AN ALTERNATE METHOD TO FIND THE CHROMATIC NUMBER OF A FINITE, CONNECTED GRAPH

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Abstract

A new algorithm to obtain the chromatic number of a finite, connected graph is proposed in this paper. The algorithm is based on contraction of non adjacent vertices.

1. Introduction

Coloring a graph with minimum colors has many applications. Hence the research in this direction is very active. It is well known that finding the chromatic number problem is NP-complete for a general graph. There are many techniques to solve the chromatic number problem. Grotschel proposed a polynomial time algorithm to find the chromatic number of a perfect graph based on ellipsoid method([1,2,3,4]). Hermann and Hertz in [5], found the chromatic number by means of critical graphs. In [6] the chromatic number is found in $O(2.4023^n)$ using maximal independent sets.

Also the decision version of graph coloring problem is to determine whether the given graph can be colored with k colors. This problem is NP hard. It is known that if one can find whether a given graph can be colored optimally with k colors in polynomial time, a polynomial time algorithm can be found for coloring this graph. In this way also graph coloring optimization problem becomes important. There are polynomial time algorithms to color some smaller classes of graphs. [8, 9, 10].

In this paper we propose an algorithm which uses contraction of nonadjacent edges which satisfy a neighborhood condition so as reach the chromatic number decision.

The remaining part of the paper is divided into 3 sections. The second section gives the basic definitions of chromatic polynomial, contraction of vertices and related operations performed using them. The third section states and proves results which are used in the justification of the algorithm and presents the algorithm. The fourth section tells about probable prospects and other related open problems

2. Basic definitions

All graphs considered here are simple, finite and connected. The vertex set of G is denoted by $V(G)$. The chromatic number of a graph is denoted as $\chi(G)$. For $v \in V(G)$, $N_G(v)$ denotes the neighborhood of vertex v in graph G. The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors. In a graph G, if there exists a pair of non-adjacent vertices then an edge is added between them.

In a graph G, if there exists a pair of non-adjacent vertices u_0 and v_0, they are contracted to form a new vertex $u_{(v_0)}$, such that if vertex is adjacent to u_0 or v_0 in G, then it is adjacent to $u_{(v_0)}$ and all the parallel edges are replaced by a single edge.

Given a graph G, we perform contraction to obtain a new graph G' . Now contraction can be performed to G' to obtain a new graph G'' and thus this process is repeated. Now instead of choosing an arbitrary pair of non-adjacent vertices, a pair which satisfies the following condition is chosen for contraction.

Let $u_0 , v_0 \in V(G)$, be a pair of non-adjacent vertices such that

$$|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)| \quad \forall\, u , v \in V(G)$$

where (u , v) being a pair of non-adjacent vertices.

This process of repeatedly fusing the pair of non-adjacent vertices which satisfies the above condition is called operation 1.

3. Results

In this section we prove two theorems which characterize the graphs obtained as a result of repeated application of operations mentioned in the previous section on an input graph. These results are helpful in justifying the proposed new algorithm for finding the chromatic number of the input graph.

3.1 Theorem 1

G is k-colorable if K-clique can be obtained through a sequence of non-edge contractions.

Proof: Since G is k-colorable, there exists k-independent sets which partitions V(G). Now, contract each partition into a single vertex, the resulting graph is a k-clique.

The next two results will tell us how to obtain the smallest K-clique.

3.2 Theorem 2

Let G be any simple graph (connected or disconnected), and let $u_0, v_0 \in G$, be a pair of non-adjacent vertices such that

$$|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)| \ \forall\, u, v \in G$$

where (u, v) being a pair of non-adjacent vertices.

Let $G_0 = G - \{u_0, v_0\}$. Let l be the chromatic number of G_0 and $A_1, A_2, \ldots A_l \in P$ where P is the chromatic partitioning of G_0.

Then the following condition holds:

$$\begin{gathered} if\ \exists\, A_{(u_0)} \in P, such\ that \\ a \notin N_G(u_0), \forall\, a \in A_{(u_0)} \\ then\ a \notin N_g(v_0), \forall\, a \in A_{(u_0)} \end{gathered}$$

The above condition is called as ‘Condition 1’.

Proof: We prove this by induction on the number of vertices.

Let $P(n)$ be a statement such that

$$P(n): For\ any\ graph\ with\ n\ vertices\ the\ theorem\ holds\,.$$

Clearly, $P(3)$ holds. Assume $P(n)$ holds.

If $|N_G(u) \cap N_G(v)| = 0 \ \forall\, u, v \in G$, (u,v) being a pair of non-adjacent vertices, then G is a complete graph or a disjoint union of complete graphs. Let $|N_G(u) \cap N_G(v)| \geq 1$ for some $u, v \in G$ .Let $G_0 = G - \{u_0, v_0\}$, where $|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)|$.

Case (i):

G_0 is a complete graph, This means that each $A_i \in P, \forall\, 1 \leq i \leq l$, is a singleton set. As

$|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)|$, $|N_G(u_0) \cap N_G(v_0)| = (n+1) - 2$ or

$|N_G(u_0) \cap N_G(v_0)| = (n+1) - 3$, as $\forall\, v \in G_0, d(v) = (n+1) - 3$.

If $|N_G(u_0) \cap N_G(v_0)| = (n+1) - 2$, then $d(u_0) = d(v_0) = (n+1) - 2$ and

$\forall\, v \in G_0, v \in N_G(u_0) \cap N_G(v_0)$ and hence condition 1 is satisfied.

If $|N_G(u_0) \cap N_G(v_0)| = (n+1) - 3$, then without loss of generality, $d(u_0) \geq d(v_0) = (n+1) - 3$. But this means that $\forall\, v \in G_0$, if $v \notin N_G(u_0)$, $then\ v \notin N_G(v_0)$. Hence Condition 1 is satisfied

Case (ii):

G_0 is not complete. We prove the theorem by the method of contradiction. Assume there exists a graph G with $n + 1$ vertices such that $P(n+1)$ does not hold.

There exists a vertex $k \in G_0$ such that $G_1 = G_0 - \{k\}$; $\chi(G_0) = \chi(G_1)$ where $\chi(G)$ is the chromatic number of G. If $A_0, A_1, \ldots, A_i, \ldots, A_l \in P$ is the chromatic partitioning of G_0 and then $A_0, A_1, \ldots, A_i', \ldots, A_l \in P'$ is the chromatic partitioning of G_1, where $k \in A_i, k \notin A_i'$, l is the chromatic number of G_0 and G_1 .

Let $G' = G - \{k\}$, where $\chi(G') = \chi(G)$. Clearly, $0 \leq |N_G(u_0) \cap N_G(v_0)| - |N_{(G')}(u_0) \cap N_{(G')}(v_0)| \leq 1$. If $|N_G(u_0) \cap N_G(v_0)| = |N_{(G')}(u_0) \cap N_{(G')}(v_0)|$, and that result 1 does not hold for G ,then it does not hold for G' . But this contradicts $P(n)$, as G' has only n vertices. If $|N_G(u_0) \cap N_G(v_0)| = |N_{(G')}(u_0) \cap N_{(G')}(v_0)| + 1$, we assume $|N_{(G')}(u) \cap N_{(G')}(v)|$ is maximum at some pair (u_0', v_0'). The following modifications are done to G'
If (u_0', v_0') have different colours, add an edge between them.

If they don't, then

$$\exists\, k' \in N_{(G')}(u_0') \cap N_{(G')}(v_0')$$
$$but\ k' \notin N_{(G')}(u_0) \cap N_{(G')}(v_0)$$

Either the edge (k', u_0') or (k', v_0') is removed but not both. By doing the above, a modified graph G' is obtained such that $|N_{(G')}(u_0) \cap N_{(G')}(v_0)|$ is maximum which implies that result 1 does not hold for G' which is a contradiction. Hence the proof is complete.

3.3 Theorem 3

A complete graph is obtained by performing operation 1 on a finite, connected graph G. It is unique and has chromatic number same as that of G.

Proof: Let G have n vertices. Let G' be formed from G by fusing vertices u_0, v_0 which satisfy the condition

$|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)|$ where u, v are non-adjacent vertices.

Claim: $\chi(G) = \chi(G')$

Let G' have chromatic number k . By the way G' was constructed, G can be properly colored with k colors. We are done if we show that G can not be properly colored with $k - 1$ colors.

Assume G can be properly colored with $k - 1$ colors. u_0, v_0 are fused to give a vertex $u_{(v_0)}$ where $u_{(v_0)} \in V(G')$. Let $G'' = G - \{u_0, v_0\}$. Clearly $G'' = G' - \{u_{(v_0)}\}$ and thus G'' is a sub graph of both G and G' . Let $\chi(G'') = l$ and $A_1, A_2, A_3 \ldots, A_l \in P$ be the chromatic partitioning of G''. We note G is formed by adding vertices u_0, v_0 to G'' and G' is formed by adding vertex $u_{(v_0)}$ to G'' .

This implies $\chi(G'') = \chi(G) = \chi(G') - 1$ (Adding a vertex can increase the chromatic number of a graph by at most 1). For the above condition to happen, the negation of condition 1 of result 1 occurs. This means that $|N_G(u_0) \cap N_G(v_0)|$ is not maximum which is a contradiction.
Hence our claim is proved.

By the way operation 1 is performed, it is found that $\chi(K_m) = \chi(G)$. This implies $m = \chi(G)$. Hence we have proved the result.

3.4 Algorithm

With the help of the theorems proved above , an algorithm is designed to find the chromatic number of a finite, connected graph.

Step1: Take a Graph G

Step2: If G is complete stop,

else find a pair of non-adjacent vertices u_0 , v_0 which satisfy the condition

$|N_G(u_0) \cap N_G(v_0)| \geq |N_G(u) \cap N_G(v)|$ where u , v are non-adjacent vertices.

Step3:Fuse $u_0 , v_0$to result in a graph G'

Step4: Repeat Step 2 to G'

Step5:The number of vertices in the complete graph is the chromatic number of the

graph G

Prospects and open problem

The application of the algorithm presented increases the motivation for its development and leads to an interesting research. It shows that to find the chromatic number, one need not find the color assignment of the vertices. We can develop the idea to find which all subclasses of graphs gives the chromatic number on performing the presented algorithm on it, except that property of the non-adjacent vertex pair is different than the one presented. For example, if property of the pair is that every induced path between them has even number of edges, we get the subclass to be the set of perfect graphs as mentioned in Haddadene's and Issaidi'spaper [7]. The complexity of the above algorithm is not given.

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