$  intersect in the interior of the unit disk. This implies  $\tau(p) \in \mathcal{H}$ .  $\square$

We now study the transitivity properties of hyperbolic transformation.

**Theorem 25.2.** *Let  $p \in \mathcal{H}$  and  $l$  be a point and a line incident to each other. Then there exists a hyperbolic transformation  $\tau$  such that  $\tau(p) = (0, 0, 1)^T$  is the origin and  $\tau(l) = (0, 1, 0)$  is the  $x$ -axis.*

*Proof.* We can base the proof on the observation of Theorem 22.2 that states that in the case of a nondegenerate fundamental conic two lines  $l$  and  $m$  are orthogonal if and only if their intersections with the fundamental conic are in harmonic position. Thus we may take the line  $l$  and its unique hyperbolic perpendicular through  $p$ , the line  $g$ . Then the intersections  $A_l, B_l, A_g$ , and  $B_g$  with  $\overline{\mathcal{H}}$  are in harmonic position:  $(A_l, B_l; A_g, B_g)_{\overline{\mathcal{H}}} = -1$ . The intersections  $A_h, B_h$  of the (horizontal)  $x$ -axis with  $\overline{\mathcal{H}}$  and the intersections  $A_v, B_v$  of the (vertical)  $y$ -axis with  $\overline{\mathcal{H}}$  are also in harmonic position. Thus the hyperbolic transformation  $\tau$  with  $\tau(A_l) = A_h$ ,  $\tau(B_l) = B_h$ ,  $\tau(A_g) = A_v$  automatically satisfies  $\tau(B_g) = B_v$ . By this  $l$  is mapped to the  $x$ -axis and  $g$  is mapped to the  $y$ -axis. Furthermore,  $p$  (the intersection of  $l$  and  $g$ ) is mapped by this transformation to the origin (the intersection of the  $x$ -axis and the  $y$ -axis).  $\square$

This theorem enables us to do many of the following derivations without loss of generality for the special case that a certain point  $p$  is located at the center of the hyperbolic disk. Furthermore, it shows that any pair of coincident point and line in  $\mathcal{H}$  can be mapped to any other such pair in  $\mathcal{H}$ . An illustration of the construction in the proof is given in Figure 25.2.

## 25.3 Angles and Boundaries

The theorem we just proved has another advantage related to angle measurement. For lines through the origin the angle measurement agrees with the usual Euclidean angle measurement, as the following theorem shows. Thus by the previous theorem we may always move to a situation with a comparatively simple angle measurement.

**Theorem 25.3.** *Let  $l$  and  $m$  be two lines that pass through the origin  $(0, 0, 1)$ . Let  $\alpha_E$  be the Euclidean angle between them and  $\alpha_H$  the hyperbolic angle between them. Then  $|\alpha_E| = |\alpha_H|$ .*

*Proof.* In our standard representation of hyperbolic geometry the measurement of angles between two lines  $l$  and  $m$  through a point  $p$  is by definition given as

$$\frac{1}{2i} \cdot \ln(l, m; X, Y).$$

Here  $X$  and  $Y$  are the tangents through  $p$  to the fundamental conic. These tangents are the lines that connect  $p$  to the intersections of the polar of  $p$  (with respect to  $\overline{\mathcal{H}}$ ) with the fundamental conic  $\overline{\mathcal{H}}$ . In the standard embedding the polar of the origin with respect to the unit circle is the line at infinity  $l_\infty = (0, 0, 1)^T$ . The intersection of this line with any circle (in particular

with the unit circle) is  $I$  and  $J$ . Hence for measurements of angles between lines through  $p$  we have  $X = I$  and  $Y = J$ , or vice versa. Thus up to a possible sign change the angle measurement agrees with Laguerre's formula (Theorem 18.9) and hence is just the Euclidean measurement.  $\square$

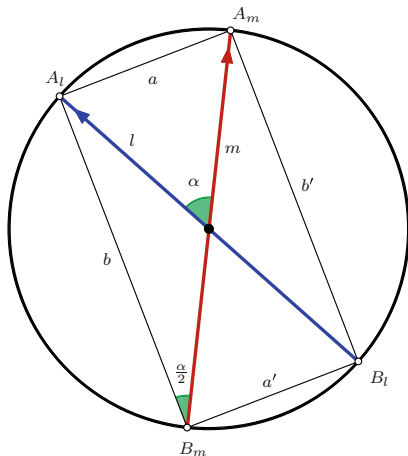
*Remark 25.1.* We have seen in Section 22.3 that if we perform measurement only for lines through one point  $p$ , then we may without any problem talk about *oriented* angle measurement. Since in this case the two tangents  $X$  and  $Y$  are always the same, we can fix their roles once and for all. Now it is possible to prove the following fact. Assume you move a point  $p_t$  continuously parameterized by a parameter  $t \in [0, 1]$  such that  $p_t$  always stays in  $\mathcal{H}$ . The motion of  $p_t$  induces two continuous paths  $X_t$  and  $Y_t$  of the two corresponding tangents. In principle, it may happen that for a round trip with  $p_0 = p_1$  the two tangents may interchange their roles (this is, for instance, possible in an analogous situation in elliptic geometry). However, if  $p_t$  stays inside  $\mathcal{H}$ , the two tangents  $X_t$  and  $Y_t$  will never coincide, and as a consequence one has  $X_0 = X_1$  and  $Y_0 = Y_1$  (the argument for this is not totally trivial and involves some elementary topology). From this it can be shown that for an arbitrary endpoint  $p_1$  the order of the tangents is independent of the concrete path. Thus we may transfer the oriented angle measurement in a unique way to a globally consistent oriented angle measurement on all of  $\mathcal{H}$ . The hyperbolic plane is orientable!

We now will give an explicit method for calculating the angle between two oriented lines. By this we mean the absolute value of the angle that is needed to rotate one of the lines to match the other in the same orientation. The fact that four points on the unit circle are in harmonic position if and only if the lines joining alternating pairs of them are perpendicular is only a special case of the following theorem. The theorem allows us to calculate the angle directly from the intersection of the two lines with the unit circle.

**Theorem 25.4.** *Let  $l$  and  $m$  be two oriented lines intersecting  $\overline{\mathcal{H}}$  in points  $A_l, B_l$ , resp.  $A_m, B_m$ . The lines are assumed to be oriented from  $B$  to  $A$ . Then the hyperbolic angle between  $l$  and  $m$  can be calculated as*

$$2 \arctan \left( \sqrt{-(A_l, B_l; A_m, B_m)_{\overline{\mathcal{H}}}} \right).$$

*Proof.* We first prove the theorem for the special case that the intersection of the two lines passes through the origin. Then we have a situation as shown in Figure 25.3. We consider the rectangle whose vertices are the intersections of the two lines with the unit circle. By Theorem 18.3 and the fact that on the unit circle the points  $A_l, B_l$  separate  $A_m, B_m$ , the cross-ratio  $(A_l, B_l; A_m, B_m)_{\overline{\mathcal{H}}}$  can be calculated as  $-\frac{a \cdot a'}{b \cdot b'}$ , where  $a, a', b, b'$  are the lengths of the sides of the rectangle. Since  $a = a'$  and  $b = b'$ , the cross-ratio becomes



**Fig. 25.3** Determining an angle from boundary points.

$$(A_l, B_l; A_m, B_m)_{\overline{\mathcal{H}}} = -\frac{a^2}{b^2} = -(\tan(\alpha/2))^2.$$

The last equation holds since the (Euclidean) angle at  $B_m$  shown in the picture is half the angle between  $l$  and  $m$ . The tangent of this angle is exactly  $a/b$ . This proves

$$\alpha = 2 \arctan \left( \sqrt{-(A_l, B_l; A_m, B_m)_{\overline{\mathcal{H}}}} \right)$$

for the special case of lines that intersect in the origin.

Now assume that  $l$  and  $m$  are arbitrary intersecting lines. By Theorem 25.2 there is a hyperbolic transformation  $\tau$  that moves the intersection of these two lines to the origin. This transformation leaves the cross-ratio  $(A_l, B_l; A_m, B_m)_{\overline{\mathcal{H}}}$  as well as the angle  $\alpha$  invariant, which proves the theorem.  $\square$

## 25.4 The Poincaré Disk

We now introduce the Poincaré disk model for hyperbolic geometry. For this we again consider a unit circle around the origin as the staging ground. However, this time we interpret it as a circle in the *complex* number plane  $\mathbb{C}$ . Again it is highly appropriate to compactify  $\mathbb{C}$  and right away consider the complex projective line  $\mathbb{CP}^1$  where the point at infinity has been added. As in  $\mathbb{RP}^2$  the region outside the unit circle helps us to understand the underlying

geometric and algebraic structure of the model. However, in this model the region outside the unit circle will play a fundamentally different role from that in the Beltrami-Klein model. Inside the unit disk both models are only deformed images of one another. The following lists give a dictionary of how objects of the hyperbolic plane are modeled in the Poincaré disk model. Roughly speaking, the ...

- ... *points* of the hyperbolic plane are represented by all points inside the unit circle,
- ... *lines* of the hyperbolic plane are represented by circular arcs inside the unit circle that intersect the unit circle orthogonally,
- ... hyperbolic *transformations* are those Möbius transformations and those anti-Möbius transformations (compare Section 17.4) that leave the unit circle invariant.

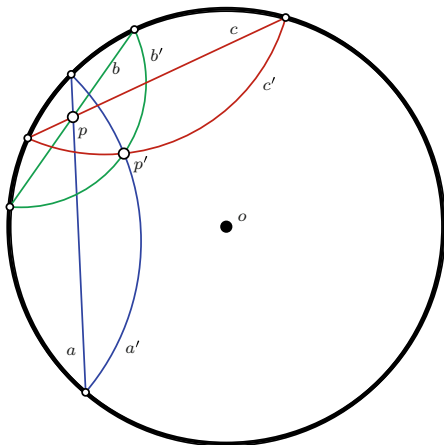
Furthermore, the Poincaré disk has several advantages related to measurements. In the Poincaré disk model the ...

- ... hyperbolic *angle* between two lines is the (Euclidean) intersection angle between the corresponding circular arcs,
- ... *circles* (as loci of real(!) constant distance to a point) are represented by Euclidean circles,
- ... hyperbolic *distances* can still be calculated as the logarithm of a certain cross-ratio.

In what follows (if not explicitly stated otherwise), by a *circle* in  $\mathbb{CP}^1$  we will always mean either a usual circle or a circle with infinite radius (i.e., a line). Lines are those circles that pass through the infinite point  $\infty$  of  $\mathbb{CP}^1$ . The possibility to translate terms from the Beltrami-Klein model that resides in  $\mathbb{RP}^2$  and that is based essentially on *linear* structures to the Poincaré disk model that resides in  $\mathbb{CP}^1$  and that is based essentially on *circular* structures in such a nice way comes from a rich interplay and network of concepts relating the two worlds. Again we will here give only a rough impression of this rich topic. Projective geometry allows us to formulate many concepts on an advanced structural level. However, it will also be helpful to apply some simple theorems from elementary geometry in this context.

We first aim at a proof that both models indeed represent structures that are isomorphic inside the unit circle. What do we have to do for this? In the Beltrami-Klein model a hyperbolic line is represented by a line segment  $a$  that connects two points of  $\bar{\mathcal{H}}$ . In the Poincaré disk model the corresponding line is represented by a circular arc  $a'$  that connects the same two points. In addition,  $a'$  has to intersect the unit circle orthogonally. This requirement makes  $a'$  uniquely defined. We now must verify that this way of mapping lines between the two models leads to a consistent specification for the mapping of points. A point  $p$  in the Poincaré disk model may be uniquely specified as the intersection of two distinct lines  $a$  and  $b$  that contain it. The corresponding circular arcs  $a'$  and  $b'$  associated to these two lines again have a unique





**Fig. 25.4** From Beltrami-Klein to Poincaré.

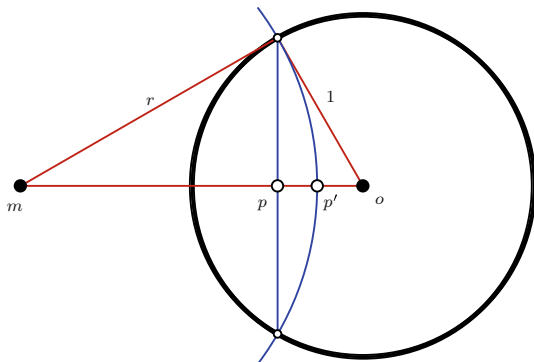
intersection  $p'$ . This point  $p'$  must be the point in the Poincaré disk model that represents  $p$ . We now must show that  $p'$  is well-defined. This means that it is independent of the specific choice of  $a$  and  $b$ . In other words, if we have a third line  $c$  (in the Beltrami-Klein model) that passes through  $p$ , then the corresponding circular arc  $c'$  must also pass through  $p'$ . Figure 25.4 illustrates the situation. Thus, in order to show the isomorphism between the two models we have to prove the following incidence theorem.

**Theorem 25.5.** *Let  $a, b, c$  be three chords of the unit circle and let  $a', b', c'$  be three circular arcs intersecting the unit circle orthogonally in the corresponding endpoints of the chords. Then  $a, b, c$  intersect in a point  $p$  if and only if  $a', b', c'$  intersect in a point  $p'$ .*

There are several ways to approach a proof of this crucial theorem. One possibility would be to simply translate it into corresponding analytic terms and do the proof by straightforward calculations (which in between will become a little messy). We will provide a proof that is closely related to the projective approaches by homogeneous coordinates used throughout this book. For this we will first aim at a nice algebraic characterization of those circles that are orthogonal to the unit circle. A circle with center  $m = (m_x, m_y)$  and radius  $r$  is represented by the equation

$$(x - m_x)^2 + (y - m_y)^2 = r^2, \quad \text{or equivalently} \quad ax + by + c + d(x^2 + y^2) = 0,$$

with  $a = -2m_x$ ,  $b = -2m_y$ ,  $c = m_x^2 + m_y^2 - r^2$ ,  $d = 1$ . We will now use the (homogeneous) coordinate quadruple  $(a, b, c, d)$  as coordinates for the circle. As usual, nonzero multiples of such a vector represent the same geometric



**Fig. 25.5** Condition for being orthogonal (red) and calculation of the map that connects the Beltrami-Klein model to the Poincaré disk model (blue).

object. The circles with  $d = 0$  cover the case of circles with infinite radius (i.e., lines).<sup>1</sup> The unit circle  $\overline{H}$  itself has coordinates  $H = (0, 0, -1, 1)$ .

Now, as a consequence of the Pythagorean theorem, a circle with center  $m$  and radius  $r$  intersects the unit circle orthogonally if  $|m|^2 - r^2 = 1$  (compare Figure 25.5). Using  $m_x = -a/2d$ ,  $m_y = -b/2d$ ,  $r^2 = -c/d + (a/2d)^2 + (b/2d)^2$ , this equation translates to

$$\underbrace{(a/2d)^2 + (b/2d)^2}_{|m|^2} + \underbrace{c/d - (a/2d)^2 - (b/2d)^2}_{-r^2} = 1 \quad \Leftrightarrow \quad c = d.$$

Thus in the  $(a, b, c, d)$  coordinate representation, being orthogonal to the unit circle simply corresponds to a linear condition. Let  $C_1 = (a_1, b_1, c_1, d_1)$  and  $C_2 = (a_2, b_2, c_2, d_2)$  be two coordinate vectors of circles. The linear combinations  $\lambda C_1 + \mu C_2$  correspond to linear combinations of the circle equations and hence represent all the circles that pass through the common intersection of the circles represented by  $C_1$  and  $C_2$ . Thus if  $C_1$  and  $C_2$  are both orthogonal to the unit circle, then *all* circles of the form  $\lambda C_1 + \mu C_2$  are also orthogonal to the unit circle. Within the bundle  $\lambda C_1 + \mu C_2$  there is also one special circle of infinite radius. If the two circles intersect in two points, then this is the unique line connecting these two points. Applying the usual Plücker's  $\mu$  trick to obtain a zero in the last coordinate, one sees that this line has the coordinates  $d_2 C_1 - d_1 C_2$ .

We now want to relate the coordinates of a circle  $C = (a, b, c, c)$  that is orthogonal to the unit circle to the coordinates of the chord that connects its two intersections with the unit circle. The circle of infinite radius in the bundle  $\lambda C + \mu H$  is

$$1 \cdot C - c \cdot H = (a, b, c, c) - (0, 0, -c, c) = (a, b, 2c, 0).$$

<sup>1</sup> The  $(a, b, c, d)$  coordinates represent circles as objects in a projective space  $\mathbb{RP}^3$ .

It represents the line  $ax+by+2c=0$  with homogeneous coordinates  $(a, b, 2c)$ . All in all, we may represent the linear map that connects circles orthogonal to  $\overline{\mathcal{H}}$  to their chords by  $\Psi((a, b, c)) := (a, b, 2c)$ .

*Proof of Theorem 25.5:* With these preparations the proof of Theorem 25.5 is simple. Three distinct circles represented by  $C_1, C_2, C_3$  pass through a common point if and only if their coordinates are linearly dependent. Since the circles are assumed to be distinct, no two of them are dependent and we have a relation  $\lambda \cdot C_1 + \mu \cdot C_2 = C_3$ . This implies  $\lambda \cdot \Psi(C_1) + \mu \cdot \Psi(C_2) = \Psi(C_3)$ . Thus also the coordinates of the corresponding chords are linearly dependent and they meet in a point. Since  $\Psi$  relates circles orthogonal to  $\overline{\mathcal{H}}$  and the possible chords bijectively, the argument applies as well in the other direction.  $\square$

So, what have we achieved so far? We have proved that the mapping of lines in the Beltrami-Klein model (the chords) to the lines in the Poincaré model extends to a well-defined mapping of the points. Since in the particular case of a diameter of the unit circle both models agree in the representation of the lines, we now can easily calculate this map (the isomorphism between the two models) explicitly. For a point  $p$  inside  $\mathcal{H}$  (considered as a Beltrami-Klein point) we consider a diameter  $l$  that passes through this point. If  $p \neq o$  this diameter is unique, but this does not matter here. Furthermore, we consider the chord  $g$  that passes through this point and is orthogonal to  $l$ . The corresponding point  $p'$  can now be calculated as the intersection of the representations of  $l$  and  $g$  in the Poincaré disk model. Since  $l$  is a diameter, it is represented again by  $l$  in the Poincaré disk. The line  $g$  becomes a circular arc perpendicular to  $\overline{\mathcal{H}}$  having the same endpoints as  $l$ . We now calculate how  $p$  and  $p'$  are related. For this again consider Figure 25.5. Assume that the distance from  $p'$  to the origin is  $z$ , and that the distance of  $p$  to the origin is  $w$ . Then  $|m| - z = r$ . We furthermore see that  $1/w = |m|/1$ . Expressing  $|m|$  by  $w$ , we get  $1/w - z = r$ . Squaring both sides and applying the Pythagorean theorem gives

$$\frac{1}{w^2} - 2\frac{z}{w} + z^2 = r^2 = |m|^2 - 1^2 = \frac{1}{w^2} - 1,$$

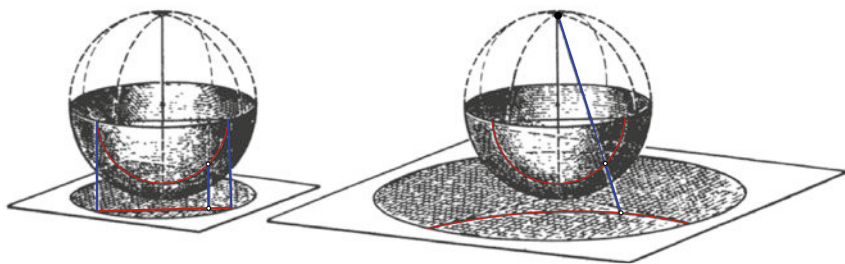
which immediately leads to

$$2\frac{z}{w} - z^2 = 1.$$

Resolving this expression for  $w$  gives

$$w = \frac{2z}{1 + z^2}.$$

In these considerations  $z$  and  $w$  were just real numbers expressing the radial distance to the origin of the points  $p'$  and  $p$ . We get a radial scaling factor of  $\frac{2}{1+z^2}$ . We may also consider  $\mathcal{H}$  embedded in the complex plane and consider  $z$



**Fig. 25.6** Projections relating the Beltrami-Klein model to the Poincaré disk.

and  $w$  as complex numbers directly encoding the positions of  $p'$  and  $p$ . We have only to be careful to replace  $z$  by  $|z|$  in the scaling factor. In this complex world the mapping becomes

$$w = \frac{2z}{1 + |z|^2}. \quad (25.1)$$

Resolving for  $z$  in an analogous way, we get (with  $z$  and  $w$  being again complex numbers representing the positions of  $p'$  and  $p$ )

$$z = \frac{w}{1 + \sqrt{1 - |w|^2}}. \quad (25.2)$$

The square root in this formula is understood to be positive. This ensures that the function maps again into  $\mathcal{H}$  (and in fact is bijective on  $\mathcal{H}$ ). Replacing the square root of the last expression by its negative gives the second intersection of the line supporting the diameter through  $p$  with the circle supporting the arc representing  $g$  in the Poincaré disk. This second intersection lies outside the unit disk.

There is also a nice way to interpret the mapping from the Beltrami-Klein model to the Poincaré disk as a sequence of two projections. For this consider Figure 25.6, which is an enhanced version of a picture that appears in Klein's *Vorlesungen über nicht-euklidische Geometrie* [68]. Consider the unit circle that represents the Beltrami-Klein model (plane in the first picture). Embed the plane supporting this circle in a three-dimensional space. On top of the circle place a sphere of radius 1 and project every point of the circle to the lower hemisphere of this sphere by an orthogonal projection parallel to the  $z$ -axis. By this a chord in the unit disk gets mapped to a half-circle whose support plane is orthogonal to the equator plane of the sphere. Now the second projection projects the lower hemisphere back to the plane by a projection whose center is the north pole of the sphere. Such a projection is the *stereographic projection* we got to know in Section 17.7. It has the property

that it maps circles to circles and preserves intersection angles of circles. The stereographic projection maps the equator to unit circle  $2\overline{\mathcal{H}}$  scaled the by a factor of two. The arcs intersecting the equator orthogonally are mapped to arcs that intersect  $2\overline{\mathcal{H}}$  orthogonally. Scaling down the region  $2\mathcal{H}$  by a factor of 2 leaves us with our version of the Poincaré disk. To see that this is indeed the right mapping, consider a point  $(x, y, 0)^T$  in the Poincaré disk embedded in  $\mathbb{R}^2$ . Scaling by a factor of 2 gives  $(2x, 2y, 0)^T$ . Mapping this point to the sphere by stereographic projection via formula (17.1) results in

$$\begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \mapsto \frac{2}{x^2 + y^2 + 1} \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}.$$

Projecting down to the plane exactly results in

$$\frac{1}{x^2 + y^2 + 1} \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}.$$

Omitting the last coordinate, this is our formula (25.1) that relates the points of the Poincaré disk to the points in the Beltrami-Klein model.

The scaling factor of the map (25.1) for points close to the center is asymptotically 2. This means that objects close to the center in the Poincaré disk model appear half the size that they appear in the Beltrami-Klein model. They thereby leave more space for the other objects that are squeezed close to the boundary. Thus the Poincaré disk model appears visually more balanced, an effect that we could observe for instance in Figure 24.5.

In our translation process one final point deserves to be mentioned: the role of the exterior of  $\mathcal{H}$  in the Poincaré disk model. Referring once more to Figure 25.6, we see that in the second projection the *upper hemisphere* of the sphere is mapped to the *outside* of the Poincaré disk. The vertical projection that relates the Beltrami-Klein model to the sphere produces in essence two intersections: one at the lower hemisphere and one at the upper hemisphere. Thus projecting also the upper hemisphere to  $\mathbb{CP}^1$ , each point inside  $\mathcal{H}$  in the Beltrami-Klein model is related to *two* points in  $\mathbb{CP}^1$ , one point inside the unit disk and one point outside the unit disk. They correspond to the choice of the sign of the square root in (25.2). Both representing points  $\frac{w}{1+\sqrt{1-|w|^2}}$  and  $\frac{w}{1-\sqrt{1-|w|^2}}$  have the same direction with respect to the origin but different distances. In fact, the radii are reciprocal, as the following calculation shows. The product of the distances of these two points to the origin evaluates to

$$\frac{|w|}{1+\sqrt{1-|w|^2}} \cdot \frac{|w|}{1-\sqrt{1-|w|^2}} = \frac{|w|^2}{1-(1-|w|^2)} = 1.$$

Both of them are related by an inversion in the unit circle  $z \mapsto \frac{1}{\bar{z}}$ . Thus the outside of the Poincaré disk is just a mirror image (by reflection in the unit circle) of the inside. The points outside the unit circle in the Beltrami-Klein model have no direct counterpart in the Poincaré disk.

## 25.5 $\mathbb{CP}^1$ Transformations and the Poincaré Disk

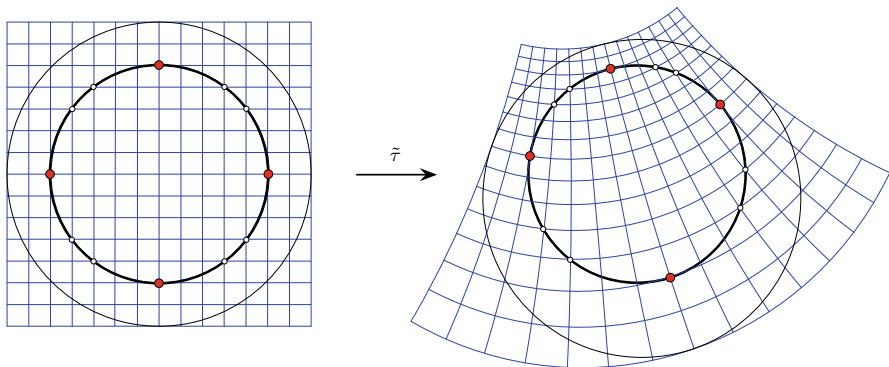
So far, we have defined hyperbolic transformations only in the world of the Beltrami-Klein model. They are the projective  $\mathbb{RP}^2$  transformations  $\tau$  that leave the unit circle as a whole invariant. Via the isomorphism of the Poincaré disk and the Beltrami-Klein model (explicitly given by equations (25.1) and (25.2)), the map  $\tau$  induces an action  $\tilde{\tau}$  in the Poincaré disk model. The map  $\tilde{\tau}$  is the corresponding hyperbolic transformation in the Poincaré disk model. It will turn out that these hyperbolic transformations  $\tilde{\tau}$  in the Poincaré disk are exactly those Möbius transformations and anti-Möbius transformations that leave the unit circle invariant. The reason for this nice interplay of (certain)  $\mathbb{RP}^2$  projective transformations and (certain)  $\mathbb{CP}^1$  projective transformations is that a hyperbolic transformation must leave the unit circle as a whole invariant. In both cases this induces a transformation of the points on the unit circle that is a projective transformation on the points of  $\overline{\mathcal{H}}$ .

Since the situation involves simultaneously several interpretations of the same objects in different mathematical spaces, it may become quite confusing. To avoid this confusion we will first specify these different spaces and their interrelations. Most of these relations have already been established in previous chapters.

**The relation of  $\mathbb{RP}^2$  and  $\mathbb{CP}^1$ :** We identify the finite part of both spaces by the map  $(x, y, 1)^T \mapsto (x + iy, 1)$ . In particular, this map relates the unit circle and its interior in both spaces bijectively. We will use the same symbols  $\overline{\mathcal{H}}$  and  $\mathcal{H}$  for the unit circle and its interior in both spaces. As before, for a point  $p \in \mathbb{RP}^2$  we will denote its counterpart in  $\mathbb{CP}^1$  by  $\tilde{p}$ .

**The unit circle in  $\mathbb{RP}^2$  and its relation to  $\mathbb{RP}^1$ :** The unit circle  $\overline{\mathcal{H}}$  is a special conic satisfying the (homogeneous) equation  $x^2 + y^2 - z^2 = 0$ . In Section 10.2 we learned that we can bijectively relate each nondegenerate conic to the space  $\mathbb{RP}^1$  by (stereographically) projecting its points to a line (compare Figure 10.7). Equivalently, this bijection can also be calculated by a cross-ratio with respect to the conic. For this we consider  $\mathbb{R} \cup \{\infty\}$  as a copy of  $\mathbb{RP}^1$  and define a specific bijection  $\Psi: \overline{\mathcal{H}} \rightarrow \mathbb{R} \cup \{\infty\} = \mathbb{RP}^1$ . We fix three mutually different points  $\mathbf{0}, \mathbf{1}, \infty$  on the conic and for every other point  $\mathbf{x} = (x, y, 1)^T$  on the conic calculate

$$\Psi(\mathbf{x}) := (\mathbf{0}, \infty; \mathbf{x}, \mathbf{1})_{\overline{\mathcal{H}}}.$$



**Fig. 25.7** A hyperbolic transformation.

In a natural way we may call a map  $f: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$  a *projective map* if the composition  $\Psi \circ f \circ \Psi^{-1}$  is a projective transformation on  $\mathbb{RP}^1$ . Theorem 10.3 states that every projective transformation on  $\mathbb{RP}^2$  that leaves  $\overline{\mathcal{H}}$  invariant induces a projective map (a  $\mathbb{RP}^1$  action) on the points of  $\overline{\mathcal{H}}$ .

**The unit circle in  $\mathbb{CP}^1$  and its relation to  $\mathbb{RP}^1$ :** We can also relate the unit circle in  $\mathbb{CP}^1$  in a natural way to the real projective line. For this we also fix three points  $\tilde{\mathbf{0}}, \tilde{\mathbf{1}}, \tilde{\infty}$  on  $\overline{\mathcal{H}}$  and consider the map  $\Phi: \overline{\mathcal{H}} \rightarrow \mathbb{R} \cup \{\infty\} = \mathbb{RP}^1$  defined by

$$\Phi(\tilde{\mathbf{x}}) := (\tilde{\mathbf{0}}, \tilde{\infty}; \tilde{\mathbf{x}}, \tilde{\mathbf{1}}),$$

the cross-ratio this time considered over  $\mathbb{CP}^1$ . This expression maps into  $\mathbb{R} \cup \{\infty\}$  since four cocircular points in  $\mathbb{CP}^1$  always produce a real or infinite cross-ratio. In fact, Theorem 18.3 tell us that both ways to relate the unit circle to  $\mathbb{RP}^1$  are identical. We have

$$\Psi(\mathbf{x}) = (\mathbf{0}, \infty; \mathbf{1}, \mathbf{x})_{\overline{\mathcal{H}}} = (\tilde{\mathbf{0}}, \tilde{\infty}; \tilde{\mathbf{1}}, \tilde{\mathbf{x}}) = \Phi(\tilde{\mathbf{x}}). \quad (25.3)$$

We may even interpret the map  $\Phi$  as a projective map that sends the circle  $\overline{\mathcal{H}}$  to the real points  $\mathbb{RP}^1 = \{(a, b)^T | a, b, \in \mathbb{R}\} - \{(0, 0)^T\}$  in  $\mathbb{CP}^1$  by sending  $\tilde{\mathbf{0}} \mapsto (0, 1)^T$ ,  $\tilde{\mathbf{1}} \mapsto (1, 1)^T$ , and  $\tilde{\infty} \mapsto (1, 0)^T$ . Again in a natural way we may call a map  $f: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$  a *projective map* if  $\Phi \circ f \circ \Phi^{-1}$  is a projective transformation on  $\mathbb{RP}^1$ . By the equivalence of the two cross-ratios in equation (25.3) the two concepts of projective transformations on  $\overline{\mathcal{H}}$  are equivalent.

Now, a projective transformation  $\tau'$  in  $\mathbb{CP}^1$  that leaves  $\overline{\mathcal{H}}$  invariant can be interpreted as a projective transformation on the points of  $\overline{\mathcal{H}}$  (considered as a  $\mathbb{RP}^1$ ) via  $\Phi \circ \tau' \circ \Phi^{-1}$ . Figure 25.7 illustrates the effect of a projective  $\mathbb{CP}^1$  transformation that leaves the unit circle invariant. Notice that the grid of lines is mapped to a grid of circular arcs. The map induces a transformation on the unit circle itself. This transformation is projective. The specific

transformation in Figure 25.7 has been chosen such that the image of three points is exactly the same as in the  $\mathbb{RP}^2$  transformation shown in Figure 21.2. This implies that both maps agree on the unit circle.

Moreover, and this is important for us, we get the following lemma.

**Lemma 25.1.** *Let  $f: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$  be a projective map. Then this map can be uniquely extended to a projective transformation  $\tau'$  in  $\mathbb{CP}^1$ .*

*Proof.* Let  $f: \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$  be a projective map on  $\overline{\mathcal{H}}$ . This implies that the map  $g := \Phi \circ f \circ \Phi^{-1}$  is a projective map  $\mathbb{RP}^1 \rightarrow \mathbb{RP}^1$  on the usual projective line. Thus  $g$  can be represented by a  $2 \times 2$  matrix with real coefficients. We may interpret this map as a projective map  $g: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ . We now can pull back this  $\mathbb{CP}^1$  action to  $\overline{\mathcal{H}}$  by setting  $\tau' = \Phi^{-1} \circ g \circ \Phi$ . In other words, every projective map on  $\overline{\mathcal{H}}$  can be written as a  $2 \times 2$  matrix and by this interpreted as a projective map on  $\mathbb{CP}^1$ .  $\square$

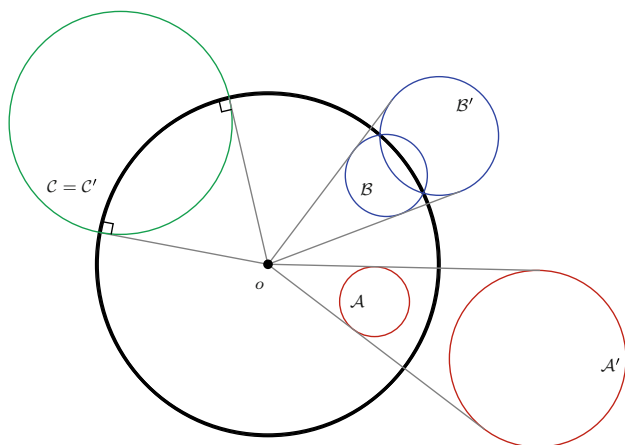
After these conceptual considerations we are ready to prove that hyperbolic transformations in the Poincaré disk are either Möbius or anti-Möbius transformations. For a given hyperbolic transformation  $\tau$  (in the Beltrami-Klein model) we want to compute the corresponding effect in the Poincaré disk model, i.e., the map  $\tilde{\tau}$ . In principle, this could be done by directly calculating the effect of a projective transformation under a conjugation by the transformation equations (25.1) and (25.2). However, we again can avoid these rather tedious calculations by performing some structural analysis of the situation. Within the Poincaré disk model we may compute the image under  $\tilde{\tau}$  of a point  $p$  inside  $\mathcal{H}$  by the following alternative procedure.

- 1: Take two distinct circular arcs  $l$  and  $g$  intersecting  $\overline{\mathcal{H}}$  orthogonally (two Poincaré lines) that both pass through  $p$ .
- 2: The two arcs intersect  $\overline{\mathcal{H}}$  in two pairs of points  $(A_l, B_l)$  and  $(A_g, B_g)$ .
- 3: Take the images  $(\tau(A_l), \tau(B_l))$  and  $(\tau(A_g), \tau(B_g))$  of these point pairs under  $\tau$  (this is the same image as under  $\tilde{\tau}(l)$ ).
- 4: They determine two new circular arcs  $\tilde{\tau}(l)$  and  $\tilde{\tau}(g)$  (orthogonal to  $\overline{\mathcal{H}}$ ) inside  $\mathcal{H}$ . These are the images of  $l$  and  $g$  in the Poincaré model.
- 5: Their intersection corresponds to the image  $\tilde{\tau}(p)$  of  $p$ .

This can be used to prove the following theorem.

**Theorem 25.6.** *Let  $\tau$  be a hyperbolic transformation. If  $\tau$  preserves the orientation of  $\overline{\mathcal{H}}$ , then  $\tilde{\tau}$  is the unique Möbius transformation that agrees with  $\tau$  on  $\overline{\mathcal{H}}$ . If  $\tau$  reverses the orientation of  $\overline{\mathcal{H}}$  then  $\tilde{\tau}$  is a composition of the unique Möbius transformation that agrees with  $\tau$  on  $\overline{\mathcal{H}}$  followed by a circle inversion in the unit circle.*

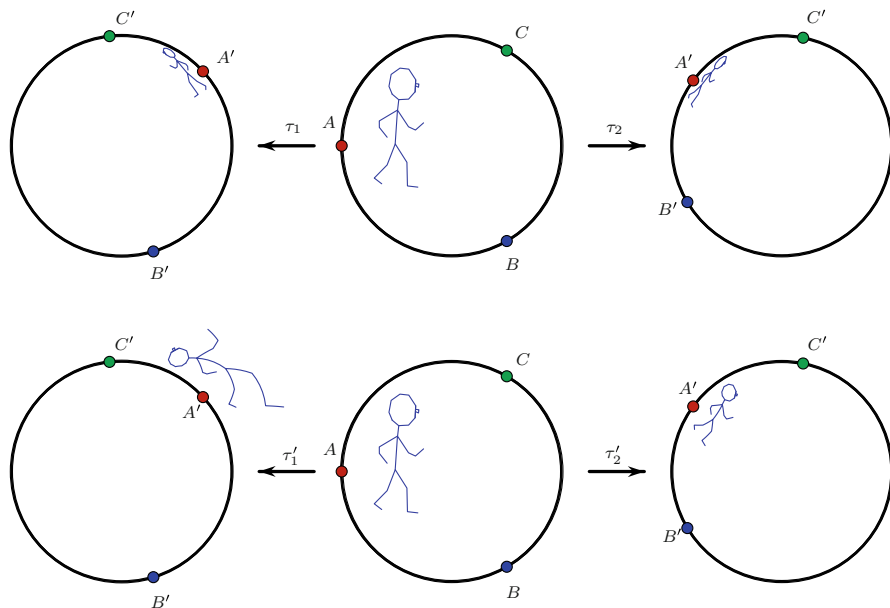




**Fig. 25.8** Circles under circle inversion.

*Proof.* Let  $\tau$  be a hyperbolic transformation in the Beltrami-Klein model. Its action on the boundary  $\overline{\mathcal{H}}$  is a projective  $\mathbb{RP}^1$  map on  $\overline{\mathcal{H}}$ , and it agrees with the action of  $\tilde{\tau}$ . By Lemma 25.1 we can extend this map on  $\overline{\mathcal{H}}$  to a projective transformation  $\tau': \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ . This transformation  $\tau'$  maps circles to circles and preserves oriented intersection angles. Therefore in particular, circles orthogonal to  $\overline{\mathcal{H}}$  are again mapped to circles orthogonal to  $\overline{\mathcal{H}}$ . If the two intersections of such a circle with  $\overline{\mathcal{H}}$  are fixed this, circle is uniquely determined.

We first study the effect of  $\tau'$  on the circular arcs representing lines in the Poincaré disk model. For this let  $l$  be such a circular arc in the interior  $\mathcal{H}$  that intersects  $\overline{\mathcal{H}}$  orthogonally. If  $\mathcal{C}$  is the circle supporting the arc representing the hyperbolic line  $l$ , then the arc representing the image  $\tilde{\tau}(l)$  is supported by the circle  $\tau'(\mathcal{C})$ . Now we have to distinguish between the two cases that  $\tau'$  preserves the orientation of  $\overline{\mathcal{H}}$  and that it reverses it. In the first case  $\tau'$  does not interchange the interior and exterior of  $\overline{\mathcal{H}}$ ; in the second case it does. In both cases we have to identify a map that maps the arc  $l$  to the arc  $\tilde{\tau}(l)$ . This arc  $\tilde{\tau}(l)$  must satisfy two requirements. Firstly, it must be supported by the circle  $\tau'(\mathcal{C})$ , since this circle is uniquely determined by the position of the intersection with  $\overline{\mathcal{H}}$  and the orthogonality condition. Secondly, it must lie in the interior  $\mathcal{H}$ . If interior and exterior are not interchanged, then we can simply set  $\tilde{\tau}(l) = \tau'(l)$ . In case that they are interchanged we have to combine  $\tau'$  with an inversion in the unit circle  $\iota$  (compare Section 17.6). The inversion in the unit circle is the specific anti-Möbius transformation  $\iota(z) = 1/\bar{z}$ . It leaves all points on the unit circle invariant and interchanges its interior and exterior. Since the intersection angle of circles is reversed by this transformation, it maps circles orthogonal to  $\overline{\mathcal{H}}$  onto themselves, thereby interchanging the interior and the exterior arcs (see Figure 25.8). Thus in the



**Fig. 25.9** The difference of projective and Möbius transformations that fix the unit circle.

second case we must set  $\tilde{\tau}(l) = (\iota \circ \tau')(l)$ . The map  $\tilde{\tau}$  depended only on the action to the points on  $\overline{\mathcal{H}}$  that was inherited from  $\tau$ . Since every point in  $\mathcal{H}$  can be determined as the intersection of two hyperbolic lines (see the procedure above), the map  $\tilde{\tau}$  is the required hyperbolic transformation in the Poincaré disk.  $\square$

Figure 25.9 demonstrates the difference between the real projective transformations and the Möbius transformations that leave the unit circle invariant. The top row shows the effect of two such real projective transformations  $\tau_1$  and  $\tau_2$  that fix the unit circle. The transformations are uniquely determined by the image of three points on the unit circle. While the transformation  $\tau_2$  preserves the orientation of the unit circle, the transformation  $\tau_1$  reverses it. In both cases the image of Dr. Stickler is inside the unit circle. The orientation under  $\tau_1$  is reversed. In contrast to this, the lower row shows the effect of the two Möbius transformations determined by the images of the same three points. The situation for  $\tau'_2$  is qualitatively the same as for  $\tau_2$ . Dr. Stickler appears inside the circle and his orientation is preserved. The situation for  $\tau'_1$  is significantly different from  $\tau_1$ . The Möbius transformation interchanges the interior and exterior of the circle but still keeps their orientation stable. Thus Dr. Stickler is mapped to the exterior of the circle with his original

orientation. The map  $\tau'_1$  followed by a circle inversion  $\iota$  would be the hyperbolic transformation that corresponds to  $\tau_1$  in the Poincaré disk model.

## 25.6 Angles and Distances in the Poincaré Disk

The hard part of the work is done. From what we have established so far it is relatively easy to determine the computation of angles and distances directly in the Poincaré disk model. The fact that hyperbolic transformations are essentially expressed by Möbius transformations and anti-Möbius transformations helps to transfer (without loss of generality) the measurement to almost trivial situations. We start with angles between hyperbolic lines. We have seen that lines in the Poincaré model correspond to circular arcs inside  $\mathcal{H}$  orthogonal to the unit circle. We again consider angles between oriented lines as in Section 25.3. First of all, the formula for computing the angles from the intersections of lines with the unit circle from Theorem 25.4 still applies to the Poincaré disk, since it agrees with the Beltrami-Klein model along the unit circle. From this we also see that the situation is most trivial for lines through the origin.

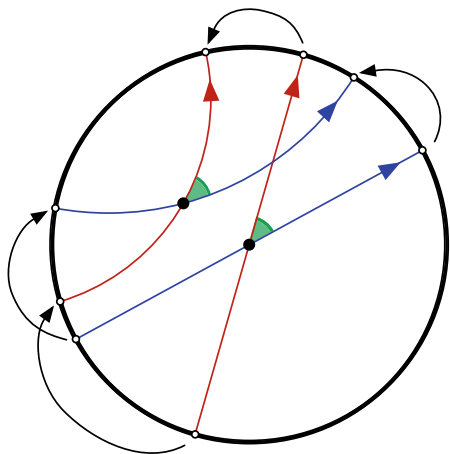
**Lemma 25.2.** *Let  $l$  and  $g$  be two oriented hyperbolic lines in the Poincaré disk that pass through the origin. Then the angle between them is exactly the Euclidean angle between them.*

*Proof.* Hyperbolic lines through the origin in the Poincaré model are just straight diameters of the unit circle. Thus the representation of lines through the origin is the same as in the Beltrami-Klein model, and the lemma follows by Theorem 25.3.  $\square$

The property of Möbius transformations to preserve the intersection angles of circles helps to transfer this result to arbitrary intersection of hyperbolic lines.

**Theorem 25.7.** *Let  $l$  and  $g$  be two arbitrary intersecting oriented hyperbolic lines in the Poincaré disk. Then the angle between them is exactly the Euclidean intersection angle between the corresponding Euclidean arcs.*

*Proof.* Apply a hyperbolic transformation  $\tau$  that maps the intersection of the two lines under consideration to the origin (such a transformation exists by Theorem 25.2). The construction in the proof of this theorem allowed us to choose this transformation to be order-preserving. In the Poincaré disk model this corresponds to a Möbius transformation by Theorem 25.6. The preservation of intersection-angles property of Möbius transformations (Corollary 17.1) ensures that the intersection angles of the circular arcs before and after the transformation are identical. Finally, Lemma 25.2 ensures that the intersection angle matches the Euclidean intersection angle.  $\square$



**Fig. 25.10** Angle between lines in the Poincaré disk.

This gives us an important property of the Poincaré disk model: The angles are represented conformally. One can simply measure the Euclidean intersection angle between the circular arcs that represent the lines to get the hyperbolic intersection angle between the lines. Figure 25.10 illustrates the situation.

It is also instructive to figure out how the hyperbolic distance between two points can be calculated directly in the Poincaré disk model. For this we use the transformation (25.1) and relate certain cross-ratios in the Beltrami-Klein model (i.e. in  $\mathbb{RP}^2$ ) and in the Poincaré disk (i.e., in  $\mathbb{CP}^1$ ). We again pull back to a situation that is algebraically nicely accessible.

**Lemma 25.3.** *Let  $P$  and  $Q$  be two points on the  $x$ -axis in the Poincaré disk. Then the hyperbolic distance between  $P$  and  $Q$  is  $|\ln((P, Q; X, Y))|$ , where  $X$  and  $Y$  are the two intersections of the  $x$ -axis with the unit circle.*

*Proof.* We may represent the points by their  $x$ -coordinates  $p$  and  $q$ . By formula (25.1) the corresponding points  $P'$  and  $Q'$  in the Beltrami-Klein model are also on the  $x$ -axis and have coordinates  $p' = \frac{2p}{1+p^2}$  and  $q' = \frac{2q}{1+q^2}$ . The absolute value of the hyperbolic distance is given by

$$\left| \frac{1}{2} \ln((P', Q'; X, Y)) \right| = \left| \frac{1}{2} \ln \left( \frac{(p' - 1)(q' + 1)}{(p' + 1)(q' - 1)} \right) \right|.$$

Inserting the expression for  $p'$  in  $\frac{(p'-1)}{(p'+1)}$  gives

$$\frac{p' - 1}{p' + 1} = \frac{2p - 1 - p^2}{1 + p^2} \cdot \frac{1 + p^2}{2p + 1 + p^2} = \left( \frac{p - 1}{p + 1} \right)^2.$$

A similar expression holds for  $q$ , and we obtain

$$\frac{(p' - 1)(q' + 1)}{(p' + 1)(q' - 1)} = \left( \frac{(p - 1)(q + 1)}{(p + 1)(q - 1)} \right)^2.$$

This shows that

$$(P', Q'; X, Y) = (P, Q; X, Y)^2.$$

Inserting this in the distance formula yields

$$\left| \frac{1}{2} \ln((P', Q'; X, Y)) \right| = \left| \frac{1}{2} \ln((P, Q; X, Y)^2) \right| = |\ln((P, Q; X, Y))|.$$

This is what is claimed by the theorem.  $\square$

We again can use this result to get a general method to calculate distances in the Poincaré model.

**Theorem 25.8.** *Let  $P$  and  $Q$  be two arbitrary points in the Poincaré disk. Then the hyperbolic distance between  $P$  and  $Q$  is  $|\ln((P, Q; X, Y))|$ , where  $X$  and  $Y$  are the intersections of the hyperbolic line through  $P$  and  $Q$  with the unit circle.*

*Proof.* Let  $P$  and  $Q$  be two arbitrary points in the Poincaré disk. By Theorem 25.2 there is a hyperbolic transformation that maps them to two points  $\tau(P)$  and  $\tau(Q)$  on the  $x$ -axis. This transformation is a Möbius transformation  $\tau$  that fixes the unit circle, and hence  $X$  and  $Y$  get mapped to two points  $\tau(X)$  and  $\tau(Y)$  that are the intersections  $X'$  and  $Y'$  of the  $x$ -axis with the unit circle. Thus we have  $(P, Q; X, Y) = (\tau(P), \tau(Q); \tau(X), \tau(Y)) = (\tau(P), \tau(Q); X', Y')$ , and Lemma 25.3 can be applied to finally prove the theorem.  $\square$

It is an amazing fact that when calculating the hyperbolic distance in the Poincaré model via the cross-ratio in  $\mathbb{CP}^1$  the factor  $1/2$  in the distance formula has to be dropped, in order to be consistent with the Beltrami-Klein model.



## Selected Topics in Hyperbolic Geometry

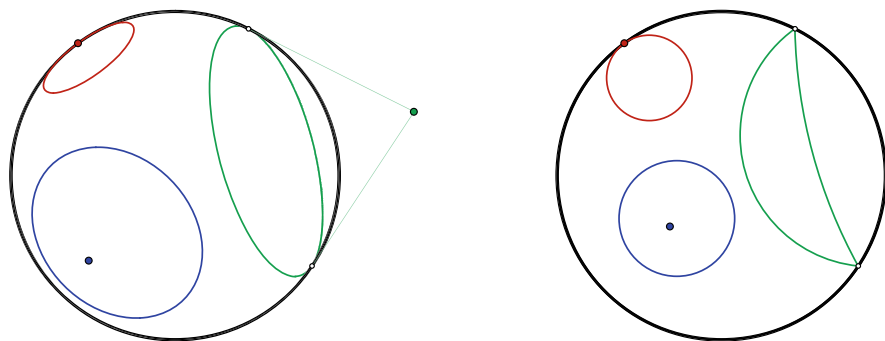
*Out of nothing I created a strange  
new universe!*

*János Bolyai*

After having mastered the basics of hyperbolic geometry, having learned how to calculate distances and angles, perform transformations, represent it in two different models, a whole world of interesting topics opens, by far more than can be covered here in this book. We now want to focus on a few of these topics that are of particular beauty and show specific relationships to our projective approaches to hyperbolic geometry.

### 26.1 Circles and Cycles in the Poincaré Disk

Without going too much into detail we want to explain a few effects concerning circles and cycles in the Poincaré disk model. For this we again have to distinguish carefully between different types of cycles, a topic we already encountered in Chapter 23. We here should again recall that the points outside the unit disk in the Beltrami-Klein model do not have a proper counterpart in the Poincaré disk. For this reason we will prominently consider those cycles that have a nonempty intersection with the interior of the unit disk (in the Beltrami-Klein model). As developed in Section 23.3, these cycles fall into three classes. Members of the the first class will be called *proper real circles*. They are a set of points inside the unit disk that have a *real* hyperbolic distance to a center *m* *inside* the unit disk. In the Beltrami-Klein



**Fig. 26.1** Circles and cycles in the Beltrami-Klein model and the Poincaré disk.

model they correspond to conics that have two complex contact points with our fundamental conic—the unit circle.

Other cycles that may have nonempty intersection with the interior of the unit disk are the *hypercycles* and the *horocycles*. Horocycles have a center that lies on the unit circle. In the Beltrami-Klein model they are conics that osculate with the unit circle of order four (i.e. the two contact points coincide). The distance of the cycle points to such a center is always infinite and cannot be used as a characterizing property for the points on the cycle. Hypercycles are those conics that have a center outside the unit circle. They have order two contact with the unit circle in two real points. The distance of the points of the cycle to the center is a complex number. Summarizing, one can say that in the Beltrami-Klein model, cycles are conics that have contact to the unit circle in two points that may be distinct complex conjugates, distinct real points, or real and coinciding points. All three types of curves have constant curvature, except for the points where they meet the unit circle.

How are these types of cycles represented in the Poincaré model? Figure 26.1 shows three cycles: a proper real circle (blue), a horocycle (red), and a hypercycle (green) in the Beltrami-Klein model and the corresponding cycles in the Poincaré model. If possible also the centers are indicated. We once more recall our discussion from Section 23.4 concerning the question whether a hypercycle should be considered *one* cycle or *two* cycles. There were good reasons for both of these viewpoints. The reason to consider the hypercycle to be *two* cycles comes from the differential geometric viewpoint: It is impossible to get from one half to the other by a finite journey along the circle in hyperbolic space. Both branches are curves of constant curvature of infinite length. The reason to consider them to be *one* branch came from the algebraic treatment. The entire cycle should be an algebraically closed curve: a complete conic. We will see that both viewpoints again have their counterparts in the Poincaré disk model. First we will briefly explain

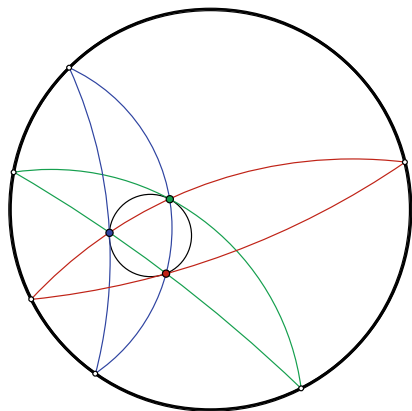


the characteristic features of the three types of cycles in the Poincaré disk model.

**Proper real circles:** In the Poincaré disk a proper real circle (blue) is represented by a curve that looks like a usual Euclidean circle that lies entirely in  $\mathcal{H}$ . This can easily be seen as follows. Assume that the center of the circle is the origin. Since the distance function with respect to the origin is rotationally symmetric, such a circle must look like a usual Euclidean circle. Now every circle can be obtained from a circle centered at the origin by a suitable hyperbolic transformation. In the Poincaré disk these hyperbolic transformations correspond to Möbius transformations, which again map circles to circles. Hence every proper real circle must in fact be represented by something that “looks like” a Euclidean circle. It is interesting to observe that although in the Poincaré disk hyperbolic circles look like Euclidean circles, their centers in general do not look like centers. Their position gets distorted by the hyperbolic metric.

**Horocycles:** Horocycles (red) do not offer much spectacularly new. In the Poincaré disk they are still represented by Euclidean circles. They arise as a limit case of proper real circles that touch the boundary. The center is then exactly located at this touching point. It is an interesting observation that the deformation function from the Beltrami-Klein model to the Poincaré disk behaves such that the osculation of order four in the Beltrami-Klein model now becomes a contact only of order two between two circles.

**Hypercycles:** A slight surprise arises for the case of hypercycles (green). The corresponding image in the Poincaré disk consists of *two* circular arcs that form a lune-shaped region. The deformation between the two models is such that the contact of order two in the Beltrami-Klein model becomes a sharp edge between two circular arcs. There is a completely different way of looking at the two circular arcs in the Poincaré disk that also explains their mutual relationship. We have seen in Section 25.4 that every point  $z$  inside the unit circle has an associated point outside the unit circle located at  $1/\bar{z} = \iota(z)$  that arises from a circular reflection at the unit circle. Structurally, both points represent the same point in the hyperbolic plane. This is the reason why we could restrict ourselves to the interior and the boundary of the unit circle. Now, if we have a proper real circle, it has (via  $z \mapsto \iota(z)$ ) an associated circle that lies completely outside the unit circle (compare for instance the red circle in Figure 25.8). Now, what happens if we move such a proper real circle to a position where it intersects the unit circle in two points (and hence is no longer a proper real circle in our sense)? Part of the original circle sticks outside the unit circle, and its reflection via  $\iota$  forms a second circular arc inside the Poincaré disk. This is exactly the second arc we see in a hypercycle. It is completely determined by the other arc and arises as circle reflection of the outside part of the first circle. To see this effect consider the blue circles in Figure 25.8. Since the center of such a hypercycle in the



**Fig. 26.2** Four  $A$ -circles through three points.

Beltrami-Klein model lies outside the unit circle, it has no visual counterpart in the Poincaré model.

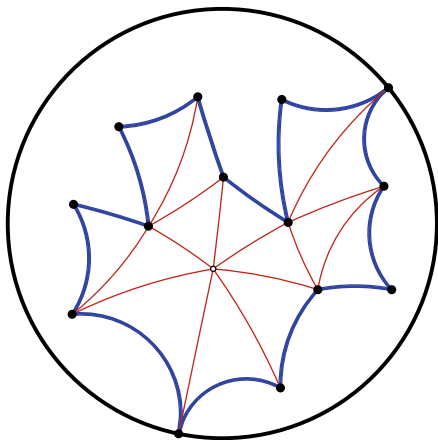
So we see that the two branches of a hypercycle in the Poincaré disk model indeed translate into two distinct (though closely related) circular arcs. Let us finally exemplify how the different viewpoints of what is a circle sensibly influence some of the fundamental statements of elementary geometry. In Euclidean geometry we are used to the fact that through three finite points there always passes a unique circle. What is the counterpart of this in hyperbolic geometry? This depends on your definition of what is a cycle. Let us take two extreme positions. We first formulate them in the Beltrami-Klein model. Let us call an  $A$ -circle ( $A$  for algebraic) for a given center  $m$  any conic described by the matrix

$$\lambda \cdot M^T B M + \mu \cdot A$$

(compare Equation (23.3)), that is, the complete conic that corresponds to a circle, a horocycle or a hypercycle. Let us call a  $C$ -circle ( $C$  for curvature) any curve of constant curvature that arises from starting at a point inside the unit disk and following with the same curvature in both directions. Thus the hypercycles get cut into two branches. In the Poincaré disk the  $A$ -circles correspond to full (Euclidean) circles in the closure of  $\mathcal{H}$  and pairs of circular arcs that are related by  $\iota$  as described above. The  $C$ -circles arise simply as intersections of ordinary Euclidean circles with the interior  $\mathcal{H}$  of the unit circle. For three points  $A, B, C$  inside  $\mathcal{H}$  we now have the following:

*There is exactly one  $C$ -circle through these three points. As long as  $A, B, C$  are not on a hyperbolic line, there are exactly four  $A$ -circles through these three points.*

In the Poincaré model the unique  $C$ -circle arises as the intersection of the unique Euclidean circle through  $A, B, C$  with  $\mathcal{H}$ . The four  $A$ -circles arise



**Fig. 26.3** Decomposing a hyperbolic polygon into triangles.

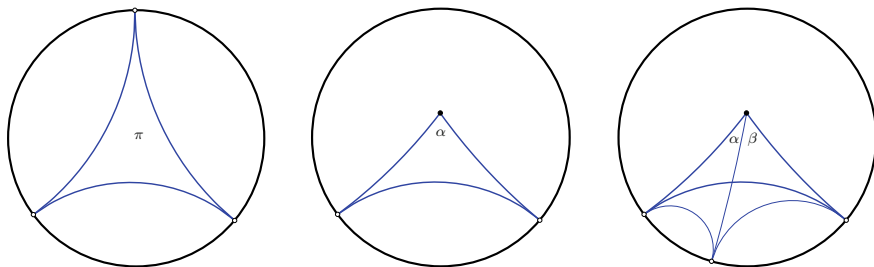
in the following way. Consider the points  $A_0 = A$ ,  $A_1 = \iota(A)$ ,  $B_0 = B$ ,  $B_1 = \iota(B)$ ,  $C_0 = C$ ,  $C_1 = \iota(C)$ . Each selection  $A_i, B_j, C_k$  with  $i, j, k \in \{0, 1\}$  defines a unique Euclidean circle. The intersections of these branches with  $\mathcal{H}$  constitute all branches of the four  $A$ -circles. Figure 26.2 shows the situation inside  $\mathcal{H}$  for three generically chosen points.

## 26.2 Area and Angle Defect

One of the most popularly known effects of hyperbolic geometry is the relation of the *angle defect* of a triangle and its area: It can be shown that in hyperbolic geometry the angle sum  $\alpha + \beta + \gamma$  of a triangle is always less than  $\pi$ .<sup>1</sup> We will now prove that the angle defect  $\pi - \alpha - \beta - \gamma$  (the deviation of the angle sum from  $\pi$ ) is directly related to the hyperbolic area enclosed by the triangle sides. The larger the angle defect, the larger the area of the triangle. So far, we have not even defined what we mean by a hyperbolic area, and we will do this now by specifying two crucial properties that a function that measures area should have. It should be *additive* and *invariant under motion*. To make our development a bit simpler we restrict ourselves to regions inside the unit circle that are bounded by a Jordan curve consisting of finitely many hyperbolic line segments. We will call such a region *simple*. An *area function* is a continuous function  $\mathbf{area}(R)$  that assigns a real number to such a simple region. The function must in addition have the following properties.

**Additivity:** This means that if we have one region  $R = U \cup V$  that is the union of two other simple regions  $U$  and  $V$  with disjoint interior, then

<sup>1</sup> If you prefer measuring angles in degrees, set  $\pi = 180^\circ$ .



**Fig. 26.4** Calculating the area of special hyperbolic triangles.

$$\mathbf{area}(R) = \mathbf{area}(U) + \mathbf{area}(V).$$

**Motion invariance:** If  $\tau$  is a hyperbolic transformation and a  $R$  a simple region, we require

$$\mathbf{area}(R) = \mathbf{area}(\tau(R)).$$

Any function that satisfies these properties will be called an area function. We will now outline a stream of arguments that shows that the above two properties already determine the area function up to a constant multiplicative factor and directly relate it to the angle deficit.

**Triangles are enough:** It can be proved that every simple region can be decomposed as a union of triangles. The additivity property implies that an area function is completely specified if its values on all triangles are determined (compare Figure 26.3). Thus it is now our task to determine the area of a triangle. In hyperbolic geometry the shape of a triangle is completely determined by its three vertex angles. Thus up to a motion (that leaves the area invariant) we can talk of *the* triangle  $\Delta_{\alpha,\beta,\gamma}$  with vertex angles  $\alpha, \beta, \gamma$ . We will now investigate how additivity and motion invariance restrict the function  $\mathbf{area}(\Delta_{\alpha,\beta,\gamma})$ . In particular, we will also admit triangles whose vertices are on the unit circle. The corresponding vertex angle at such a vertex is 0. All accompanying pictures for the following explanations will be in the Poincaré disk model, since there it is possible to directly perceive the vertex angles.

**The infinite triangle  $\Delta_{0,0,0}$ :** We will start by fixing the area function for one specific triangle. Conceptually, there are two extremes that are a good starting point for such a convention: *infinitesimally small triangles* and the *largest triangles possible*. In fact, we want infinitesimally small regions of the hyperbolic plane to behave asymptotically like the Euclidean plane. Accordingly, our definition of length measurement

$$|\mathbf{dist}(p, q)| = \left| -\frac{1}{2} \ln(p, q, X, Y) \right|$$

determines the area of infinitesimally small triangles. We want (if possible) to construct our area function such that a right triangle whose legs have length  $a$  has the area  $a^2/2$  as  $a \rightarrow 0$ . Starting with this assumption is a possible way to go, but leads to rather messy calculations. We will go the other way around and fix the area for a triangle that is as large as possible and at the end show that our choice is consistent with the above requirement for infinitesimal triangles. The largest triangle possible is a triangle all of whose vertices are “at infinity” (i.e., they are on the unit circle). For such a triangle all vertex angles are 0. We define

$$\mathbf{area}(\Delta_{0,0,0}) := \pi.$$

We will see that this choice uniquely determines all other triangle areas. It is clear that all triangles with all vertices on the unit circle are hyperbolically congruent, since a projective  $\mathbb{RP}^1$  transformation on the unit circle can map every triple of points to every other one. The left picture in Figure 26.4 illustrates such an infinite triangle.

**Triangles of shape  $\Delta_{\alpha,0,0}$ :** We next determine the area for triangles with all but one vertex at infinity (see Figure 26.4 in the middle). The function  $f(\alpha) := \mathbf{area}(\Delta_{\alpha,0,0})$  should be continuous in  $\alpha$ . Furthermore it should satisfy  $f(0) = \pi$  (the area of the infinite triangle) and  $f(\pi) = 0$  (a degenerate triangle has zero area). Figure 26.4 (right) reveals another important identity of the function  $f$ . It must satisfy

$$f(\alpha) + f(\beta) = f(\alpha + \beta) + \pi.$$

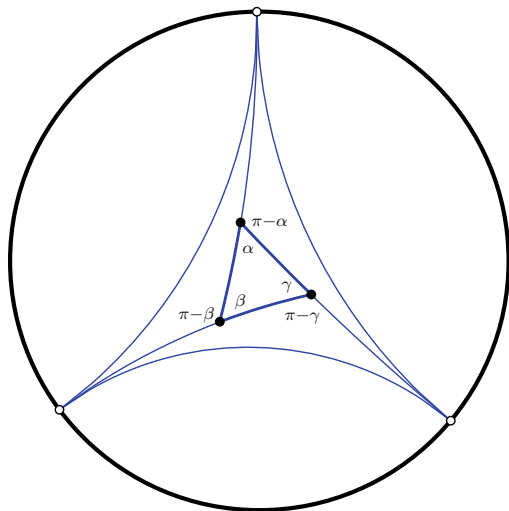
This can easily be seen by considering the quadrangle  $Q$  formed by the four vertices in this pictures decomposed in two different ways. Taking the derivative of both sides with respect to  $\alpha$ , we get

$$f'(\alpha) = f'(\alpha + \beta),$$

which implies that  $f'(\alpha)$  is constant and hence  $f(\alpha)$  must be a linear function. Together with  $f(0) = \pi$  and  $f(\pi) = 0$  this implies

$$\mathbf{area}(\Delta_{\alpha,0,0}) = \pi - \alpha.$$

**Triangles of general shape  $\Delta_{\alpha,\beta,\gamma}$ :** Now it is easy to determine the area of a general triangle  $\Delta_{\alpha,\beta,\gamma}$ . For this consider Figure 26.5. It shows how the infinite triangle can be decomposed into a triangle of shape  $\Delta_{\alpha,\beta,\gamma}$  and three other triangles of shapes  $\Delta_{\pi-\alpha,0,0}$ ,  $\Delta_{\pi-\beta,0,0}$ , and  $\Delta_{\pi-\gamma,0,0}$ . The additivity property implies that



**Fig. 26.5** Calculating the area of a general hyperbolic triangle.

$$\begin{aligned} \mathbf{area}(\Delta_{0,0,0}) = & \mathbf{area}(\Delta_{\alpha,\beta,\gamma}) \\ & + \mathbf{area}(\Delta_{\pi-\alpha,0,0}) \\ & + \mathbf{area}(\Delta_{\pi-\beta,0,0}) \\ & + \mathbf{area}(\Delta_{\pi-\gamma,0,0}) \end{aligned}$$

Inserting the area formulas we already established, we get

$$\pi = \mathbf{area}(\Delta_{\alpha,\beta,\gamma}) + \alpha + \beta + \gamma$$

or in other words,

$$\mathbf{area}(\Delta_{\alpha,\beta,\gamma}) = \pi - \alpha - \beta - \gamma.$$

The area of a triangle equals precisely the angle defect!

**Infinitesimally small triangles:** We want to conclude this section with the observation that our initial choice  $\mathbf{area}(\Delta_{0,0,0}) := \pi$  is consistent with the requirement that infinitesimally small triangles have asymptotically the same area as in Euclidean geometry. It is sufficient to show this for one particular (small) triangle. Within our standard embedding with the hyperbolic plane located inside the unit disk we consider the right triangle with vertices located at  $(0, 0)$ ,  $(a, 0)$ ,  $(0, a)$ . It is of shape  $\Delta_{\pi/2,\alpha,\alpha}$  with  $\alpha \rightarrow \pi/4$  as  $a \rightarrow 0$ . We want to express  $\alpha$  in terms of  $a$  and compare the desired asymptotic Euclidean area formula to the angle defect formula  $\mathbf{area}(\Delta_{\pi/2,\alpha,\alpha}) = \pi - \pi/2 - 2\alpha$ . We only sketch these calculations, since they are fairly standard.

Let us first derive the asymptotic Euclidean area formula. For this we must calculate the hyperbolic side length of the two legs of the triangle. By symmetry it is sufficient to calculate the distance between  $o = (0, 0)$  and

$p = (a, 0)$ . For the line supporting these two points we get  $X = (-1, 0)$  and  $Y = (1, 0)$ . Thus we get with choice of our constant  $c_{\text{dist}} = -1/2$  (which now really enters as a significant constant)

$$\mathbf{dist}_{\text{hyp}}(o, p) = -1/2 \ln \left( \frac{1-a}{1+a} \right).$$

We already studied this function in Section 20.3. Figure 20.2 shows a plot of this function that demonstrates that for small  $a$  this function is asymptotically close to the identity (this is how  $c_{\text{dist}}$  was chosen). Thus in the asymptotic situation we can replace the hyperbolic distance simply by  $a$  itself. So we get a right triangle whose legs (asymptotically) are of length  $a$ . The corresponding Euclidean area function for such a triangle is  $a^2/2$ . We now must show that asymptotically the angle defect approximates closely this value. The value of  $\alpha$  can be calculated from  $a$  by the formula

$$\cos(\alpha) = \frac{\Theta_{lm}}{\sqrt{\Theta_{ll}\Theta_{mm}}},$$

with  $l = (0, 1, 0)^T$  and  $m = (-1, -1, a)$  being the homogeneous coordinates of a leg and the hypotenuse. Inserting in the above formula gives

$$\alpha = \arccos \left( \frac{1}{\sqrt{2-a^2}} \right)$$

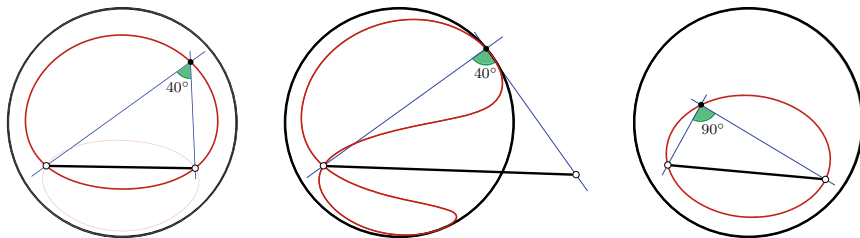
Inserting this in the angle deficit formula yields

$$\mathbf{area}(\Delta_{\pi/2, \alpha, \alpha}) = \frac{\pi}{2} - 2 \arccos \left( \frac{1}{\sqrt{2-a^2}} \right).$$

By standard techniques from a first-year calculus course it can be shown that for as  $a \rightarrow 0$  we have

$$\frac{a^2}{2} \approx \frac{\pi}{2} - 2 \arccos \left( \frac{1}{\sqrt{2-a^2}} \right).$$

It is an amazing effect (and one of the beautiful coincidences that make mathematics a fascinating subject) that the choice  $\mathbf{area}(\Delta_{0,0,0}) := \pi$  of the area of the largest possible triangle leads to the simplest possible formulas for the area in general and even leads to formulas for very small triangles that approximate the Euclidean area function.



**Fig. 26.6** The peripheral angle configuration (left and middle) and Thales' theorem in hyperbolic geometry (right).

## 26.3 Thales and Pythagoras

As we did in Chapter 19 for Euclidean geometry, we now want to have a brief look at elementary geometry in the hyperbolic plane from a projective perspective. We already supplied many results in this direction in Chapter 22 on the level of general Cayley-Klein geometries. There we proved theorems on altitudes, angle bisectors, medians and an analogue of the law of sines for triangles. We here will look at some of the peculiarities that are more or less special to hyperbolic geometry.<sup>2</sup> We will focus on two well-known theorems from Euclidean geometry and look for their counterparts in hyperbolic geometry: The theorems of Thales and Pythagoras.

Thales' theorem can be considered a special case of the peripheral angle theorem for circles. The peripheral angle theorem states that (modulo  $\pi$ ) every point of a circle sees a fixed chord under a fixed angle. Thales' theorem is the special case in which the chord becomes a diameter and the angle is a right angle. Let us first see what still remains of the peripheral angle theorem in hyperbolic geometry. The left and middle pictures of Figure 26.6 show (in the Beltrami-Klein model) a segment (black) and the locus of all points that see this segment under a hyperbolic angle of  $40^\circ$ . The left picture illustrates that the locus is topologically still of the type of a circle. However, metrically it is not a circle at all, not even a conic. It turns out to be a branch of an algebraic curve of degree four. The light curve shown in this picture is the locus of points that see the segment under an angle of  $-40^\circ$ . This locus forms another branch of the same curve. The picture in the middle shows the analogous situation of a segment that intersects the fundamental conic. Algebraically, for this situation it is still feasible to speak of the locus of all points that see the segment under an angle of  $40^\circ$ . In this case the two branches merge and form a more complicated curve. The picture on the right

<sup>2</sup> To be more precise, it would perhaps be better to say “are more or less special to Cayley-Klein geometries with nondegenerate fundamental conic,” since on the algebraic side hyperbolic and elliptic geometry are essentially equivalent and each statement in one geometry has its counterpart in the other.



now shows the locus of all points that see the segment under a right angle (this is the case of a hyperbolic analogue of Thales' theorem). In this case the two branches of the curve overlap and form *one* curve of degree 2: a conic. Thus Thales' theorem survives in the following restricted form, and with a projective way of thinking it is almost trivial to prove it.

**Theorem 26.1 (Hyperbolic Thales).** *The points that see a given segment in hyperbolic geometry under a right angle all lie (in the Beltrami-Klein model) on a conic.*

*Proof.* Let  $A$  and  $B$  be the two endpoints of the segment. Let  $l$  be a line through  $A$ . We calculate the (unique) line through  $B$  perpendicular to  $l$ . This is  $g = B \times l^*$  with  $l^*$  being the polar of  $l$  with respect to the fundamental conic. If we consider the two line bundles  $\mathcal{L}_A$  and  $\mathcal{L}_B$  through  $A$  and  $B$  and equip each of them with a projective basis, then the transition from  $l$  to  $g$  is just a projective transformation. Theorem 10.2 implies that the locus of all intersections of  $l$  and  $g$  is a conic.  $\square$

Let us stay with right triangles and come to a hyperbolic version of perhaps the most famous theorem in geometry: the *Pythagorean theorem*. The essence of the Pythagorean theorem is that it allows one to calculate the side length of the hypotenuse of a right triangle from the side lengths of the two legs. In hyperbolic geometry the Pythagorean theorem takes the following surprising form:

**Theorem 26.2 (Hyperbolic Pythagoras).** *Let  $a, b, c$  be the three hyperbolic side lengths of a hyperbolic right triangle with  $c$  being the hypotenuse and  $a, b$  being the legs. Then*

$$\cosh(a) \cdot \cosh(b) = \cosh(c).$$

*Proof.* We will again attempt to prove this theorem in essentially projective terms. By this we will also see more projective reformulations of the same statement. Recall from Section 22.5 (see the summary table there at the end) that with respect to hyperbolic distance measurement we have

$$\cosh^2(\mathbf{dist}(p, q)) = \frac{\Omega_{pq}^2}{\Omega_{pp}\Omega_{qq}}.$$

We assume that the three vertices of the triangle are  $p, q, r$  such that  $c = \mathbf{dist}(p, q)$  and the right angle is at point  $r$ . Since for real  $x$  the function  $\cosh(x)$  is always positive, the statement of the theorem is equivalent to

$$\cosh^2(a) \cdot \cosh^2(b) = \cosh^2(c).$$

Applying the above identity, this translates to

$$\frac{\Omega_{qr}^2}{\Omega_{pp}\Omega_{rr}} \cdot \frac{\Omega_{rq}^2}{\Omega_{rr}\Omega_{qq}} = \frac{\Omega_{pq}^2}{\Omega_{pp}\Omega_{qq}}.$$

Canceling terms that occur on both sides and clearing denominators gives

$$\Omega_{pr}^2 \cdot \Omega_{rq}^2 = \Omega_{pq}^2 \cdot \Omega_{rr}^2.$$

Thus we are done if we prove

$$\Omega_{pr} \cdot \Omega_{rq} = \Omega_{pq} \cdot \Omega_{rr},$$

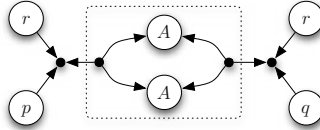
or equivalently

$$(p^T Ar) \cdot (r^T Aq) = (p^T Aq) \cdot (r^T Ar), \quad (26.1)$$

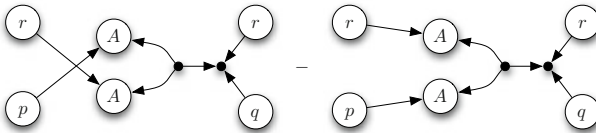
for a triangle  $p, q, r$  with right angle at  $r$ . The condition for the orthogonality of the lines **join**( $r, p$ ) and **join**( $r, q$ ) can according to Theorem 22.1 be expressed by

$$(r \times p)^T B (r \times q) = 0. \quad (26.2)$$

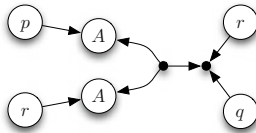
Proving the hyperbolic Pythagorean theorem thus results in proving the equivalence of equations (26.1) and (26.2). Perhaps the simplest and most direct proof of this equivalence goes via the use of tensor diagrams (see Chapter 13 and Chapter 14). So we will make one small exception to our rule not to use diagram techniques. The matrix  $B$  may be chosen to be the adjoint  $A^\Delta$  of the primal fundamental conic  $A$ . Thus the diagram of  $(r \times p)^T B (r \times q)$  is given by



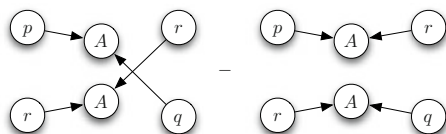
Here the dotted part plays the role of  $B = A^\Delta$ . Applying the  $\varepsilon$ - $\delta$ -rule reduces this diagram to



which is nothing but twice the diagram



Applying the  $\varepsilon$ - $\delta$ -rule again leaves us with twice



Thus we have

$$(r \times p)^T A^\Delta (r \times q) = 2 \left( (p^T Aq) \cdot (r^T Ar) - (p^T Ar) \cdot (r^T Aq) \right),$$

and the theorem follows.  $\square$

The reader should again go over the proof and see how every single argument involved in the proof is just a straightforward step in a projective setup of the statement.

*Remark 26.1.* The reader might wonder why in hyperbolic geometry the Pythagorean theorem takes a multiplicative form instead of the usual additive structure  $a^2 + b^2 = c^2$ . The reason for this is that the  $\cosh(\dots)$  functions are *not* the analogues of the Euclidean lengths. This analogue is played by the  $\sinh(\dots)$  functions. Using  $\cosh^2(x) - \sinh^2(x) = 1$ , Theorem 26.2 translates to

$$(1 + \sinh^2(a))(1 + \sinh^2(b)) = (1 + \sinh^2(c)).$$

Simplification yields

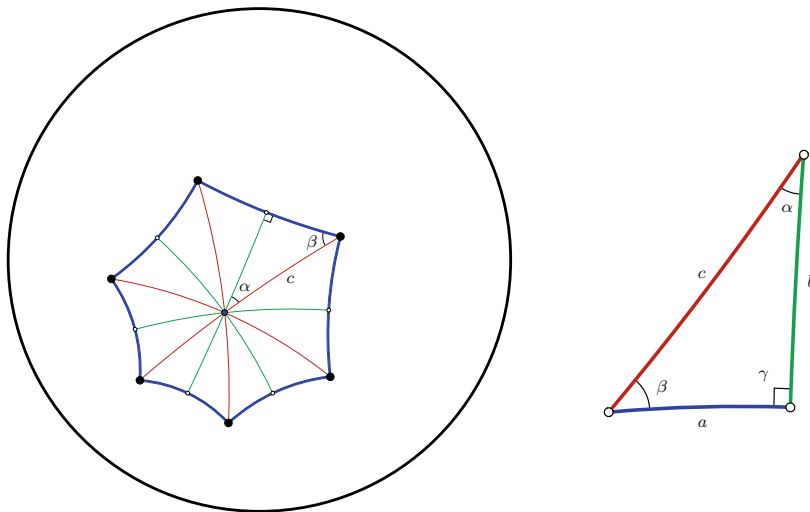
$$\sinh^2(a) + \sinh^2(b) + \sinh^2(a) \sinh^2(b) = \sinh^2(c).$$

In this form the theorem is very close to the form  $a^2 + b^2 = c^2$ . However, in hyperbolic geometry a correction term  $\sinh^2(a) \sinh^2(b)$  is needed.

## 26.4 Constructing Regular $n$ -Gons

A regular hyperbolic  $n$ -gon is a polygon whose  $n$  sides are hyperbolic line segments of equal length  $d$  and with identical angles  $\psi$  at each vertex. In contrast to in the Euclidean case (where regular polygons can be arbitrarily scaled) in hyperbolic geometry the angle  $\psi$  and the number  $n$  already completely determine the side length  $d$ . It is the aim of this section to give a procedure that once  $n$  and  $\psi$  are known allows us to determine  $d$  and the radius (the distance of the vertices to the center of symmetry) of the  $n$ -gon.

This task can be reduced to the following problem. *Letting  $\alpha$  and  $\beta$  be the two leg angles of a right triangle, compute the corresponding hypotenuse.* To see how a solution of this problem leads to a construction of a regular  $n$ -gon with prescribed vertex angle, consider Figure 26.7. This picture shows



**Fig. 26.7** A hexagon with only right angles decomposed into right triangles.

a hexagon (thus  $n = 6$ ) all whose vertex angles are  $\psi = 90^\circ$ . It can be decomposed into a ring of  $12 = 2n$  right triangles. The two leg angles are  $\pi/n = 30^\circ$  for those angles that meet in the center of the hexagon and  $\beta = \psi/2 = 45^\circ$  for the angle that meets at the vertex of the  $n$ -gon. Conversely, a collection of  $2n$  right triangles with leg-corner angles  $\beta = \psi/2$  and  $\alpha = \pi/n$  can be used to assemble an  $n$ -gon with vertex angle  $\psi$ .

So, how can we solve the above problem and construct the length  $c$  of the hypotenuse from  $\alpha$  and  $\beta$ ? This is a relatively simple task if apply some hyperbolic trigonometry. For the labeling we refer to Figure 26.7 (right). From the Pythagorean theorem we know that

$$\cosh(a) \cdot \cosh(b) = \cosh(c).$$

We can rewrite this as

$$\sinh^2(a) + \sinh^2(b) + \sinh^2(a) \sinh^2(b) = \sinh^2(c).$$

Furthermore, the sine theorem tells us that

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}.$$

Since  $\gamma = 90^\circ$ , we have  $\sin(\gamma) = 1$ , and we get

$$\sin(\alpha) \sinh(c) = \sinh(a) \quad \text{and} \quad \sin(\beta) \sinh(c) = \sinh(b).$$

Inserting this in the above equation gives

$$\sin^2(\alpha) \sinh^2(c) + \sin^2(\beta) \sinh^2(c) + \sin^2(\alpha) \sin^2(\beta) \sinh^4(c) = \sinh^2(c).$$

Canceling  $\sinh^2(c)$  yields

$$\sin^2(\alpha) + \sin^2(\beta) + \sin^2(\alpha) \sin^2(\beta) \sinh^2(c) = 1.$$

Resolving for the remaining  $\sinh^2(c)$  gives

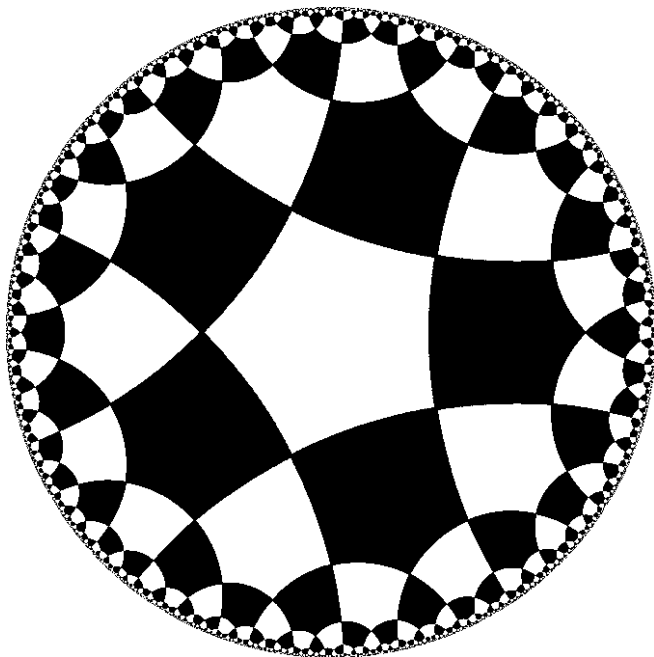
$$\sinh^2(c) = \frac{1 - \sin^2(\alpha) - \sin^2(\beta)}{\sin^2(\alpha) \sin^2(\beta)} = \frac{1}{\tan^2(\alpha) \tan^2(\beta)} - 1.$$

The above equation is a consequence of simple trigonometric identities. Given the last identity, it is an easy task to compute the length  $c$  of the hypotenuse  $c$  from the two angles  $\alpha$  and  $\beta$ .

The fact that the angle sum in a hyperbolic triangle is always less than  $180^\circ$  is reflected by the following fact. Let  $p$  and  $q$  be the endpoints of the hypotenuse  $c$ . They are both inside the unit disk (the fundamental conic) if the cross-ratio  $\Xi = (p, q; X, Y)$  is positive. Since  $\sinh(c) = (1/\Xi + \Xi - 2)/4$ , the term  $\sinh(c)$  must be positive for the hypotenuse to lie entirely in the unit disk. This implies that in this case  $\frac{1}{\tan^2(\alpha) \tan^2(\beta)} - 1$  must be positive. This is the case only if  $|\alpha| + |\beta| < 45^\circ$ .

## 26.5 Symmetry Groups

No exposition on hyperbolic geometry would be complete without mentioning the fascinating symmetry properties regular hyperbolic tessellations can have. The reason for this richness of structure comes from the fact that in the hyperbolic plane one can have polygons all of whose vertex angles are arbitrary small divisors of  $360^\circ$ . Consider, for instance a regular hyperbolic pentagon whose vertex angles are all  $90^\circ$ . One can take four such pentagons and arrange them such that they meet tightly in a common vertex, like the squares in a Euclidean checkerboard. One can infinitely repeat this process and fill the entire hyperbolic plane with an infinite collection of such right-angled pentagons, four of them meeting at every vertex. Figure 26.8 shows such an arrangement where in addition the pentagons are colored alternately black and white. It is possible to create such a tiling with arbitrary hyperbolic right-angled  $n$ -gons. The considerations of our last section imply that such  $n$ -gons exist whenever  $n > 4$ . We then only have to choose  $\alpha = 180^\circ/n$  and  $\beta = 45^\circ$ . Such tilings by regular  $n$ -gons are not only restricted to right angles. As long as the parameters  $n$  and  $\beta$  satisfy  $360^\circ/n + \psi < 90^\circ$  and  $\psi = 2\beta$  is a divisor of  $360^\circ$ , the corresponding polygon will seamlessly tile the hyper-



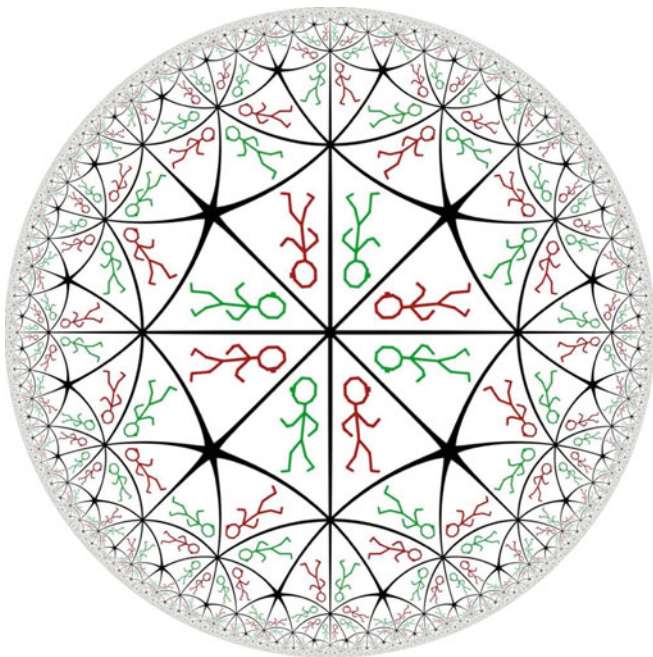
**Fig. 26.8** A pentagonal hyperbolic checkerboard.

bolic plane. In other words, the tiling condition is  $\alpha = 180^\circ/n$ ,  $\beta = 180^\circ/m$  together with the inequality  $1/n + 1/m < 1/2$ .

One should be aware that this is in sharp contrast to the case of Euclidean geometry, where the only possible tilings by regular polygons are those created by regular triangles, squares, and hexagons. This is the case since as tiling condition in the Euclidean plane the equation  $1/n + 1/m = 1/2$  for the angles  $\alpha = 180^\circ/n$ ,  $\beta = 180^\circ/m$  has to be satisfied sharply.

Also other effects that are well known in Euclidean geometry have their counterpart in hyperbolic geometry and lead there to an infinite variety of different objects. One example is kaleidoscopes. A triangular kaleidoscope (*kaleido* = beauty, *scope* = viewer) consists of three mirrors that meet at angles  $\alpha = 180^\circ/a$ ,  $\beta = 180^\circ/b$ ,  $\gamma = 180^\circ/c$  that are divisors of  $180^\circ$  (i.e.,  $a, b, c \in \mathbb{N}$ ). In Euclidean geometry there are only three such arrangements of mirrors. They have vertex angles  $(60^\circ, 60^\circ, 60^\circ)$ ,  $(90^\circ, 60^\circ, 30^\circ)$ , and  $(90^\circ, 45^\circ, 45^\circ)$ , since in Euclidean geometry the angle sum must be exactly  $180^\circ$ . Translated in terms of the integers  $a, b, c$ , this gives the condition

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$



**Fig. 26.9** Dr. Stickler in a hyperbolic mirror cabinet.

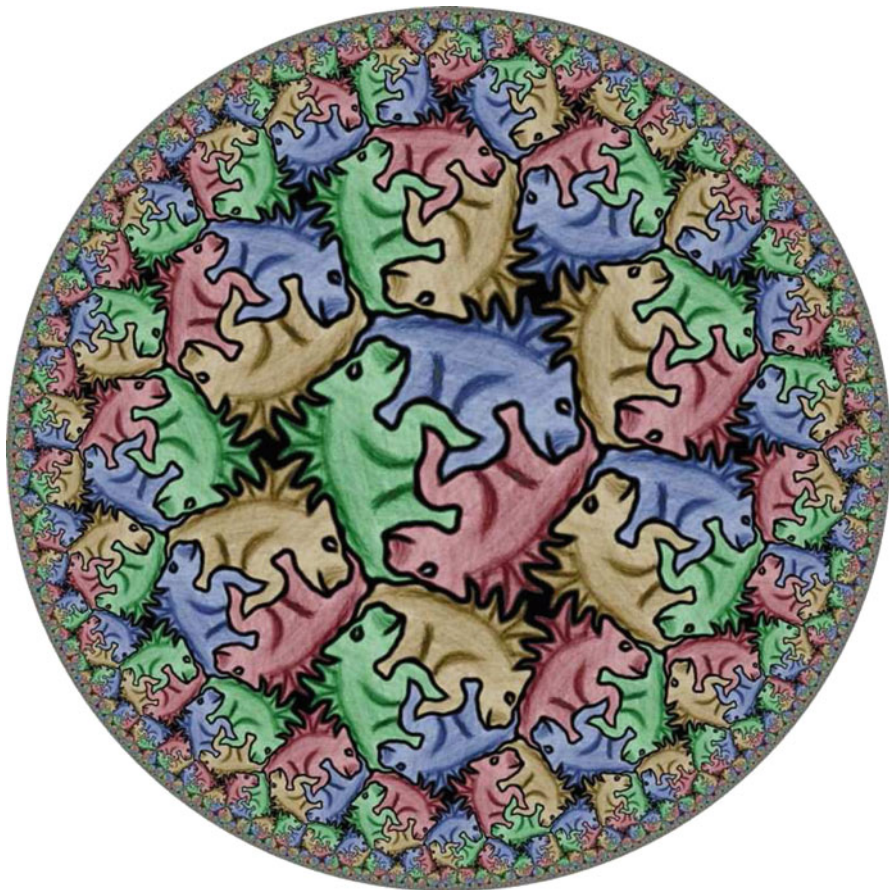
In hyperbolic geometry the angle sum is less than  $180^\circ$ , and consequently there are infinitely many possibilities for such triangles. All integer solutions of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

lead to an appropriate kaleidoscope. Why are such triangles whose vertex angles are divisors of  $180^\circ$  good candidates for a kaleidoscopic cell? For this assume that the sides of the triangle are made of a reflecting material. Around each vertex the iterated reflections of the triangle in the two mirrors that meet at this vertex produce a ring of triangles that closes up tightly, since the vertex angle was chosen to be a divisor of  $180^\circ$ . Thus by iteratively reflecting the triangle in its own edges, the entire hyperbolic plane gets filled seamlessly and free of overlaps with infinitely many copies of the original triangle.

Figure 26.9 demonstrates this effect for a hyperbolic triangle with vertex angles  $(60^\circ, 45^\circ, 45^\circ)$ . We placed Dr. Stickler in the middle of one kaleidoscope cell. One can observe how the entire hyperbolic plane is covered by copies of Dr. Stickler (the copies are alternately red and green to indicate the handedness of the mirror images).

Finally, there are also more complicated symmetric patterns than those coming from reflection groups. For this we may consider a (possibly infinite) group  $G$  consisting of a set of isometries acting on a certain metric space  $X$



**Fig. 26.10** The parade of dragoncats (artwork by the author).

(for instance a Cayley-Klein geometry). Each point  $p \in X$  has an orbit under the group  $\bar{p} = \{g \circ p \mid g \in G\}$ . The group is said to be a *discrete group on  $X$*  if the orbit  $\bar{p}$  of every point  $p \in X$  does not have an accumulation point with respect to the distance measure in  $X$ . This rather abstract concept has nice visual representations. In particular, the reflection groups generated by kaleidoscopic mirror triangles are discrete groups. In practice, this means that each point of the orbit of every point can be surrounded by a small circle that does not contain any other point of the orbit. Discrete groups are by far more general than reflection groups. In Euclidean geometry the discrete groups are classified into two infinite classes (the *rosette groups*, having a single center of rotational symmetry), the seven *frieze groups* (they have a translational symmetry in only one direction), and the 17 *wallpaper groups*



(they have translational symmetry in two directions). Most prominent are the 17 wallpaper groups forming the mathematical basis of most symmetric planar geometric structures, such as the ornamental patterns of Islamic art and many drawings by the famous Dutch artist M.C. Escher. A fundamental region of such a group is a connected subset  $F$  of  $X$  such that the orbits of  $F$  form a disjoint union of  $X$ . In Euclidean geometry the wallpaper groups are exactly those with a finite fundamental region. In a wallpaper ornament each fundamental region could be considered a tile whose symmetric repetition according to the rules of the group forms the entire ornament.

Also here the structure in hyperbolic geometry is by far richer. The analogues of the Euclidean wallpaper groups are those discrete hyperbolic groups having a finite fundamental region, and (no surprise) there are infinitely many of them. (The structure has a fascinating richness.) Figure 26.10 shows such a hyperbolic ornament, a hyperbolic tiling where each of the tiles is animal shaped and all the tiles fit seamlessly together to cover the hyperbolic plane. Recall that in the hyperbolic plane each of these animals indeed has exactly the same size and shape. Notice also the subtle color symmetries of this drawing.<sup>3</sup>

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<sup>3</sup> Figures 26.8, 26.9, 26.10 were generated with the software *morenaments*, which is part of a joint project of Martin von Gagern and the author of this book [43, 42].



## What We Did Not Touch

*This is not the end. It is not even the beginning of the end.  
But it is, perhaps, the end of the beginning*

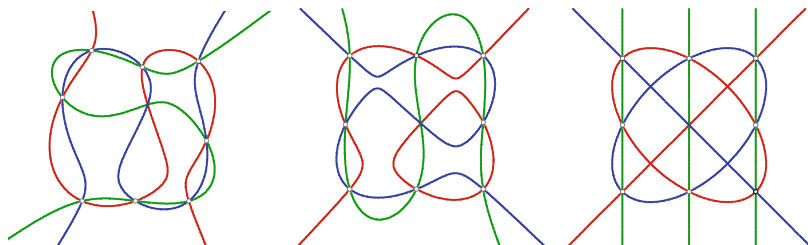
*Winston Churchill*

This book now comes to an end, and it became considerably fatter than originally intended. Still there are many interesting, amazing, deep, esthetic topics that we did not touch at all. This final chapter is intended to give a very brief overview of a loose selection of topics that I think are among the most fascinating ones among them.

### 27.1 Algebraic Projective Geometry

Throughout this book we have dealt mainly with linear objects (lines, planes, flats) and quadratic objects (conics). Many fascinating effects occur when one deals with algebraic curves of *higher degree* in a projective framework. New types of incidence theorems, interesting singularities, and a surprising theory of duality arise (see for instance [19, 40]). Also the interplay of complex and real parts of curves becomes an interesting subject. We will scratch the surface of a few of these topics. An algebraic curve is the solution set of a homogeneous polynomial equation in three variables  $x, y, z$ . For instance, an algebraic cubic (a curve of degree three) is the set of all points  $(x, y, z)^T$  that satisfy an equation

$$p_1x^3 + p_2x^2y + p_3x^2z + p_4xy^2 + p_5xyz + p_6xz^2 + p_7y^3 + p_8y^2z + p_9yz^2 + p_{10}z^3 = 0.$$



**Fig. 27.1** An incidence theorem on cubics.

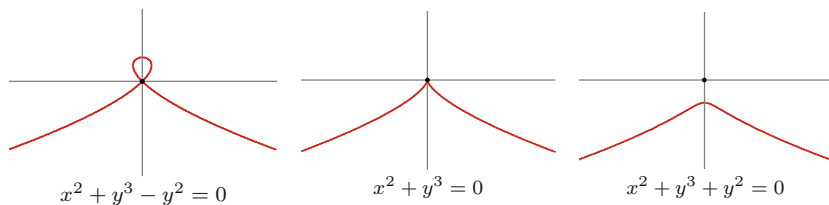
The parameter vector  $(p_1, p_2, \dots, p_{10})$  determines the cubic uniquely. In general, an algebraic curve of degree  $d$  has  $\binom{d}{2}$  parameters. The parameters themselves are homogeneous coordinates of the curve. So the set of all cubics has nine degrees of freedom. This implies that in general, a cubic is uniquely determined by the position of *nine* points (in suitably “general” position) that lie on it. This is analogous to the fact that a conic is determined by five points. In the same way as conics may degenerate into a pair of lines, cubics can also degenerate into curves of lower degree. They may degenerate into either three lines or a pair of a line and a conic.

**Incidence theorems:** As for conics and lines, there are also incidence theorems involving higherorder algebraic curves. Since algebraic curves carry much more information than conics or lines, there exist already incidence theorems with relatively few such curves. Perhaps the smallest one is the following:

**Theorem 27.1.** *Let  $C_1$ ,  $C_2$ , and  $C_3$  be three cubic curves. If  $C_1$  and  $C_2$  have nine distinct points in common and if  $C_3$  passes through eight of these points, then it automatically passes through the last point as well.*

We will not prove this result here. This theorem is known as the *Cayley-Bacharach-Chasles theorem* (for two nice comprehensive overviews see [38, 63]), and a variation of it was given in Section 1.5. Figure 27.1 illustrates a few instances of this theorem. It is particularly interesting to consider degenerate cases of the Cayley Bacharach theorem in which some of the cubics degenerate. There are many of them, and it is a good exercise to enumerate and classify all possible degenerate situations. The rightmost drawing in Figure 27.1 illustrates such a degenerate situation that involves five lines and two conics. The middle picture shows a situation that is close to this degenerate situation. If all three conics degenerate into line triples, we obtain (surprise) Pappos’s theorem (see also Figure 1.17).

The proof of the Cayley Bacharach theorem relies heavily on another beautiful (and powerful) result about algebraic curves: Bézout’s theorem. This theorem gives a precise account of the number of intersections two algebraic



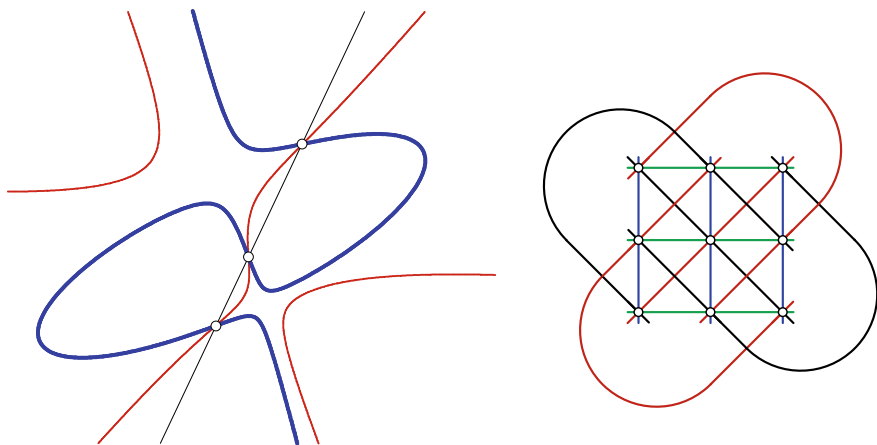
**Fig. 27.2** Singularities of a cubic

curves can have. Here one has to be careful to get the right concept of intersections and *intersection multiplicity*. For instance, the point at which a line touches a conic in tangential position has to be counted as a double intersection, since an  $\varepsilon$ -perturbation of the situation immediately generates two intersections. Also, complex intersections and intersections at infinity have to be counted. So here again it is inevitable to have a framework that is both projective and complex: see [19]. With such an advanced understanding of the concept of intersection, Bézout's theorem can be stated as follows:

**Theorem 27.2.** *Two algebraic curves of degree  $n$  and  $m$  either meet in  $n \cdot m$  points (counted with correct multiplicity) or have a whole component in common.*

We have encountered several special cases of this theorem already. Two lines meet in one point, a line and a conic meet in two points, and two conics meet in four points. Concerning cubics we now see that a general line meets a cubic in three points (this corresponds to the fact that a polynomial of degree three has three solutions), a conic meets a cubic in six points, and two cubics meet in nine points.

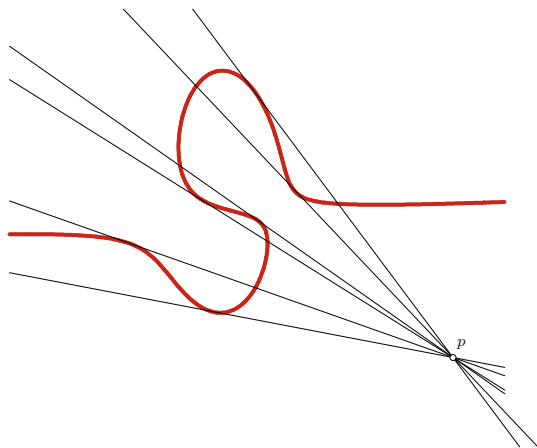
**Special points on curves:** Algebraic curves may have very interesting points with special properties. We have seen only one type of such special points: a conic may intersect itself in a point if the conic degenerates into two lines. Already starting with degree three, other interesting special points occur. For this consider Figure 27.2, in which several interesting examples are shown. All curves shown there are of degree three and in fact belong to a one-parameter continuous family of curves (implicit equations of the curves in inhomogeneous coordinates are given). In the first picture the curve intersects itself. Such a situation in which one part of the curve passes transversally through another part of the curve is called a *simple singularity* or *double point*. For every nonsingular point of the curve it is possible to assign a unique tangent. This is no longer the case for a double point. In a certain sense (which involves the technique of a *blowup* of a singularity) one can assign two well-defined tangents at this point of the curve, one for every smooth branch of the curve passing through it. The second picture shows the limit situation, in which the loop that exists in the first picture is just



**Fig. 27.3** Real inflection points of a (blue) cubic and the structure of all inflection points.

about to vanish. Considering the tangent situation such a curve has a sharp edge. A point traversing the curve would have to change its direction in such a singularity. Such a situation is called a *cusp*. In a sense, cusps are degenerate double points. In the third picture the double point has (seemingly) completely vanished. The red curve seems to be perfectly smooth. Care has to be taken here, since the interplay of complex and real numbers comes into play. First observe that the point  $(x, y) = (0, 0)$  still satisfies the equation of the curve although it is no longer on the red curve. The point  $(0, 0)$  is an isolated point of the curve and in fact has to be counted as a double point, since it is the real (!) intersection of two complex conjugate tangents that may be associated at this point. Going from the left to the right in Figure 27.2, the point  $(0, 0)$  has first two real tangents. In the middle picture these two tangents coincide, and in the last picture they became complex conjugates. Singularity theory is a very wide field, and much more complicated situations than shown above may arise. However, also here a projective viewpoint is always the right starting point.

There is one more type of special point on curves that is not singular but still plays an important role for the theory: the *inflection points*. Roughly speaking, a point of an algebraic curve is an inflection point if the curvature in this point changes its sign. To return to our metaphor of riding a bike along a curved road, an inflection point is at the position at which your handle bar is in momentarily straight position and back and front wheel are aligned. Figure 27.3 shows the three (real) inflection points of a cubic (the blue curve). As we will soon see the interplay of real and complex again offers some surprises. There is an amazingly elegant way to calculate the position of the inflection points. All inflection points lie on the so-called *Hessian* of



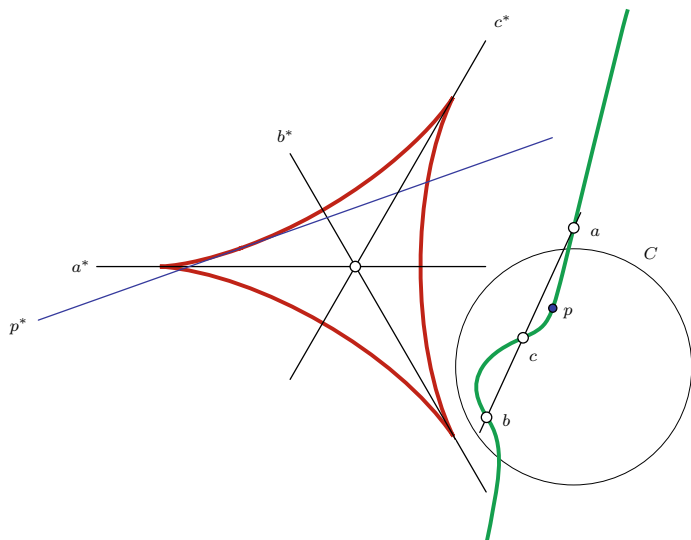
**Fig. 27.4** Tangents to a cubic.

the algebraic curve  $f(x, y, z) = 0$  (named after Otto Hesse). The Hessian of a curve that is the zero set of a polynomial  $f$  is the algebraic curve given by the following equation:

$$\det \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z \partial z} \end{pmatrix} = 0.$$

The Hessian is again an algebraic curve. If  $f$  has degree  $d$ , then each of the partial derivatives occurring in the above determinant has degree  $d - 2$ . Thus the Hessian has degree  $3d - 6$ . The inflection points are the intersections of the original curve with its Hessian. By Bézout's theorem there are (with proper counting of multiplicities, complex and infinite points)  $(3d - 6)d$  inflection points. In the case of a cubic there are exactly nine of them. In the left picture of Figure 27.3 we see three of them. The other six are complex. The Hessian is drawn in red.

One might wonder whether there is a better example in which one can see all inflection points. In fact, it turns out that this is impossible. For a suitably generic cubic always exactly three inflection points are real. Again an incidence theorem occurs that shows that these inflection points are even collinear. The right picture of Figure 27.3 indicates the combinatorial structure of the collinearities of the nine inflection points. It turns out that they form an amazingly symmetrical configuration such that there are even twelve collinearities among them. On every line there are three points, and through every point pass four lines. If in the picture one deletes all lines of one color, one is left with Pappos's configuration. This very special point configuration cannot be represented such that all points are real.



**Fig. 27.5** A cubic curve (green) and its polar (red).

**Duality:** As in every field of projective geometry there is also a duality theory for the realm of algebraic curves. We briefly recall the situation for conics. The polar of a conic is again a conic. This could be derived in the following way. We consider all tangents to a conic  $f(x, y, z) = 0$ . The collection of them describes the polar of the conic. The algebraic equation  $f^*(a, b, c) = 0$  of a polar of a conic is a polynomial that has the homogeneous coordinates of these tangents as zeros. From each point not on the conic there are exactly two tangents (perhaps complex). Hence, the polar of a conic is again a conic. The situation changes drastically if one increases the degree of the algebraic curve  $f(x, y, z) = 0$ . From a generic point there are in general  $d(d-1)$  different tangents to an algebraic curve  $f(x, y, z) = 0$  of degree  $d$  (see Figure 27.4 for an example of tangents to a cubic). This implies that the equation  $f^*(a, b, c) = 0$  for the tangents is a curve of degree  $d(d-1)$ . Thus the polar of a general cubic is a curve of degree 6. Stop! There is a problem. From usual polarity structures we know that we can go back and forth and we would expect  $(f^*)^* = f$ . The above formula, however, indicates that the polar  $f^*$  of a cubic  $f$  is of degree 6 and the polar  $(f^*)^*$  of this curve is of degree 30. In the early days of projective geometry this issue was for a long time known as the *polarity paradox*. It was nicely resolved by Julius Plücker (see [101]). It led to the *Plücker formulas* for the degrees of polar curves. An excellent account of the historical background can be found in [48].

The point is that the presence of singularities in the primal curve results in an overcounting of the degree of the polar curve (we here do not explain why). The presence of a double point leads to an overestimation of 2, and the presence of a cusp leads to an overestimation of 3. Thus the correct formula



for the degree of a polar of a curve  $f$  of degree  $d$  is given by

$$\deg(f^*) = d(d-1) - 2 \cdot \#(\text{double points of } f) - 3 \cdot \#(\text{cusps of } f).$$

How does this help in the case of a cubic? It turns out that the presence of an inflection point in  $f$  causes the presence of a cusp in  $f^*$ . Thus a (non singular) cubic describe, by  $f$  has degree 3 and possesses nine inflection points. The curve polar  $f^*$  has indeed degree 6 and possesses nine cusps (six of them are complex). The degree of the polar of  $f^*$  then amounts to

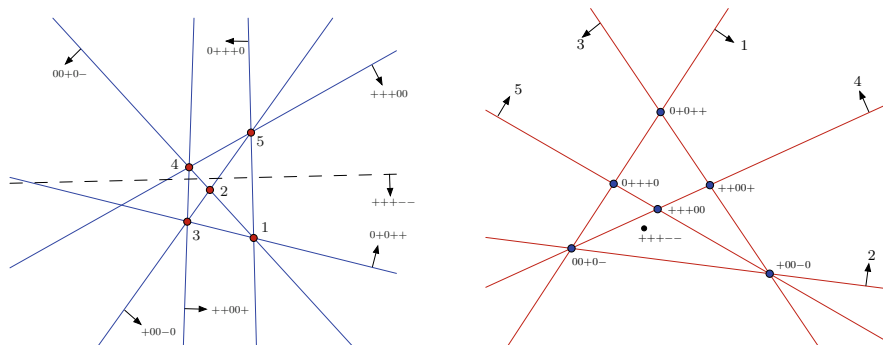
$$6 \cdot 5 - 2 \cdot 0 - 3 \cdot 9 = 30 - 27 = 3.$$

The polar of a polar of a cubic has again degree three and is the original cubic.

Figure 27.5 illustrates this. There a cubic (green) and its polar (red) are shown. The polar is also drawn as a point curve. The polar is of degree 6 and possesses three visual cusps. The other six are complex. The three tangents in these visual cusps indeed meet in a point, since the corresponding inflection points are collinear. The two curves shown in the picture are related in the following way. The circle shown in the picture is used as a conic with respect to which a concrete polarity is defined. The polars of the tangents to one curve are the points of the other curve and vice versa.

## 27.2 Projective Geometry and Discrete Mathematics

After this little excursion to the mathematics of continuous structures such as algebraic curves and their singularities let us go to the other extreme and study the underlying *combinatorial* properties of linear configurations. Throughout this book we have dealt with geometric configurations mostly on a very concrete level. Points and lines were represented by certain coordinates; the incidence relation of points and lines could be tested by algebraic calculations. The hypotheses and the conclusions of theorems, however, were often of a more combinatorial nature: “Let these and these points be incident with these and these lines ... then these and these points are collinear as well.” In fact, there are two powerful purely combinatorial theories that deal with projective configurations on a mainly incidence-geometric level: the theories of *matroids* [92] and of *oriented matroids* [7, 13, 116]. These theories encapsulate the combinatorial essence of projective point and line configurations. While matroid theory deals only with incidences, the theory of oriented matroids also carries relative position information (such as how a line spanned by two points subdivides some other points). We will give here a brief introduction to the (richer) theory of oriented matroids. We will only briefly touch matroid theory. We will also restrict ourselves to the planar case only.



**Fig. 27.6** An arrangement of points and some of its covectors (left) and the dual situation, an arrangement of lines and its covectors (right). In the primal case covectors correspond to point partitions by hyperplanes; in the dual case they correspond to signatures of regions.

The theory of oriented matroids has several facets, and in fact it was “invented” independently at several places by researchers working in projective geometry, linear programming [2], graph theory, topology [41], structural chemistry, computational geometry [71], and polytope theory [80]. In all these different fields certain combinatorial axiom systems were found to be at the core of the research problems. Later on, it turned out that all these different axiom systems were essentially cryptomorphic (this means that the terms of each one of them could be translated into the terms of the others), which made oriented matroids a useful link between quite different branches of mathematics. We will here briefly sketch different approaches and give hints to why they are equivalent.

A common feature of many of the different approaches to oriented matroid theory is that they are most easily expressed when one considers *oriented projective geometry* (see [124]) instead of projective geometry. In projective geometry we express points by homogeneous coordinates where nonzero scalar multiples are considered to be equivalent. In contrast to this, in oriented projective geometry one considers only *positive scalar multiples* to be equivalent. So in a sense we study points on the sphere  $S^2 \subset \mathbb{R}^3$ , and in contrast to in our usual setup, this time we do *not* identify antipodal points. It is relatively easy to get from oriented projective geometry to projective geometry by forming suitable 2-element equivalence classes. If we want to think projectively, we may consider each point in the projective plane to be equipped with a “+” or a “−” sign encoding its orientation. We now consider a sequence of vectors  $V = (v_1, \dots, v_n) \in \mathbb{R}^{3 \times n}$  (they may represent oriented points in the projective plane). From these vectors we will now extract combinatorial information in various ways.

**Covectors of  $V$ :** We first study the so-called covectors of  $V$ . They encode all possible ways in which an (oriented) line can separate the set of points

associated to the vectors. Equivalently, these are the partitions of the vectors in  $V$  by linear hyperplanes. Formally, the set of covectors  $\mathcal{L}(V)$  may be defined as follows.

$$\mathcal{L}(V) := \{(\text{sign}(v_1^T h), \dots, \text{sign}(v_n^T h)) \mid h \in \mathbb{R}^3\}.$$

The vector  $h$  plays the role of the normal vector of the dividing hyperplane. As only exception we also admit the vector  $h = (0, 0, 0)^T$ , which leads to the all-zero covector. Figure 27.6 shows an example of a configuration of five points  $p_1, \dots, p_5$ . The corresponding vectors in the standard embedding have coordinates  $v_1 = (x_1, y_2, 1), \dots, v_5 = (x_5, y_5, 1)$ . The picture on the left indicates (in blue) all lines that are spanned by the points in the configuration. Each one is equipped with an orientation. This leads to a sign vector that indicates the relative position of the points w.r.t this line. For instance, the oriented line through the points 1 and 2 leads to the the signvector  $(00+0-)$ . This indicates that also point 4 is on this line, that 3 is on the positive side, and 5 is on the negative side of this line (an opposite orientation would lead to a negated signvector). In the picture, besides the lines spanned by points, also one more general line is given (in black and dashed). This line passes through none of the points and has signvector  $(+++-)$ . Notice that there are many more such covectors of this point configuration (altogether 58). They encode characteristic relative position and incidence information of the configuration. For instance, the fact that  $(00+0-)$  contains zeros at positions 1, 2, and 4 indicates that these three points are collinear. The covector  $(++00+)$  tells us that the points 3 and 4 span a segment of the convex hull.

There is also a nice dual interpretation of the covectors. Consider Figure 27.6 on the right. It shows the dual situation. Now every vector  $v_i$  is interpreted as homogeneous coordinates of an (oriented) line. The labeling in the picture indicates which line corresponds to which point. Now each covector corresponds to the signature of a point w.r.t the lines in this diagram of oriented lines. The (blue) lines spanned by points in the primal picture correspond to the (blue) points that are intersections of the lines in the dual picture. For instance, in the dual picture the intersection of 1 and 2 generates the covector  $(00+0-)$ , indicating that also line 4 passes through it and that it lies on the positive side of 3 and the negative side of 5. The black point in the drawing corresponds to the point dual to the black dotted line in the left picture. Again the sign vector indicates the relative position with respect to the lines. In the dual picture the covectors are in a sense easier to visualize. Within a region not separated by lines the sign vector does not change. Thus the sign vectors encode the signatures of every full-dimensional cell, every segment or ray not separated by a point, and the intersection points as well. From the signatures of the covectors it is possible to completely reconstruct the topological structure of the line arrangement (this is like a little puzzle).

The covectors satisfy several interesting and characteristic combinatorial properties. They are best explained if we first introduce a few notions. We

denote by  $E = \{1, \dots, n\}$  the index set of the point configuration and set  $\mathcal{L} = \mathcal{L}(V)$ . Furthermore, we set for  $C, D \in \mathcal{L}$   $S(C, D) := \{e \in E \mid C_e = -D_e \neq 0\}$  (the so-called separating set) and define a composition operator  $C \circ D$  by

$$(C \circ D)_e := \begin{cases} C_e & \text{if } C_e \neq 0, \\ D_e & \text{otherwise.} \end{cases}$$

For instance, if  $C := (+, +, -, 0, -, +, 0, 0)$  and  $D := (0, 0, -, +, +, -, 0, -)$ , then we have

$$C \circ D = (+, +, -, +, -, +, 0, -), \quad S(C, D) = \{5, 6\}.$$

The following properties are satisfied by all sets of covectors coming from a vector configuration. (It is not important now what they exactly mean or why they are true. We will only refer to the mere existence of these properties.)

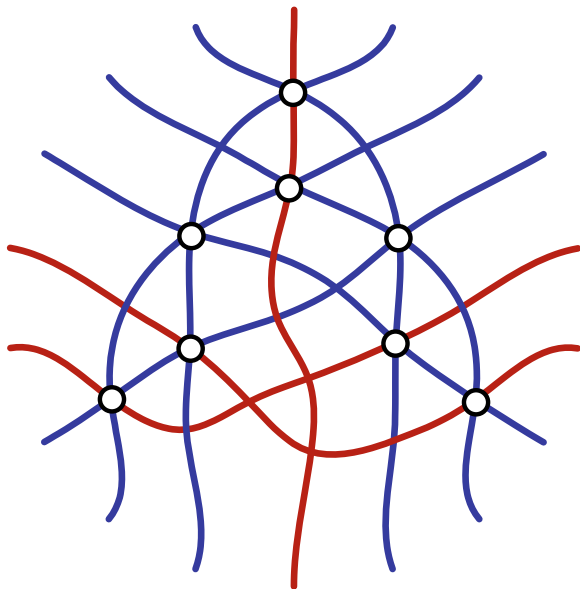
- (CV0)  $\mathbf{0} \in \mathcal{L}$ ,
- (CV1)  $C \in \mathcal{L} \implies -C \in \mathcal{L}$ ,
- (CV2)  $C, D \in \mathcal{L} \implies C \circ D \in \mathcal{L}$ ,
- (CV3)  $C, D \in \mathcal{L}, e \in S(C, D) \implies$   
           there is a  $Z \in \mathcal{L}$  with  $Z_e = 0$  and with  $Z_f = (C \circ D)_f$   
           for  $f \in E \setminus S(C, D)$ .

The first property states that the zero vector is a covector. The second states that with each covector also its negative is a covector. The third property describes a kind of  $\varepsilon$ -perturbation. If out start at a covector  $C$  and perturb in the direction of a covector  $D$ , the result is the covector  $C \circ D$ . The last property encodes a kind of elimination property.

If a collection  $\mathcal{L}$  of sign vectors on  $E$  satisfies the properties (CV0–CV3), then the pair  $(E, \mathcal{L})$  is called an *oriented matroid*. One thing is important to be mentioned. Not every oriented matroid comes from a vector configuration. Oriented matroids  $(E, \mathcal{L}_V)$  that come from a vector configuration  $V$  are called *realizable*.

At first sight this definition of an oriented matroid may seem a bit arbitrary and very abstract. However, in a very precise sense it encodes the combinatorial essence of an arrangement of points or an arrangement of lines, as we will see now.

**Arrangements of pseudolines:** We have mentioned that from the informations carried by the covectors of  $V$  one can reconstruct the combinatorics of the associated arrangements of lines. Without giving technical details, this procedure goes roughly as follows. One starts with a covector  $C \neq \mathbf{0}$  that has a maximal number of zeros. Such a covector corresponds to some intersection  $p_C$  of some of the lines involved. Next one collects all covectors that differ from  $C$  in just one sign that was a zero in  $C$ . This must be the segments incident with the point  $p_C$ . The cyclic order of these segments can be reconstructed by identifying the covectors that form the two-dimensional



**Fig. 27.7** A smallest nonstretchable arrangement of pseudolines.

cells between these segments. Also, the other endpoints of the segments can easily be identified. Proceeding inductively one obtains a rough sketch of the cell complex cut out by the original lines. The only problem is that the lines are now decomposed into a chain of segments, which may not necessarily be aligned. Finding a picture where the lines are really straight is another problem: the so-called *realizability problem* (or *stretchability problem*). One could also phrase this in rather intuitive terms. We will do this now.

A *pseudoline* in  $\mathbb{RP}^2$  is a smooth curve that is topologically equivalent to a line in  $\mathbb{RP}^2$ . This means that it is obtained by a smooth deformation of a line in  $\mathbb{RP}^2$ . In particular, a pseudoline is free of self-intersections. An *arrangement of pseudolines* is a collection of many pseudolines such that any two of them cross exactly once. Thus an arrangement of pseudolines behaves just like an arrangement of lines only that the pseudolines do not have to be straight (compare [50, 85, 117]). Figure 27.7 shows such an arrangement of pseudolines. The lines have to be considered as reaching out to infinity to both ends. By assigning an orientation to each of the pseudolines, each of the regions, segments, and points in the arrangement gets a signature. Although the lines in this example are not straight, we get again a collection of covectors of an oriented matroid. In fact, it turns out that both concepts are essentially equivalent:

*From each oriented matroid an (up to smooth deformation) unique arrangement of oriented pseudolines can be reconstructed. Each arrange-*

*ment of oriented pseudolines generates the covectors of an oriented matroid.*

The arrangement in Figure 27.7 is a nonstretchable one. The reason for this is Pappos's theorem. This can be seen as follows. If one compares this arrangement of pseudolines with the lines of Pappos's theorem as shown in Figure 6.3, one recognizes in essence the same structure with one significant difference. The coincidence of the three red lines has been perturbed and replaced by a triangle. Still, each of the red lines meets the others in exactly one point—but they do not all meet in a single point. Assume that all pseudolines were realized by straight lines, then Pappos's theorem would imply that also the three red lines had to be concurrent—a contradiction.

In fact, it turns out that this example with nine lines is the smallest nonstretchable arrangement of pseudolines. All arrangements with eight or fewer pseudolines are indeed stretchable.

**Chirotopes:** Here comes another approach to oriented matroids again based on vectors and on the algebraic properties of determinants.

Let again  $V = (v_1, \dots, v_n) \in \mathbb{R}^{3 \cdot n}$  be a collection of vectors (considered as oriented homogeneous coordinates of a point configuration with  $n$  points). Let  $E_n = \{1, \dots, n\}$  be the index set of the points. We define a map  $\chi_V : E^3 \rightarrow \{-1, 0, +1\}$  according to

$$\chi_V(i, j, k) = \text{sign}(\det(v_i, v_j, v_k)).$$

Thus to each triple  $(i, j, k)$  we assign the orientation of the basis given by the vectors  $v_i, v_j, v_k$  if they are linearly independent; otherwise, we assign 0. Such a map is called a *chirotope* of  $V$ . It is sufficient to provide the values for strictly ordered sequences of indices. The rest is determined by the alternating determinant rules. For instance, the chirotope for the configuration in Figure 27.6 is determined by the following list of signs:

$$\begin{aligned} \chi(1, 2, 3) = +, \chi(1, 2, 4) = 0, \chi(1, 2, 5) = -, \chi(1, 3, 4) = -, \chi(1, 3, 5) = -, \\ \chi(1, 4, 5) = -, \chi(2, 3, 4) = -, \chi(2, 3, 5) = 0, \chi(2, 4, 5) = -, \chi(3, 4, 5) = -. \end{aligned}$$

From the chirotope  $\chi_V$  the set of covectors  $\mathcal{L}_V$  can be reconstructed. Conversely, if one fixes the sign of only one basis in  $\chi_V$ , the signs of the remaining bases are determined by  $\mathcal{L}_V$ .

What are the conditions that a given sign list  $\chi$  corresponds to the covectors of a general oriented matroid? It turns out that this question is closely related to the Grassmann-Plücker relations. In general, a map  $\chi : E^3 \rightarrow \{-1, 0, +1\}$  is called a chirotope if it satisfies the following two properties.

(Chi0) The map is alternating.

(Chi1) It is in “no obvious contradiction” to the Grassmann-Plücker relations.

The first statement means that interchanging two indices reverses the sign. The second slightly loosely formulated statement means the following. Consider a specific Grassmann-Plücker relation; as an example we take

$$[1, 2, 3][1, 4, 5] - [1, 2, 4][1, 3, 5] + [1, 2, 5][1, 3, 4] = 0.$$

The chirotope  $\chi$  now proposes a sign for each of the brackets. By this we can deduce the signs of the summands. There would be an obvious contradiction to this equation if one summand turned out to be positive and no other summand compensated for this by being negative. This is not allowed to happen for any Grassmann-Plücker relation.

It turns out that whenever the map  $\chi$  is a chirotope, then it is consistent with an oriented matroid and vice versa. Thus chirotopes, arrangements of pseudolines, and oriented matroids are essentially the same objects.

**Realizability:** There are many fascinating theorems on oriented matroids and still many interesting research areas. We want to briefly mention one of the most fascinating concepts, the *realizability problem* (see [7, 16, 50]). It is perhaps most easily stated in terms of chirotopes. Given a chirotope  $\chi$ , when does it come from a vector configuration  $\chi = \chi_V$ ? In a sense, the axiomatization of chirotopes rules out the “stupidest reasons” for a sign map  $E^3 \rightarrow \{-1, 0, +1\}$  not to come from a vector configuration. At least it must be alternating, since it is modeling the behavior of determinants. And it must at least be consistent with the Grassmann-Plücker relations. These are exactly the two chirotope axioms. They are checkable in polynomial time. Since planar Grassmann-Plücker relations involve at most six points, we can also say that we require that at least all subconfigurations with at most six points be realizable.

From this point on, things get amazingly hard. It turns out that this local realizability implies that the smallest nonrealizable chirotope comes from a perturbed Pappos’s configuration. In fact, every nonrealizable chirotope is related to some perturbed incidence theorem. Thus knowing about nonrealizable oriented matroids means knowing about incidence theorems. The proof methods for incidence theorems from Chapter 15 had their origins in automated nonrealizability proofs for chirotopes (see [15, 105]). One can even prove (and this is a famous theorem in oriented matroid theory) the so-called universality theorem:

*Deciding whether a chirotope is realizable is as hard as solving an arbitrary system of polynomial inequalities and equations.*

At the core of the proof of this theorem are the von Staudt constructions we used in Chapter 5 to encode addition and multiplication on the level of projective incidence configurations (for details see [96, 97, 107, 121]). Also with projective methods a similar statement can be proved for the realization spaces of polytopes [106, 108].

## 27.3 Projective Geometry and Quantum Theory

It is always an amazing phenomenon when a mathematical field has applications to subjects that at the first sight seem to be not related at all to it. Many of the methods presented in this book have very close relationships to various branches of physics (among them quite modern ones). For instance, Cayley-Klein geometries can be used as a very good framework to model space-time coordinate systems (and their invariants) in special relativity theory. A bit more surprising is the relation of projective geometry to quantum physics. Again we can give only a glimpse of the subject. For matters of brevity we limit ourselves to the rather restricted area of *quantum information theory* (see for instance [90]). This subject deals with the concept of information on the quantum level. While in the usual framework of information theory the fundamental unit of information is a *bit* (a single 0/1 decision), in quantum information theory the fundamental unit of information is a *qubit* (short for quantumbit). A carrier of such a single piece of information can be, for instance, an elementary particle such as an electron or a photon.

**Qubits and  $\mathbb{CP}^1$ :** As a first step we have to clarify how an electron or a photon can carry information. For this we have to deal with quantum states and measurements. A quantum state completely describes the physical properties of an elementary particle such as an electron or a photon. Typically these states can be expressed as the (complex) superposition of several mutually exclusive elementary states. These elementary states will be represented by an orthogonal basis of vectors. For the situation we are interested in, the electron spin, we will need two such elementary states. In the physics literature these base states are usually denoted by  $|0\rangle$  and  $|1\rangle$ . The superposition is

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle.$$

Here  $\alpha_0, \alpha_1 \in \mathbb{C}$ . For reasons related to the measurement of quantum properties one furthermore requires  $|\alpha_0|^2 + |\alpha_1|^2 = 1$ . From a mathematical point of view we can simply identify the base states with two unit vectors of  $\mathbb{CP}^2$  and get

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}.$$

The normalization  $|\alpha_0|^2 + |\alpha_1|^2 = 1$  says that the vector length of  $|\psi\rangle$  considered as composed of four real components (the two real and imaginary parts of  $\alpha_0$  and  $\alpha_1$ ) is normalized to 1. In addition (also due to the properties of measurements explained later), states that differ only by a complex unit  $e^{it}$ ,  $t \in \mathbb{R}$ , are considered to represent the same state:  $|\psi\rangle \sim e^{it}|\psi\rangle$ . From a projective point of view, after considering states differing by a phase shift to be equivalent, the normalization is superfluous. We may equally well represent a state simply by a linear combination  $\alpha_0|0\rangle + \alpha_1|1\rangle$  and identify nonzero complex multiples. So a state vector is nothing but an element of



$\mathbb{CP}^1$ . Nevertheless, the normalization will be technically helpful when we consider measurements later on. This identification of states and points in  $\mathbb{CP}^1$  can be made very concrete for the case of an electron spin or photon polarization. For reasons of brevity we will restrict ourselves to the case of electron spin (and by neglecting the photon miss another geometrically fascinating interpretation of qubits).

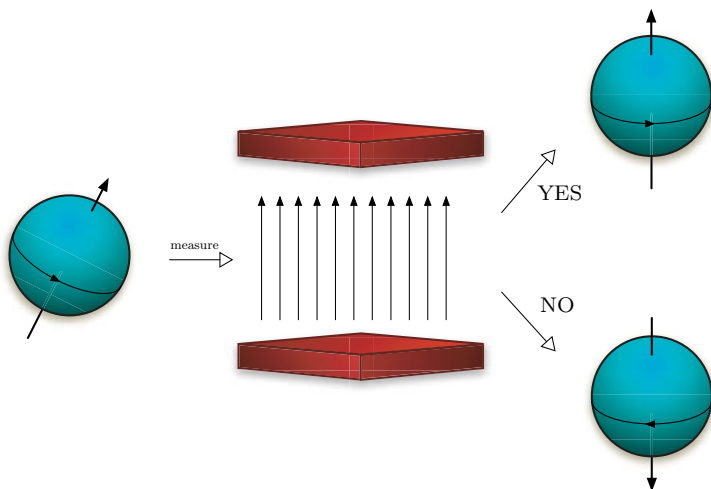
In the case of the electron, the state  $|\psi\rangle$  encodes all information about the *spin* of the electron. The spin is a quantum phenomenon that has no classical analogue. Nevertheless, a good way (at least good enough for our purposes) to think about it is to visualize the electron as a spinning sphere. The spin corresponds to the direction of the rotation axes. Since the rotation has a direction, also the axis is directed by a kind of right-hand rule (the fingers indicate the rotation and the thumb points in the direction of the axis).<sup>1</sup> So the spin is essentially a point on the unit sphere. A vector connecting the origin to this point indicates the directed axis. This is the moment when projective geometry comes into play. In Section 17.7 we have seen how the complex projective line  $\mathbb{CP}^1$  can be identified with the unit sphere via stereographic projection (see Figure 17.9). As a consequence of this interpretation, by stereographic projection two basic states  $|0\rangle$  and  $|1\rangle$  correspond to two opposite directions of the spin pointing to the southpole and to the northpole. More generally, any pair of antipodal points on the projected sphere becomes a pair of homogeneous coordinates whose inner product vanishes.

The linear combinations of the basic states  $\alpha_0|0\rangle + \alpha_1|1\rangle$  resemble the space of all possible states. For instance, all states of the form  $\frac{1}{\sqrt{2}}(e^{it}|0\rangle + \alpha_1|1\rangle)$  with  $t \in \mathbb{R}$  represent electron spins whose rotation axis meets the equator, the great circle of all points halfway between the north and southpoles.

To understand why firstly the normalization  $|\alpha_0|^2 + |\alpha_1|^2 = 1$  and secondly the identification of states differing only by a phase make sense, it is essential to understand the measurement process that transfers quantum states in classical observables. It is one of the fundamental properties of quantum states that they are not directly accessible by classical observations. They only influence the probability of the outcome of certain measurements. In the case of the electron spin, the situation is as follows. Typically, when measuring the electron spin one has to decide *before* the measurement with respect to which axis one wants to measure. So when an electron is exhibited to a measuring device, the device performs the measurement and outputs “yes” or “no” depending on the direction of the spin of the electron. If the electron spin points in the direction of the measurement, the answer is surely “yes.” If it points in the opposite direction of the measurement, the answer is surely “no.” For all other spin directions the probability of a “yes” or “no” depends on the proximity of the spin direction and the measurement direction. Measurements have the strange property that *after* the measurement, the electron is indeed

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<sup>1</sup> Do not think about all the other properties one would classically associate to a rotation such like speed and acceleration. The electron spin just has a direction; that’s it.



**Fig. 27.8** Measurement of electron spin.

in the state that has been measured. Thus the measurement influences the quantum state of the electron.<sup>2</sup> Figure 27.8 roughly illustrates this process. An electron with an arbitrary spin enters the measurement device that tests for a spin pointing upward. After it leaves the device there are two possible results of the measurement. Either the output is “yes” and the spin then really points upward, or the output is “no” and the spin then really points downward.

Mathematically speaking, measurements are expressed in a similar way as states. A measurement is a linear combination of two directions

$$\langle\phi| = \beta_0\langle 0| + \beta_1\langle 1|$$

of two mutually exclusive unit states  $\langle 0| = (1, 0)$ ,  $\langle 1| = (0, 1)$ . Note that this time we use row vectors. We make an important twist to the representation of states. The complex numbers  $\beta_0$  and  $\beta_1$  are the complex conjugates of the numbers  $\alpha_0$  and  $\alpha_1$  that would encode the same direction as a spin. So if  $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$  is the state (i.e., direction) of a spin, then

$$\langle\psi| = \overline{\alpha_0}\langle 0| + \overline{\alpha_1}\langle 1| = (\overline{\alpha_0}, \overline{\alpha_1})$$

encodes the measurement in exactly the same direction. In other words, directions of measurement are represented by the Hermitian conjugates of the corresponding directions of states. The Hermitian conjugate is derived by

<sup>2</sup> Do not try to understand in classical terms how this can happen. It is one of the (more harmless) mysteries of quantum mechanics.

simultaneously transposing and conjugating. If we denote the Hermitian conjugate of a vector  $v$  by  $v^\dagger$ , then we get  $\langle\psi| = |\psi\rangle^\dagger$ .

The states  $\langle 0| = (1, 0)$  and  $\langle 1| = (0, 1)$  represent opposite directions in the measurement space. Again every other direction can be expressed as a linear combination of the basis states. The probability that a measurement  $\langle\phi|$  responds with “yes” to a state  $|\psi\rangle$  is calculated by

$$|\langle\phi|\cdot|\psi\rangle|^2 = |\alpha_0\beta_0|^2 + |\alpha_1\beta_1|^2.$$

Measuring a state  $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$  with respect to its own direction  $\langle\psi|$  results in the probability

$$|\langle\psi|\cdot|\psi\rangle|^2 = \alpha_0\overline{\alpha_0} + \alpha_1\overline{\alpha_1} = |\alpha_0|^2 + |\alpha_1|^2 = 1.$$

This a posteriori justifies the normalization  $|\alpha_0|^2 + |\alpha_1|^2 = 1$  and  $|\beta_0|^2 + |\beta_1|^2 = 1$ . Also an easy calculation shows that a phase shift (i.e., replacing  $|\psi\rangle$  by  $e^{it}|\psi\rangle$ ) does not influence the probability of the measurement, and by this will have no effect on the classical world.

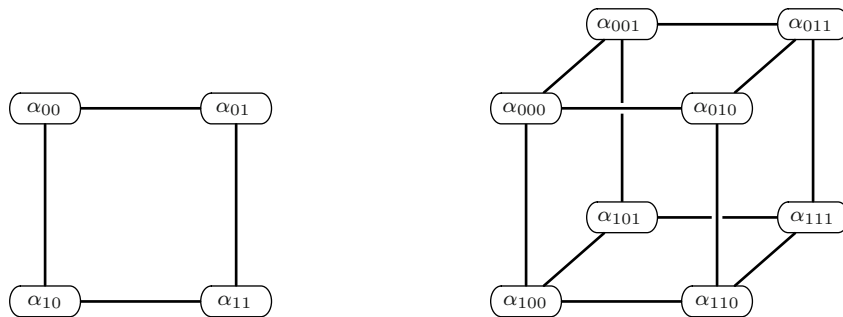
Let us see what happens if one measures one of the equatorial states  $|\psi\rangle = \frac{1}{\sqrt{2}}(e^{it}|0\rangle + \alpha_1|1\rangle)$  with respect to the south direction  $\langle 1|$ . For the probability of getting “yes” we get

$$|\langle 1|\cdot|\psi\rangle|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 (|0\cdot 1|^2 + |1\cdot e^{it}|^2) = \frac{1}{2}.$$

This corresponds to the physically reasonable fact that measurements of an equatorial spin with respect to the south direction produces a “yes” equally often as it produces a “no.”

There is one important point on possible local transformations of a qubit. Our choice of the basis  $|0\rangle, |1\rangle$  was somehow random. We could equally well have chosen a pair of other opposite directions as basis. In other words there are some basis transformations of the mathematical representation that do not alter the physical reality. Let  $|\tilde{0}\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$  be one such state (with  $|\alpha_0|^2 + |\alpha_1|^2 = 1$ ). Then  $|\tilde{1}\rangle = -\overline{\alpha_1}|0\rangle + \overline{\alpha_0}|1\rangle$  represents the state with opposite spin direction. Thus a basis transformation from the  $(|\tilde{0}\rangle, |\tilde{1}\rangle)$ -basis to the  $(|0\rangle, |1\rangle)$ -basis is given by a transformation matrix  $\begin{pmatrix} \alpha_0 & -\overline{\alpha_1} \\ \alpha_1 & \overline{\alpha_0} \end{pmatrix}$ . This is a matrix from the matrix group  $SU(2)$  of unitary matrices with determinant 1. Such an  $SU(2)$  matrix can also be used to describe rotation of the measurement device. From a projective point of view such transformations are nothing but special projective transformations of  $\mathbb{CP}^1$ . Similarly to the fact that we could describe hyperbolic transformations in the Poincaré model by certain  $\mathbb{CP}^1$  transformations, the  $SU(2)$  transformations represent transformations of elliptic geometry in a suitable  $\mathbb{CP}^1$  representation.

One note on notation. From a mathematical point of view the states and the measurements are nothing but a fancy notation for column and row



**Fig. 27.9** The parameters of systems of two and three qubits

vectors. Still this notion turns out to be extremely compact and insightful in many practical applications. The expression  $\langle\phi|\cdot|\psi\rangle$  corresponds to a usual Hermitian bilinear product  $\langle\phi, \psi\rangle$  on a complex vector space, sometimes also called a *bracket*. This is the reason why the measurement  $\langle\phi|$  is often called a *bra* and the state  $|\psi\rangle$  is called a *ket*. A bra and a ket together make a bra-(c)-ket. It is also often written as  $\langle\phi|\psi\rangle$ .

One note on physical interpretation. An expression like  $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$  can be interpreted in various ways. It can be interpreted, for instance, as an electron with a spin in one specific direction. When a measurement comes, this direction influences the probability of the outcome of the measurement. It could also be interpreted as a (complex) *superposition* of the two states  $|0\rangle$  and  $|1\rangle$ . The electron is “at the same time” in both states. When a measurement arises, it “decides” in which of its two schizophrenically superimposed states it prefers to be. It is one of the secrets of dealing with quantum theory that both viewpoints are essentially equivalent and lead to the same classical observations.

**Many qubits:** What happens if we have not only one qubit, but two, three or more. Imagine that there are  $n$  electrons whose spins are independently measured. Each measurement results in a classical “yes” or “no” answer. So the classical experiment has  $2^n$  possible outcomes. Each outcome excludes the other. These classical alternatives form the basis vectors of a vector space of suitably high dimension. A quantum state of the system of  $n$  electrons is a (complex) superposition of these classical alternatives.

We will exemplify this concept by systems of two and of three qubits. We assume that we measure each of these qubits with respect to the same direction, say the north direction. Such a measurement of a two-qubit system has four possible outcomes, whose states are denoted by

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle.$$

The superposition of these states is given by

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$

Since the outcomes are mutually exclusive, we may again represent them as mutually orthogonal unit vectors and get

$$|\psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}.$$

As before, we have to assume that this vector is normalized, i.e., we have  $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$ . The obvious combinatorics of the indices of the four different entries indicates that it might be reasonable to arrange these indices not as a vector but as a matrix:

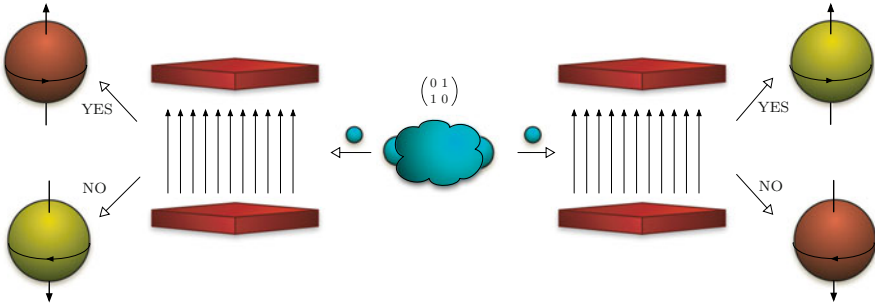
$$|\psi\rangle = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}.$$

Again a common phase shift is irrelevant. This together with the normalization implies once more that we may equivalently skip normalization and phase-shifting and instead consider the states that differ by a complex scalar to be equivalent. Thus the state space is nothing but  $\mathbb{CP}^3$ .

If we perform the same considerations with three instead of two qubits, we will get eight possible classical outcomes of the measurements. We will have eight controlling (complex) parameters. This time it is best to locate these eight parameters at the eight vertices of a cube. Again complex multiples represent the same state and we get a projective space  $\mathbb{CP}^7$  as state space. One might consider such a state, as a  $2 \times 2 \times 2$  tensor (in fact tensor diagram techniques are rather common in quantum physics [23, 123]).

So a system of  $n$  qubits is described by a  $2^n$ -dimensional vector representing the complex (homogeneous) coordinates. It can also be considered a  $2 \times 2 \times \cdots \times 2$  tensor. From an information-theoretic viewpoint this is remarkable. A single qubit is an element of  $\mathbb{CP}^1$  and by this has in essence one complex parameter. A system of two qubits is an element of  $\mathbb{CP}^3$  and has three complex parameters. A system of three qubits is an element of  $\mathbb{CP}^7$  and has essentially seven complex parameters. So the collection of the parts carries significantly (exponentially) more information than all individual parts taken together.

What does this mean and how could this happen? The key to this seemingly paradoxical situation is a concept called *entanglement*, which has no classical counterpart at all. We again start our explanations with the two-qubit case. Consider some kind of particle decay that emits two electrons in opposite directions. The two electrons form a 2-qubit quantum ensemble, and their quantum state is described by  $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ .



**Fig. 27.10** Measurement of electron spin.

Now imagine that both electrons travel and at some point in time become considerably separated by space and reach two observers Alice and Bob, who measure their spins. For Alice and Bob the two electrons appear as individuals, although they are related by a joint quantum state. The measurements of Alice and Bob must be such that they could be at the same time interpreted by the electrons having an individual quantum state and Alice and Bob ignoring the existence of each other and on the other hand as having a common quantum state. We will exemplify the situations with two concrete quantum states that have in a sense extreme properties. These two quantum states  $X$  and  $Y$  correspond to the matrix representations

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the case of the joint quantum state  $X$  we have a superposition with each of the four basis states equally weighted. Thus measuring the state of both electrons with respect to the measurement  $\langle 00|$  has a probability of  $1/4$ . If Alice alone measures the spin of her electron with respect to  $\langle 0|$ , then she will get a probability  $1/2$ . In half of the situations where Alice get a “yes” Bob also will get a “yes.” In a sense, for state  $X$  both electrons are independent. The outcome of the measurement of one is not correlated to the outcome of the measurement of the other. The situation is very different for state  $Y$ . For this state, measuring the joint state w.r.t  $\langle 00|$  or  $\langle 11|$  will always result in a “no.” The other two measurements  $\langle 01|$  and  $\langle 10|$  will each have probability  $1/2$ . For Alice (at least as long as she does not know of the other electron and Bob’s existence), the situation did not change. Measuring with respect to  $\langle 0|$  will still have a probability  $1/2$  of a “yes.” However, this time if she gets a “yes,” Bob will surely get a “no” and vice versa. The states are strongly negatively correlated. A state like  $Y$  is called a *totally entangled* state. A state like  $X$  is called *totally independent*. Intermediate situations of partial entanglement are also possible. Figure 27.10 illustrates the situation

for state  $Y$ . A measurement of both electrons results either in situation of the two yellow electrons or in the situation of the two red electrons.

One might say, “So what, where is the problem? Both electrons already have their spin when they leave their particle source, like two coins in two closed boxes that are opened only when the measurement is performed. The outcome of measurement of the electrons is governed by some hidden variables.” Although from a classical perspective this seems to be a reasonable explanation, one can show that this is definitely *not the case*. This effect is known as the Einstein-Podolsky-Rosen paradox (EPR) and has been confirmed by experiments [37, 12]. The outcome of the measurement by Alice is really completely undetermined unless the measurement has been performed. By some mysterious principle of nonlocality, the outcome of the measurement is somehow transported to Bobs measurement device and influences his measurement. How this happens is still an open problem.<sup>3</sup>

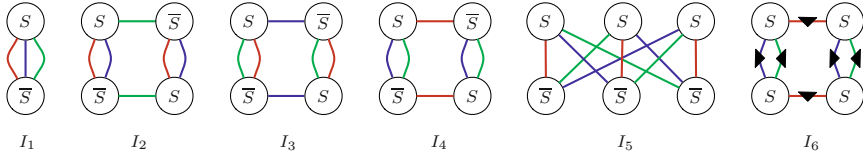
**Entanglement invariants:** Understanding the mathematics of entanglement brings us again back to projective geometry and invariant theory (see [127]). We here will be able to give only a rough sketch of what happens. We again start with the 2-qubit case. The transformations acting on the combined space of the two electron spins within the space  $\mathbb{CP}^3$  is given by all matrix operations in  $SU(4)$ . This is a certain subgroup of all projective transformations of  $\mathbb{CP}^3$ . Compared to this, Alice and Bob have only very restricted influence on the state space. Each of them can act with an  $SU(2)$  on their individual electrons. Thus after the electrons have been separated, only  $SU(2) \times SU(2) \subset SU(4)$  actions are possible. Alice’s action corresponds to a left multiplication of the state matrix with an  $SU(2)$  matrix. Bob’s action corresponds to a right multiplication of the matrix with an  $SU(2)$  matrix. Such actions cannot change the degree of entanglement. For instance, it can be shown that for state  $Y$ , if Alice decides to measure her electron with respect to another basis, there is a corresponding basis for Bob such that he still will always get the opposite answer to Alice’s. Similarly, in case of state  $X$  there is no choice of the bases such that the measurements become correlated.

In fact, it turns out that completely independent states correspond to rank-1 state matrices  $(\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)^T$ . They are always completely unentangled. The rank-1 property cannot be influenced by the local perturbations. It turns out that the absolute value of the determinant of the state matrix is an *entanglement invariant*. It cannot be altered by local transformations. After normalization this absolute value of the determinant can take values between 0 and 1. The state  $X$  results in a value of 0 (nonentangled); the value  $Y$  results in a value 1 (fully entangled). The space of all nonentangled states forms a conic in the state space  $\mathbb{CP}^3$ .

The situation becomes more complicated if more than two qubits are involved. In principle, one is interested in all

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<sup>3</sup> Do not try to understand in classical terms how this can happen. It is one of the (less harmless) mysteries of quantum mechanics.



**Fig. 27.11** The entanglement invariants of a three-qubit system.

$$\underbrace{\mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2)}_n \subset \mathrm{SU}(2^n)$$

invariants of a combined system of  $n$  qubits that lives in  $\mathbb{CP}^{2^n-1}$ . The search of such invariants can be attacked by diagram techniques described in Chapter 13 and Chapter 14. One considers each qubit a tensor of rank  $n$  and considers closed diagrams involving  $\varepsilon$ -tensors, the qubit tensor, and a special tensor that encodes that we are interested in  $SU$  transformations (in principle, this tensor can “forget” the direction of the arrows). Each arc coming from a quantum-state tensor in a diagram has a color (which encodes the integrity of the corresponding electron), and one has to make sure that only arcs of equal color are connected. Figure 27.11 shows diagrams for a generating set of the entanglement invariants of a three-qubit system [127, 114]. Many of these invariants can be directly interpreted in projective terms [5]. For instance,  $I_2$  has the following projective interpretation. Take the cube in Figure 27.9 that encodes the parameters of the state. Slice it in a top and a bottom plane. Consider each plane as a four-dimensional vector  $v_1$  and  $v_2$ . The invariant is the square of absolute value of the Plücker product of the two vectors:

$$I_2 = \|v_1 \vee v_2\|^2.$$

Similar interpretations hold for  $I_3$  and  $I_4$ . The invariant  $I_6$  is the well-known hyperdeterminant of a  $2 \times 2 \times 2$  tensor, which also plays an important role in the theory of projective geometry of algebraic curves.

## 27.4 Dynamic Projective Geometry

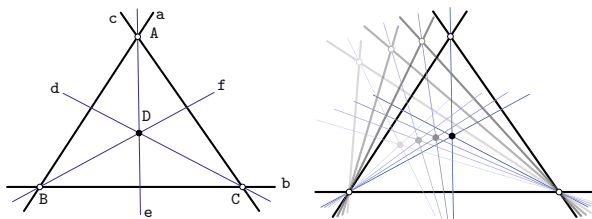
The connection of geometric configurations and complex numbers is even deeper than may be apparent so far. Until now we have considered most of our geometric scenarios as a kind of static snapshot capturing a certain geometric effect. However, very often it is reasonable to consider geometric constructions as *dynamic constructions* in which certain free elements are allowed to move freely while other elements of the constructions (the dependent elements) are moved according to certain relations with respect to the free elements.



```

1: A=FreePoint;
2: B=FreePoint;
3: C=FreePoint;
4: a=Join(A,B);
5: b=Join(B,C);
6: c=Join(A,C);
7: d=AngleBisector(b,c);
8: e=AngleBisector(a,c);
9: f=AngleBisector(a,b);
10: D=Meet(d,e);

```

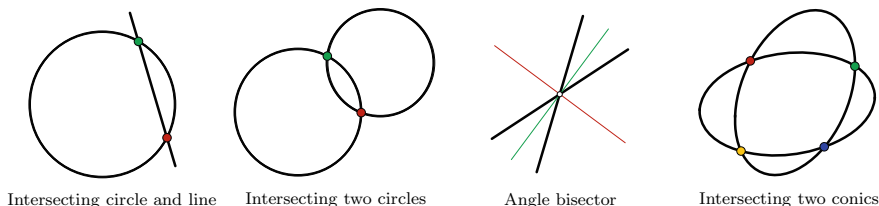


**Fig. 27.12** A geometric construction, a static instance, and a dynamic movement.

A geometric theorem can very often be considered a statement of a certain property that is invariant within the configuration space of the construction (see [72, 73, 74]). Almost all drawings in this book were constructed by this paradigm of treating a geometric construction as a dynamic picture. Usually, first a rough sketch was created in which care was taken only about the logical relations between the elements of the construction. Then in a second step the free elements of the construction were moved such that the whole picture became esthetically pleasing and showed the intended effect in a clear way.

The pictures were constructed with a “dynamic geometry system” (DGS). The specific program that was used is the program *Cinderella* [112, 113], which was developed by Ulrich Kortenkamp and the author of this book. During the period of developing this program (which is a kind of never-ending story) many fascinating and interesting problems arose with sometimes surprising solutions. This section is about one of these problems, whose solution exemplifies once more the close connection between geometry and complex numbers. It should be emphasized at the beginning that the following expositions can be also interpreted in a context of pure mathematics having no direct relation to the implementation of a DGS.

**Typical tasks of a DGS:** We start by sketching the typical tasks that a dynamic geometry system should be able to perform. With such a system it should be possible to do elementary geometric constructions. Typically, such constructions start with a bunch of freely positioned elements (very often points) and proceed by a sequence of construction steps that generate geometric elements depending on the already constructed ones. The construction steps may include operations like *join of two points*, *meet of two lines*, *mid-point of two points*, *circle by center and boundary point*, *conic through five points*, *intersection of a conic and a line*, *intersection of two conics*, *angle bisector of two lines*. Once a construction is created, it should be possible to select the free elements with the mouse and drag them while the entire construction moves according to the rules of the construction. Sometimes also *half-free* elements are admitted, such as a point moving on a circle or a line through an already existing point. Figure 27.12 shows a *construction*



**Fig. 27.13** The entanglement invariants of a three-qubit system.

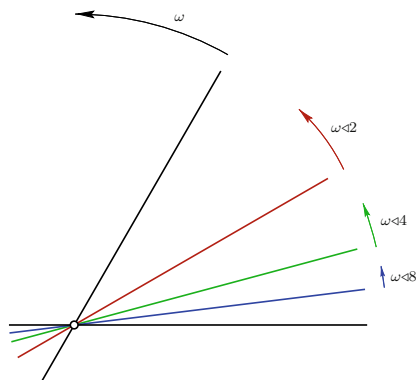
*sequence* (leftmost picture) and (in the middle) an image of a static instance of this sequence. An *instance* of the construction sequence is a concrete assignment of geometric objects to each construction step that is consistent with the construction sequence. The rightmost picture indicates a dynamic scenario in which point A is moved by some mouse interaction.

Many DGS are implemented in a straightforward way. Points are implemented by real  $xy$ -coordinates, lines by a Hesse normal form, circles by center and a radius. The explanations in this book make clear that already at a very elementary level it is reasonable to rely on more advanced techniques, i.e., to represent points and lines by homogeneous coordinates, to represent circles and conics by quadratic forms, etc. By this projective approach a proper treatment of infinite elements and of many degenerate situations is possible. It is also a reasonable choice to allow for the use of complex coordinates. In particular, this simplifies the treatment of various measurements, since the techniques using the points I and J as well as general Cayley-Klein geometries become accessible.<sup>4</sup>

In a sense, the problems resolved by the use of homogeneous and/or complex coordinates are still of a *static* nature. These approaches may become relevant for certain instances of a construction sequence, for instance when lines whose intersection is formed become parallel. However, even beyond these static issues there are certain questions in the modeling of DGS that need advanced mathematical techniques that are intimately related to the *dynamic* nature of a DGS. The requirement that constructions remain stable under a continuous movement of the free elements once more indicates the use of complex numbers.

**The problem of jumping elements:** If one analyzes the behavior of many of the available DGS programs, one observes an amazing effect. If a construction becomes more and more complicated, it may happen that a small change of the free elements may suddenly result in a large jump of some of the dependent elements. The reason for this lies in the fundamental

<sup>4</sup> It still remains a modeling decision how far calculations with homogeneous and complex coordinates should be transparent to a user. We will not elaborate on this topic here, since it involves educational, software, ergonomic, and mathematical considerations.



**Fig. 27.14** In iterated angle bisector.

nature of certain geometric primitive operations. Some operations, such as intersecting a line with a circle, intersecting two circles, calculating the angle bisector of two lines, and intersecting two conics, are intrinsically *ambiguous*. In these cases the input elements do not uniquely determine the output elements. Intersecting a circle with a line in general has *two* solutions, as well as intersecting a circle and a circle. Two lines have *two* angle bisectors. Two conics in general intersect in *four* different points. We dealt with each of these primitives in the context of this book. For the first three cases a square-root operation (which carries an intrinsic ambiguity) is involved in the computation of the output (compare Section 11.3 and Section 19.2). For intersection of two conics solving a polynomial of degree three and square roots are involved (compare Section 11.4). Figure 27.13 shows some instances of these ambiguous operations together with the different outputs of the primitive operations (indicated by different colors).

For an initial instance of a geometric construction in a DGS it is usually clear (by user interaction) which of the several outputs of an ambiguous operation has to be taken. However, when the free points of a construction are dragged, *the computer* has to decide which of the outputs for ambiguous operations has to be taken. It may happen that the computer makes a decision in which a dependent element suddenly jumps from one position to the other.

One might think that there are appropriate simple heuristics that depending on the position of the free elements allow one to choose the dependent elements such that no such discontinuous jumping arises. In fact, one can prove by a relatively simple argument that every implementation that avoids discontinuous behavior of the dependent elements must take the history of the movement into account [74]. For this consider Figure 27.14. It shows the iteration of an angle bisector. The red line is the angle bisector of the two black lines (one of them is the  $x$ -axis). The green line is the angle bisector of the red line and the  $x$ -axis. The blue line is the angle bisector of the green

line and the  $x$ -axis. Now imagine that the steep black line is rotated about the intersection of the lines with an angle velocity of  $\omega$ . Then the red, green and blue lines rotate with angle velocities  $\omega/2$ ,  $\omega/4$ ,  $\omega/8$ , respectively. This means that if the black line is rotated by  $360^\circ$ , then the red line makes a  $180^\circ$  turn, the green line a  $90^\circ$  turn, and the blue line a  $45^\circ$  turn. Although the (half-)free elements of the construction are back at their original positions, the green and the blue lines continuously moved to another position. Thus if one wants to have a continuous behavior, one has to take the history of the movement into account.

The mathematical modeling of the situation is not completely elementary and involves the consideration of complex ambient spaces of the objects, Riemann surfaces for the configuration spaces involved, and analytic continuation. Again we can only give a rough impression of the underlying theory. For more details see [72, 73, 74].

**Constructions and movements:** The key to a mathematically satisfactory resolution of the jumping elements problem is the observation that the coordinates of the output of a geometric primitive operation are analytic functions in the coordinates of the input. For instance, the intersections of a line and a circle can be calculated using the four basic arithmetic operations and a square-root operator (one can even exclude division if homogeneous coordinates are used). The intrinsic ambiguity of the square-root operator (the solution of  $x^2 = y$  for given  $y$ ) leads to the intrinsic ambiguity of the geometric operations. By embedding the entire geometric construction in a complex ambient space (as we did throughout this book) it is possible to trace the different branches generated by the ambiguities of a moving construction by analytic continuation—at least if no degenerate situations arise during the movement. In a very sketchy way the basic strategy for the resolution of the jumping elements problem can be phrased as follows:

- admit complex coordinates,
- trace elements by analytic continuation,
- resolve singularities by taking complex detours.

The detailed and exact mathematical modeling of a DGS is a very subtle issue and full of small hidden technical pitfalls. Nevertheless, we will give an abridged version of it that neglects the problem of modeling the concrete geometric primitive operations. A construction sequence  $\mathbf{\Gamma} = (\Gamma_0, \dots, \Gamma_k)$  consists of a sequence of  $k$  geometric primitive operations  $\Gamma_i$ . For simplicity we restrict ourselves to the following operations: *free point*, *join of two points*, *meet of two lines*, *circle by center and a perimeter point*, *intersection of a circle and a line*, *intersection of two circles*.

Except for free points, the input elements of each operation must be specified in the previous construction steps. Notice that each operation specifies the type of the corresponding output element. It is important that each of the construction steps be considered a *relation* rather than a function to

handle the possibility of ambiguous output. Now, an *instance* of a geometric construction is a sequence  $X_0, X_1, \dots, X_k$  of geometric objects whose types are consistent with the geometric operations specified by the construction. Furthermore, in an instance of  $\Gamma$  each object  $X_i$  must be consistent with the geometric requirements of its corresponding relation  $\Gamma_i$  specified by the  $i$ -th construction step.

Now assume that a construction sequence  $\Gamma$  and an appropriate start instance  $\mathbf{X} = (X_0, X_1, \dots, X_k)$  are given. We want to model the dynamic behavior of the construction under the movement of a free element. Without loss of generality assume that  $X_0$  is a free point that is moved from a position  $A$  to another position  $B$ . We may assume that the coordinates of  $X_0$  are given by homogeneous coordinates  $A, B \in \mathbb{RP}^2$ . The movement  $X_0(t)$  of the point may be parameterized by a parameter  $t \in [0, 1]$ . We assume that  $X_0(t)$  is a coordinatewise analytic function with  $X_0(0) = A$  and  $X_0(1) = B$ . A *continuous movement* given by  $(\Gamma, \mathbf{X}, X_0(t))$  is an assignment of continuous functions  $\mathbf{X}(t) = (X_0(t), X_1(t), \dots, X_k(t))$  such that

- (i)  $\mathbf{X}(0) = \mathbf{X}$ ,
- (ii) for each  $t \in [0, 1]$  the values  $\mathbf{X}(t)$  are an instance of  $\Gamma$ ,
- (iii) all other free elements except for  $X_0$  remain constant.

It is not clear in advance whether a given triple  $(\Gamma, \mathbf{X}, X_0(t))$  admits a continuous movement at all. In particular, this depends on the detailed modeling of the semantics of the geometric primitive operations and on the special path  $X_0(t)$ . If we allow complex geometric elements, then the major problem is degenerate solutions for which some of the primitive operations may not be performable.

If the path  $X_0(t)$  is fixed in advance, it may be unavoidable to come across degenerate situations during a movement (for instance, two points involved in a join operation may pass through each other). However, if one is interested only in the initial position  $X_0(0) = A$  and in the end position  $X_0(1) = B$  and if one in particular allows also the point  $X_0$  to become complex, one can always find a path  $X_0(t)$  such that the construction avoids all critical situations during the move (except perhaps for the endpoint  $B$ ). We first specify what exactly we mean by *critical*. A (partial) instance  $X_0, \dots, X_j$ ,  $j \leq k$ , of a construction  $\Gamma = (\Gamma_0, \dots, \Gamma_k)$  is called *critical* if either the operation in step  $\Gamma_{j+1}$  becomes degenerate or if in one of the ambiguous operations among  $\Gamma_0, \dots, \Gamma_j$  some of the possible outputs coincide. So critical situations arise, for instance, if a join of two identical points is requested or if one wants to construct the intersection of a circle and a line tangent to it.

One can prove the following fact: *Let as before  $\mathbf{X}$  be a (noncritical) initial instance of  $\Gamma$ . If  $X_0(t)$  is a coordinatewise analytic path of the control point  $X_0$ , then we can uniquely assign a continuous (even analytic) movement  $(X)(t)$  given by  $(\Gamma, \mathbf{X}, X_0(t))$ .* The proof of this result essentially goes by induction on the number of construction steps. It uses the fact that the

branches of the output of any geometric primitive operation can be calculated by analytic functions (for details see [74]). By the absence of critical situations it is always possible to distinguish the branches of an ambiguous operation. To stay with the example of intersecting a line and a circle: if no critical situations are involved, the two intersections never coincide (however, they might become complex). By this every noncritical movement of the line and the circle can be used to reconstruct a unique path for the two branches of possible intersections. The assumption of being noncritical is essential to get a uniqueness of the branches of the intersection. If the line becomes tangent to the circles, the two branches will eventually meet.

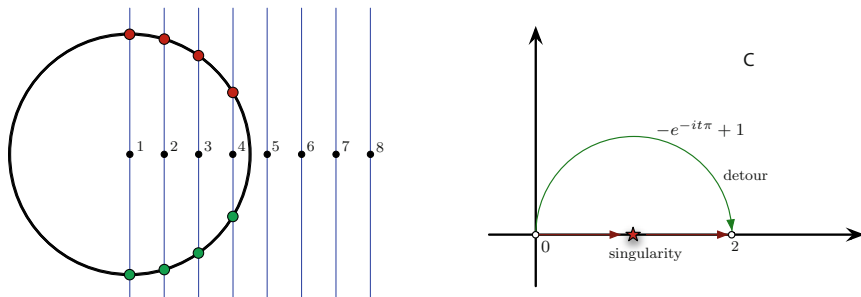
So the main issue becomes the avoidance of critical points. As long as we are interested only in the initial point and in the endpoint of  $X_0(t)$  we have sufficient freedom to choose the path such that all critical points are avoided. For this we consider the linear interpolation

$$X_0(t) := (1 - \lambda(t)) \cdot A + \lambda(t) \cdot B,$$

where now  $\lambda: [0, 1] \rightarrow \mathbb{C}$  is an analytic function with  $\lambda(0) = 0$  and  $\lambda(1) = 1$  on the interval  $[0, 1]$ . It is possible—by methods of complex function theory—to prove that  $\lambda$  can always be chosen in a way such that the resulting path of the construction is noncritical for  $t \in [0, 1]$ . In fact, the critical situations arise only as pointlike singularities in  $\mathbb{C}$ , the space in which  $\lambda$  is defined. For that reason it is always possible to circumvent critical situations by a suitable complex detour. Allowing complex values for  $X_0(t)$  is essential. If one requires  $\lambda(t)$  to be real, it may be unavoidable to pass through critical situations.

One might wonder why one allows the *free choice* of a path  $X_0(t)$  and even with complex coordinates. This resembles very much the information situation in a real dynamic geometry system implemented on a computer. In dragging a free point, the only information on the user input is the mouse interaction. This information, however, consists of a sequence of discrete sampling points (the mouse-drag information). The above strategy uses this freedom to interpolate this information by noncritical analytic paths. All in all, this method allows for a continuous movement of the dependent elements for any specific mouse movement and by this resolves the jumping elements problem.

**A simple example:** We will end with a minimal example that illustrates the complex tracing process. We consider a unit circle  $c$  and a vertical line  $l$  whose position is controlled by some free point (it is easy to construct such a situation in a DGS). We want to analyze the behavior of the two intersection points. Since all relevant elements stay finite, it is admissible to do the analysis by considering the usual Euclidean  $xy$ -coordinates. The points of the circle are all points  $\{(x, y) \mid x^2 + y^2 = 1\}$ . Let  $l_a$  be the vertical line with  $x$ -coordinate  $a$ . The two intersections of  $c$  and  $l_a$  have the coordinates  $p_{\pm} = (a, \pm\sqrt{1 - a^2})$ . For all  $a$  in the open interval  $(-1, 1)$  we have two real solutions. For  $a = \pm 1$  we

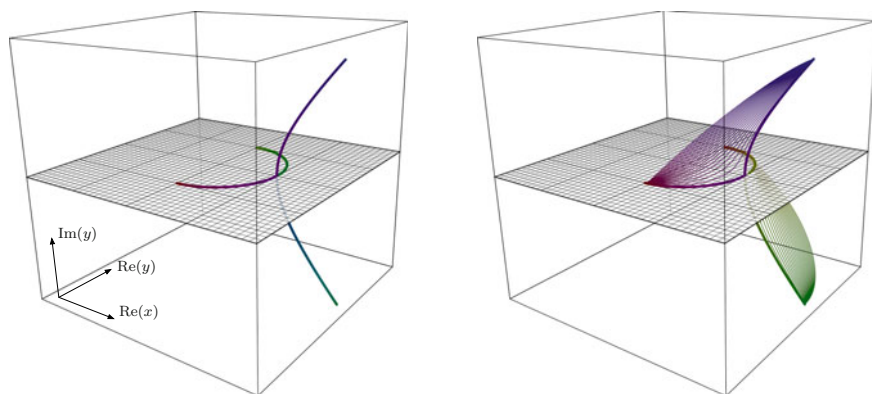


**Fig. 27.15** Intersections of a vertical line and a circle under a movement (left). The space of the controlling parameter of the line (right).

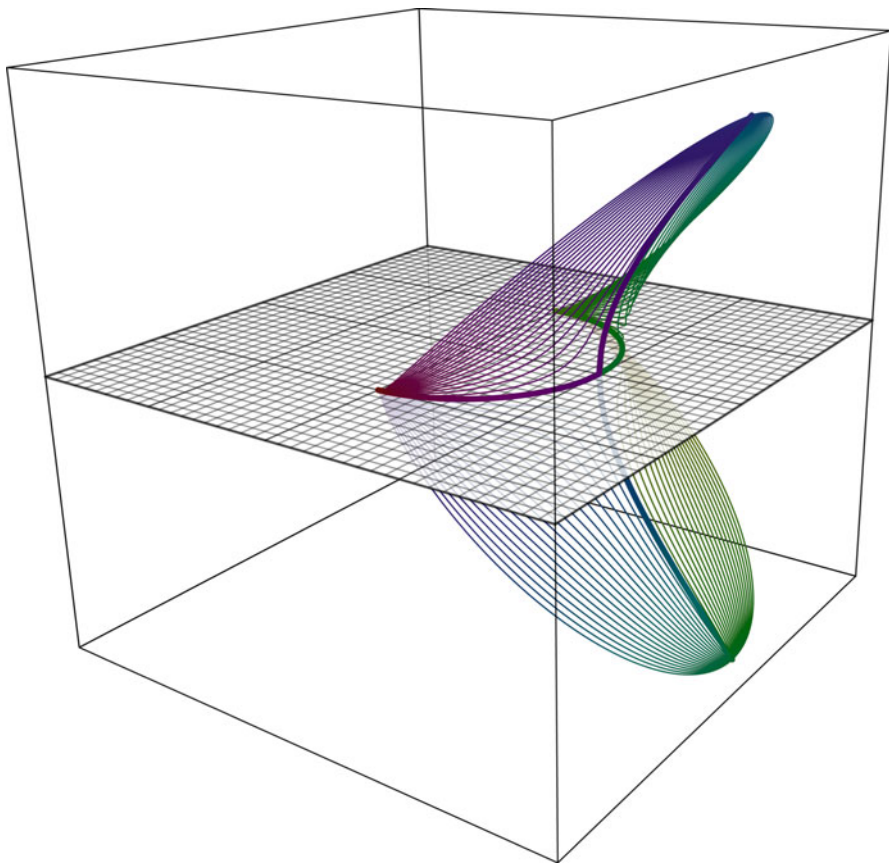
have critical situations. For all other  $a$  the two intersections have a complex  $y$ -coordinate. If we move the controlling parameter  $a$  continuously from 0 to 2 along the real axis, we will necessarily pass through the critical situation at  $a = 1$ . However, we may take a detour through complex space (for instance  $a = -e^{-it\pi} + 1$ ,  $t \in [0, 1]$ ) that easily avoids the singularity. Inserting this path into the  $y$ -coordinate of  $p_{\pm}$ , we get

$$y = \pm \sqrt{(1 - (-e^{-it\pi} + 1))^2} = \pm \sqrt{-e^{-2it\pi} + 2e^{-it\pi}}.$$

The expression under the square root describes a cycloid (the sum of two cyclic motions) whose absolute value is always greater than or equal to 1. So



**Fig. 27.16** The intersections of a line and a circle with complex  $y$ -coordinate. The left picture shows the situation for a real path of  $a$ ; the right picture shows the path of the intersections under a controlling path  $a(t) = 1 + \cos(t) + ib \sin(t)$ ,  $t \in [0, \pi]$ , for various values of  $b$ .



**Fig. 27.17** Now the control parameter  $a(t) = 1 + \cos(t) + ib\sin(t)$  is taken to traverse a full cycle  $t \in [0, 2\pi]$  for various values of  $b$ .

the two branches will never meet. Figure 27.15 (left) illustrates the geometric situation of the intersection, while the right picture shows the situation in the parameter space of  $a$ . The green path is the detour described above that avoids the singularity.

Figure 27.16 shows the situation in a way that visualizes the  $y$ -coordinates of the intersection in the full complex plane. The left picture shows the situation where the parameter  $a$  is moved continuously from 0 to 2 along the real axis. The two real branches that move along the circle meet at  $(1, 0)$  and branch off in two complex conjugate values. There is no reasonable way to assign the complex paths to the individual branches of the intersections, since the situation is completely symmetric. Choosing a path that avoids the singularity breaks the symmetry. The right picture shows the situation where the



parameter  $a$  is moved along a closed path  $a(t) = 1 + \cos(t) + ib \sin(t)$ ,  $t \in [0, 2\pi]$  for the values of  $b \in \{\frac{1}{20}, \frac{2}{20}, \dots, \frac{19}{20}, 1\}$ .

Finally, Figure 27.17 demonstrates what happens if the control parameter  $a$  describes a full round trip around the singularity. It is a remarkable fact that in this case, such a circular path of the control parameter causes the two intersections to continuously interchange their roles, without ever meeting.



# References

1. S. APEL, & J. RICHTER-GEBERT, *Cancellation patterns in Atutonated Geometric Theorem Proving*, extended abstract ADG 2010, 14p, 2010.
2. A. BACHEM & W. KERN, *Linear Programming Duality. An Introduction to Oriented Matroids*, Universitext, Springer-Verlag, Berlin Heidelberg 1992.
3. H. BEHNKE, F. BACHMANN, K. FLADT, & H. KUNLE, (EDS.), *Grundzüge der Mathematik – Band II Geometrie*, Vandenhoeck & Ruprecht, Göttingen, 1971. Also available as translated version *Fundamentals of Mathematics Vol. 2, Geometry*, MIT Press, 1974.
4. A. BELOW, V. KRUMMECK, & J. RICHTER-GEBERT, *Matroids with complex coefficients – phirotopes and their realizations in rank 2*, in *Discrete and Computational Geometry – The Goodman-Pollack Festschrift* B. Aronov, S. Basu, J. Pach, M. Sharir (eds), Algorithms and Combinatorics **25**, Springer Verlag, Berlin (2003), pp. 205-235.
5. R. BISCHOFF, *Geometry of Qubits*, Diploma Thesis, ETH Zurich (2003)
6. W. BLASCHKE: *Projective Geometry*, Birkhäuser, Basel, 1954.
7. A. BJÖRNER, M. LAS VERGNAS, B. STURMFELS, N. WHITE, G. M. ZIEGLER, *Oriented Matroids*, Cambridge University Press 1993, second ed. 1999
8. J.F. BLINN, *Lines in space*,  
Computer Graphics and Applications, IEEE  
*Part 1: The 4D cross product*, Volume 23, Issue 2 (March-April 2003) 84-91,  
*Part 2: The line formulation*, Volume 23, Issue 3 (May-June 2003) 72-79,  
*Part 3: The two matrices*, Volume 23, Issue 4 (July-Aug. 2003) 96-101,  
*Part 4: Back to the diagrams*, Volume 23, Issue 5 (Sept.-Oct. 2003) 84-93,  
*Part 5: A tale of two lines*, Volume 23, Issue 6 (Nov.-Dec. 2003) 84-97,  
*Part 6: Our Friend the Hyperbolic Paraboloid*, Volume 24, Issue 3 (May-Jun 2004) 92-100,  
*Part 7: The Algebra of Tinkertoys*, Volume 24, Issue 4 (July-Aug. 2004) 96-102,  
*Part 8: Line(s) through four lines*, Volume 24, Issue 5 (Sept.-Oct. 2004) 100-106.
9. J.F. BLINN, *Polynomial discriminants*,  
Computer Graphics and Applications, IEEE  
*Part 1: Matrix magic*, Volume 20, Issue 6 (Nov.-Dec. 2000) 94-98,  
*Part 2: Tensor diagrams*, Volume 21, Issue 1 (Jan-Feb 2001) 86-92.
10. J.F. BLINN, *Quartic discriminants and tensor invariants*, Computer Graphics and Applications, IEEE Volume 22, Issue 2 (March-April 2002) 86-91.
11. J.F. BLINN, *Uppers and downers*,  
Computer Graphics and Applications, IEEE  
*Part 1: Volume 12, Issue 2 (March 1992) 85-91,*  
*Part 2: Volume 12, Issue 3 (May 1992) 80-85.*

12. D. BOHM, A. AHARONOV, *Discussion of Experimental Proof for the Paradox of Einstein, Rosen and Podolsky*, Phys. Rev. **108** (1957), 1070 – 107.
13. J. BOKOWSKI Oriented matroids, Chapter 2.5 in: *Handbook of Convex Geometry* (eds. P. Gruber, J. Wills), North-Holland, Amsterdam, 1993, 555–602.
14. J. BOKOWSKI, J. RICHTER & B. STURMFELS, *Nonrealizability proofs in computational geometry*, Discrete Comput. Geometry, **5**, (1990), 333–350.
15. J. BOKOWSKI, & J. RICHTER, *On the finding of final polynomials*, Europ. J. Combinatorics, **11**, (1990), 21–34.
16. J. BOKOWSKI & B. STURMFELS, *Computational Synthetic Geometry, Lecture Notes in Mathematics 1355*, Springer-Verlag, Berlin Heidelberg 1989.
17. F. BOTANA & T. RECIO, (EDS.), *Automated Deduction in Geometry - ADG 2006 Revised Papers*, LNAI 4969. Springer-Verlag, Berlin Heidelberg, 2008.
18. M. BOUTIN & G. KEMPER, *On Reconstructing Configurations of Points in  $P^2$  from a Joint Distribution of Invariants*, Appl. Algebra Engrg. Comm. Comput. **15** (2005), 361–39.
19. E. BRIESKORN & H. KNÖRRER, *Plane Algebraic Curves*, Birkhäuser Verlag Basel, 1986.
20. R. H. BRUCK & H. J. RYSER, *The non-existence of certain finite projective planes*, Can. J. Math. **1** (1949), p. 88–93.
21. A. CAYLEY, *Sixth Memoire upon Quantics*, Philosophical Transactions of the Royal Society of London, **159**, (1859), p. 61–91.
22. S.C. CHOU, *Mechanical Geometry Theorem Proving*, D. Reidel Publishing Company, Dodrecht, Holland, 1988.
23. P. CVITANOVIĆ, *Group Theory Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press (2008).
24. W. CLIFFORD, *Extract of a Letter to Mr. Sylvester from Prof. Clifford of University College*, Am. J. Math., vol. 1, 1878, pp. 126–128.
25. H.S.M. COXETER, *Projective Geometry*, Springer, New York, Berlin, 1994 (orig. 1963).
26. H.S.M. COXETER, *The Real Projective Plane*, Springer, New York, 1992 (orig. 1949).
27. H.S.M. COXETER, *The Beauty of Geometry: Twelve Essays*, Dover 1968.
28. H.S.M. COXETER & S.L. GREITZER, *Geometry Revisited*, Mathematical Association of America, Washington, DC, 1967.
29. H. CRAPO, *Invariant-Theoretic Methods in Scene Analysis and Structural Mechanics*, J. Symb. Comput., **11**, (1991), 523–548.
30. H. CRAPO & J. RICHTER-GEBERT, *Automatic proving of geometric theorems*, in: “Invariant Methods in Discrete and Computational Geometry”, Neil White ed., Kluwer Academic Publishers, Dodrecht, (1995), 107–139.
31. H. CRAPO & J. RYAN, *Spatial realizations of linear scenes*, Structural Topology, **13**, (1986), 33–68.
32. G. DESARGUES, *Oeuvres de Desargues réunies et analysées par M. Poudra*, 2 vols, 1864, Paris.
33. P. DOUBILET, G.-C. ROTA & J. STEIN, *On the foundations of combinatorial theory. IX.*, Combinatorial methods in invariant theory, Studies in Appl. Math. **53** (1974), 185–216.
34. A.W.M. DRESS & W. WENZEL, *Endliche Matroide mit Koeffizienten*, Bayreuth. Math. Scr., **24** (1978), 94–123.
35. A.W.M. DRESS & W. WENZEL, *Geometric Algebra for Combinatorial Geometries*, Adv. in Math. **77** (1989), 1–36.
36. A.W.M. DRESS & W. WENZEL, *Grassmann-Plücker Relations and Matroids with Coefficients*, Adv. in Math. **86** (1991), 68–110.
37. A. EINSTEIN, B. PODOLSKY & N. ROSEN, *Can quantum-mechanical description of physical reality be considered complete?* Physical Review, **41** (1935), 777 – 180.
38. D. EISENBUD, M. GREEN, & J. HARRIS, *Cayley-Bacharach Theorems and Conjectures*, Bulletin (New Series) of the AMS **33**, 3, (1996), 295–324.

39. D. FEARNLEY-SANDER, *Plane Euclidian Reasoning*, in GAO, X.-S., YANG L. & WANG, D. (EDS.), *Automated Deduction in Geometry - ADG 1998 Proceedings*. LNAI 1669. Springer-Verlag, Berlin Heidelberg, 1999, 86-110.
40. G. FISCHER, *Plane algebraic curves*, *Student Mathematical Library*, 15, American Mathematical Society, 2001.
41. J. FOLKMAN & J. LAWRENCE, *Oriented matroids*, *J. Combinatorial Theory Ser. B* **25** (1978), 199-236.
42. M. V. GAGERN, *Morenaments*, [www.morenaments.de](http://www.morenaments.de).
43. M. V. GAGERN & J. RICHTER-GEBERT, *Hyperbolization of Euclidean Ornaments*, In *The Anders Björner Festschrift*, Hultman, A., Linusson, S., Ziegler, G.M (eds) *The Electronic Journal of Combinatorics* **16** (2009), R12.
44. L.E. GARNER, *An Outline of Projective Geometry*, North Holland, New York, Oxford, 1981.
45. X.-S. GAO, L. YANG & D. WANG, (EDS.), *Automated Deduction in Geometry - ADG 1998 Proceedings*, LNAI 1669. Springer-Verlag, Berlin Heidelberg, 1999.
46. H. GRASSMANN, *Die lineare Ausdehnungslehre*, Verlag Otto Wigand Leipzig 1844, 2. Aufl., 1878.
47. H. GRASSMANN, *Die Ausdehnungslehre*, Berlin 1862.
48. J. GRAY, *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*, *Springer Undergraduate Mathematics Series*, Springer 2006.
49. M.J. GREENBERG, *Euclidean and non-Euclidean Geometries*, (3rd ed.), Freeman and Company, New York, 1996 (orig. 1974).
50. B. GRÜNBAUM, *Arrangements and Spreads*, Amer. Math. Soc. Regional Conference Series in Mathematics **10**, Rhode Island 1972.
51. B. GRÜNBAUM & G.C. SHEPHARD, *Ceva, Menelaus, and the Area Principle*, *Mathematics Magazine*, **68** (1995), 254-268.
52. B. GRÜNBAUM & G.C. SHEPHARD, *A new Ceva-type theorem*, *Math. Gazette* **80** (1996), 492-500.
53. B. GRÜNBAUM & G.C. SHEPHARD, *Ceva, Menelaus, and Selftransversality*, *Geometriae Dedicata*, **65** (1997), 179-192.
54. B. GRÜNBAUM & G.C. SHEPHARD, *Some New Transversality Properties*, *Geometriae Dedicata*, **71** (1998), 179-208.
55. G.B. GUREVICH, *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff, Groningen (1964).
56. TH. HAWKINS, *Hesses's principle of transfer and the representation of lie algebras* *Archive for History of Exact Sciences*, **39**, 1, 1988, 41-73.
57. L.O. HESSE, *Ein Uebertragungsprinzip*, *Jl. für reine u. angew. Math.*, **66**, (1866), 15-21.
58. D. HILBERT, *Grundlagen der Geometrie*, Leipzig 1899, reprinted Teubner, Stuttgart 1999.
59. D. HILBERT & S. COHN-VOSSEN, *Geometry and the Imagination*, original *Anschauliche Geometrie*, Springer 1932, reprinted, AMS Chelsea Publishing, 1999.
60. H. HONG & D.WANG (EDS.), *Automated Deduction in Geometry - ADG 2004 Revised Papers*, LNAI 3763. Springer-Verlag, Berlin Heidelberg, 2006.
61. D. JÖRGENSEN, *Der Rechenmeister*, Aufbau dv, 1999.
62. L. KAUFFMAN, *Knots and Physics: Series on Knots and Everything (Series on Knots and Everything, Vol.1)*, World Scientific Publishing Co Pte Ltd, 1991.
63. G. KATZ, *Curves in Cages: An Algebro-Geometric Zoo*, *American Mathematical Monthly*, **113**, Number 9, 2006, 777-791.
64. F. KLEIN, *Über die Transformationen der allgemeinen Gleichung des zweiten Grades zwischen Linien-Coordinaten auf eine canonische Form*, Inauguraldissertation, Bonn, 1968.
65. F. KLEIN, *Das Erlanger Programm : Vergleichende Betrachtungen ber neuere geometrische Forschungen*, 1871, reprinted in Verlag Harry Deutsch, Frankfurt/ Main 1995.

66. F. KLEIN, *Über die so-genannte nicht-euklidische Geometrie*, Mathematische Annalen, **4** (1871), 573–625.
67. F. KLEIN, *Über die so-genannte nicht-euklidische Geometrie*, Mathematische Annalen, **6** (1873), 112–145.
68. F. KLEIN, *Vorlesungen über nicht-euklidische Geometrie*, Springer, Berlin, reprinted 1968 (orig. 1928).
69. F. KLEIN, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Springer, Heidelberg, 1928.
70. F. KLEIN, *Development of mathematics in the 19th century* (1928), reprinted and translated in Math Science Press, 1970.
71. D. E. KNUTH, Axioms and Hulls, *Lecture Notes in Computer Science* **606**, Springer-Verlag, Berlin Heidelberg 1992.
72. U. KORTENKAMP, *Foundations of dynamic geometry*, PhD-thesis, ETH Zürich, 1999, <http://www.inf.fu-berlin.de/~kortenka/Papers/diss.pdf>.
73. U. KORTENKAMP & J. RICHTER-GEBERT, *Complexity issues in dynamic geometry*, In *Festschrift in the honor of Stephen Smale's 70th birthday*, M. Rojas, F. Cucker (eds.), World Scientific, pp 355–404, (2002).
74. U. KORTENKAMP AND J. RICHTER-GEBERT: *Grundlagen dynamischer Geometrie*, Zeichnung - Figur - Zugfigur (2001) 123–144.
75. G. KOWOL, *Projektive Geometrie und Cayley-Klein Geometrien der Ebene*, Birkhäuser, 2009.
76. B. KUTZLER & S. STIFTER, *On the application of Buchberger's algorithm to automated geometry theorem proving*, J. Symb. Comput., **2**, (1986), 389–297.
77. E. LAGUERRE, *Sur la théorie des foyers*, Nouv. Ann. Math., **12** (1853) 57–66.
78. C. W. H. LAM, L. H. THIEL & S. SWIERCZ, *The Non-existence of Finite Projective Planes of Order 10*, *Canad. J. Math.*, **41** (1989), 1117–1123.
79. C. W. H. LAM, L. H. THIEL, AND S. SWIERCZ, *The search for a finite projective planes of order 10*, *Amer. Math. Monthly*, **98** (1991), 305–318.
80. M. LAS VERGNAS, *Convexity in oriented matroids*, J. Combinatorial Theory, Ser. B **29** (1980), 231–243.
81. R. LAUFFER, *Die nichtkonstruierbare Konfiguration (10<sub>3</sub>)*, Math. Nachr., **11**, (1954), 303–304.
82. P. LEBMEIR & J. RICHTER-GEBERT, *Recognition of Computationally Constructed Loci*, Lecture Notes in Computer Science/Artificial Intelligence, Vol. 4869 (2007) 52–68.
83. P. LEBMEIR & J. RICHTER-GEBERT, *Rotations, Translations and Symmetry Detection for Complexified Curves*, Computer Aided Geometric Design (special issue *Classical Techniques for Applied Geometry*), **25**, 2008, 707–719.
84. P. LEBMEIR & J. RICHTER-GEBERT: *Diagrams, tensors and geometric reasoning*, Discrete Comput. Geom. **42**, No. 2, (2009), 305–334.
85. F. LEVI, *Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade*, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss., **78** (1926), 256–267.
86. D. LIEBSCHER, *Einsteins Relativitätstheorie und die Geometrien der Ebene*, Teubner Verlag, 1999.
87. S.B. MAURER, *Matroid basis graphs I*, J. Combin. Theory B, **26** (1979), 159–173.
88. J. EDMONDS & A. MANDEL, *Topology of oriented matroids*, Ph.D. Thesis of A. Mandel, University of Waterloo 1982, 333 pages.
89. G. MONGE, *Traité de géométrie descriptive*, 1811, Paris.
90. M. NIELSEN & I. CHUANG, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
91. A.L. ONISHCHIK & R. SULANKE, *Projective and Cayley-Klein Geometries*, Springer Monographs in Mathematics Springer, 2006.
92. J. OXLEY, *Matroid Theory*, Oxford University Press, Oxford 1992.
93. R. MACPHERSON, & M. MCCONNELL, *Classical projective geometry and modular varieties*, in Algebraic Analysis, Geometry and Number Theory: Proceedings of the JAMI Inaugural Conference, ed. Jun-Ichi Igusa, Hohn Hopkins U. Press, (1989), 237–290.

94. D. MUMFORD, C. SERIES & D. WRIGHT, *Indras Pearls – the Vision of Felix Klein*, Cambridge University Press 2002.
95. C.W. O'HARA & D.R. WARD, *An Introduction to Projektive Geometry*, Oxford at the Clarendon Press, 1937.
96. N.E. MNĚV, *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties*, in: Viro, O.Ya. (ed.): *Topology and Geometry — Rohlin Seminar*, Lecture Notes in Mathematics **1346**, Springer, Heidelberg 1988, 527–544.
97. N.E. MNĚV, *The universality theorems on the oriented matroid stratification of the space of real matrices*, in: *Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift* (P. Gritzmann, B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc., Providence, RI, **6** (1991), 237–243.
98. P. J. OLVER, *Classical Invariant Theory*, London Mathematical Society Student Texts 44, Cambridge: Cambridge University Press, 1999.
99. J. PLÜCKER, *Über ein neues Coordinatensystem*, *Journal für reine und angewandte Mathematik*, **5**, (1830), 1–36.
100. J. PLÜCKER, *System der analytischen Geometrie*, Berlin, 1935.
101. J. PLÜCKER, *System der algebraischen Curven*, Bonn, 1939.
102. J. PLÜCKER, *Neue Geometrie des Raumes gegndet auf die Betrachtung der geraden Linie als Raumelement*, Teubner, Leibzig, 1868.
103. J.V. PONCELET, *Traité des propriétés projectives des figures*, Gauthier-Villars, Paris 1822.
104. H. REICHHARDT, *Gauss und die Anfänge der nicht-euklidischen Geometrie*, B.G. Teubner ; Wien ; New York : Distributed by Springer, c1985.
105. J. RICHTER, *Kombinatorische Realisierbarkeitskriterien für orientierte Matroide*, *Mitteilungen Math. Seminar Gießen* **194** (1989).
106. J. RICHTER-GEBERT, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics **1643**, Springer, Heidelberg 1996.
107. J. RICHTER-GEBERT, *Mněv's universality theorem revisited*, *Proceedings of the Séminaire Lotharingien de Combinatoire* 1995, 211–225.
108. J. RICHTER-GEBERT, *Universality Theorems for Oriented Matroids and Polytopes*, *Contemporary Mathematics*, **223**, 1999, 269–292.
109. J. RICHTER-GEBERT, *Mechanical theorem proving in projective geometry*, *Annals of Mathematics and Artificial Intelligence* **13** (1995), 139–172.
110. J. RICHTER-GEBERT, *Meditations on Ceva's Theorem*, In *The Coxeter Legacy: Reflections and Projections* (Eds. Chandler Davis & Eric Ellers, American Mathematical Society, Fields Institute), 227–254, 2006.
111. J. RICHTER-GEBERT & TH. ORENDT, *Geometrikalküle*, Springer, 2009.
112. J. RICHTER-GEBERT & U. KORTENKAMP, *Cinderella - The interactive geometry software*, Springer 1999.
113. J. RICHTER-GEBERT & U. KORTENKAMP, *Cinderella - The interactive geometry software*, 2006 <http://www.cinderella.de>.
114. J. Richter-Gebert & P. Lebmeir, *Diagrams, tensors and geometric reasoning*. *Discrete Comput. Geom.* **42**, No. 2, (2009), 305–334.
115. J. RICHTER-GEBERT & D.WANG (EDS.), *Automated Deduction in Geometry - ADG 2000 Revised Papers*, LNAI 2061. Springer-Verlag, Berlin Heidelberg, 2001.
116. J. RICHTER-GEBERT & G.M. ZIEGLER, *Oriented Matroids*, *CRC Handbook on "Discrete and Computational Geometry"* (J.E. Goodman, J. O'Rourke, eds.) CRC Press, Boca Raton, New York, (1997), 111–132.
117. G. RINGEL, *Teilungen der Ebene durch Geraden oder topologische Geraden*, *Math. Zeitschrift* **64** (1956), 79–102.
118. A. SAAM, *Ein neuer Schließungssatz für die projektive Ebene*, *Journal of Geometry* **29** (1987), 36–42.

119. A. SAAM, *Schließungssätze als Eigenschaften von Projektivitäten*, Journal of Geometry **32** (1988), 86–130.
120. G.C. SHEPHARD, *Cyclic Product Theorems for Polygons (I) Constructions using Circles*, Discrete Comput. Geom., **24** (2000), 551–571.
121. P. SHOR, *Stretchability of pseudolines is NP-hard*, in: *Applied Geometry and Discrete Mathematics – The Victor Klee Festschrift* (P. Gritzmann, B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Amer. Math. Soc., Providence, RI, **4** (1991), 531–554.
122. P. STÄCKEL ED., *Geometrische Untersuchungen mit Unterstützung der Ungarischen Akademie der Wissenschaften*, Magyar Tudományos Akadémia, Leipzig, Berlin: B.G. Teubner, 1913.
123. G.E. STEDMAN, *Diagram techniques in group theory*, Cambridge University Press, New York, 1990.
124. J. STOLFI, *Oriented Projective Geometry*, Academic Press, 1991.
125. J. STILWELL, *Sources of Hyperbolic Geometry*, (History of Mathematics, V. 10), American Mathematical Society, 1996.
126. B. STURMFELS: *Algorithms in Invariant Theory*, Springer-Verlag Wien New York, 1993.
127. A. SUDBERY, *On local invariants of three-qubit states*, J. Phys. A, 34:643–652, 2000. (quant-ph/0001116).
128. J. SYLVESTER, *On an Application of the New Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics, with Three Appendices*, Am. J. Math., vol. 1, 1878, pp. 64–125.
129. B. L. VAN DER WAERDEN, *Zur algebraischen Geometrie. XV. Lösung des Charakteristikenproblems für Kegelschnitte*, Mathematische Annalen, **115**, 1938, 645–655.
130. W. WENZEL, *A Group-Theoretic Interpretation of Tutte's Homotopy Theory*, Adv. in Math. **77** (1989), 27–75.
131. H. WEYL, *The Classical Groups*, Princeton University Press, Princeton, New Jersey, 1939.
132. N. WHITE, *The Bracket Ring of Combinatorial Geometry I*, Transactions AMS **202** (1975), 79–95.
133. WINKLER, F. (ED), *Automated Deduction in Geometry - ADG 2002 Revised Papers*, LNAI 2930. Springer-Verlag, Berlin Heidelberg, 2004.
134. W.T. WU, *On the decision problem and the mechanization of theorem-proving in elementary geometry*, Contemp. Math. **29**, (1984), 213–234.
135. W.T. WU, *Basic principles of mechanical theorem proving in elementary geometries*, J. Syst. Sci. Math. Sci., **4**, (1984), 207–235.
136. I.M. YAGLOM, *A simple non-Euclidean geometry and its physical basis : an elementary account of Galilean geometry and the Galilean principle of relativity*, Abe Shenitzer (trans.), Springer-Verlag New York, 1979.
137. I.M. YAGLOM, Felix Klein and Sophus Lie - Evolution of the Idea of Symmetry in the Nineteenth Century, Birkhäuser, Boston, Basel, 1988.
138. G. M. ZIEGLER, *Lectures on Polytopes*, Graduate Texts in Mathematics **152**, Springer-Verlag, New York 1995; *Updates, corrections, and more* available per WWW from <http://www.math.tu-berlin.de/~ziegler>



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