# Chapter 8

# Graph colouring

## 8.1 Vertex colouring

A *(vertex) colouring* of a graph $G$ is a mapping $c : V(G) \to S$. The elements of $S$ are called *colours*; the vertices of one colour form a *colour class*. If $|S| = k$, we say that $c$ is a *k-colouring* (often we use $S = \{1, \dots, k\}$). A colouring is *proper* if adjacent vertices have different colours. A graph is *k-colourable* if it has a proper $k$-colouring. The *chromatic number* $\chi(G)$ is the least $k$ such that $G$ is $k$-colourable. Obviously, $\chi(G)$ exists as assigning distinct colours to vertices yields a proper $|V(G)|$-colouring. An *optimal colouring* of $G$ is a $\chi(G)$-colouring. A graph $G$ is *k-chromatic* if $\chi(G) = k$.

In a proper colouring, each colour class is a stable set. Hence a *k-colouring* may also be seen as a partition of the vertex set of $G$ into $k$ disjoint *stable sets* $S_i = \{v \mid c(v) = i\}$ for $1 \leq i \leq k$. Therefore $k$-colourable are also called *k-partite graphs*. Moreover, 2-colourable graphs are very often called *bipartite*.

Clearly, if $H$ is a subgraph of $G$ then any proper colouring of $G$ is a proper colouring of $H$.

**Proposition 8.1.** *If $H$ is a subgraph of $G$, then $\chi(H) \leq \chi(G)$.*

**Proposition 8.2.** $\chi(G) = \max\{\chi(C), C$ *connected component of* $G\}$.

*Proof.* Proposition 8.1 gives the inequality $\chi(G) \geq \max\{\chi(C), C$ connected component of $G\}$ because every connected component of $G$ is a subgraph of $G$.

Let us now prove the opposite inequality. Let $C_1, C_2, \dots, C_p$ be the connected components of $G$. For $1 \leq i \leq p$, let $c_i$ be a proper colouring of $C_i$ with colours $1, 2, \dots, \chi(C_i)$. Let $c$ be the union of the $c_i$ that is the colouring of $G$ defined by $c(v) = c_i(v)$ for all $v \in C_i$. Since there is no edge between two vertices in different connected component, $c$ is a proper colouring of $G$ with colours $1, 2, \dots, \max\{\chi(C_i) \mid 1 \leq i \leq p\}$. Hence $\chi(G) \leq \max\{\chi(C), C$ connected component of $G\}$. □

Often in the following, we will consider connected graphs.

### 8.1.1 Complexity

On the algorithmic point of view, one may wonder what is the complexity of computing the chromatic number of graph. Moreover, for evey fixed $k$, we might ask for the complexity of deciding if a graph is $k$-colourable.
$k$-COLOURABILITY
Input: A graph $G$.
Question: Is $G$ $k$-colourable?

The 1-colourable graphs are the empty graphs (i.e. graphs with no edges). The 2-colourable graphs are the bipartite graphs and can be characterized and recognized in polynomial time (See Section 2.3). Hence for $k \leq 2$, $k$-COLOURABILITY is polynomial-time solvable. In constrast, we now prove that $k$-COLOURABILITY is $\mathcal{NP}$-complete for all $k \geq 3$.

**Theorem 8.3** (Garey, Johnson and Stockmeyer [12]). 3-COLOURABILITY *is $\mathcal{NP}$-complete.*

*Proof.* 3-COLOURABILITY is clearly is $\mathcal{NP}$ as every proper 3-colouring is a certificate.

To prove its hardness, we use a reduction from 3-SAT.

Let $\Phi$ be a 3-SAT formula with $n$ variables $x_1, \dots, x_n$ and $m$ clauses $C_1, \dots, C_m$. Let us create a graph $G_\Phi$ as follows:

- Create a 3-cycle with vertices $v_{true}$, $v_{false}$ and $v_{base}$;
- For each variable $x_i$, create two vertices $x_i$ and $\bar{x}_i$, and create the 3-cycle $(v_{base}, v_i, \bar{v}_i)$;
- For each clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, add a small gadget graph $G_j$ containing the vertices $\ell_1$, $\ell_2$, $\ell_3$, $v_{false}$ and $v_{base}$ as depicted in Figure 8.1.

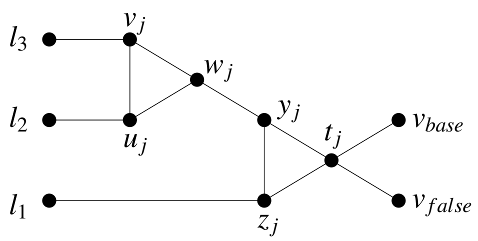



Figure 8.1: The clause gadget $G_j$

We claim that $\Phi$ is satisfiable if and only if $G_\Phi$ is 3-colourable.

Suppose first that $G_\Phi$ is 3-colourable. Let $c$ be proper colouring of $G_\Phi$ with the three colours $true$, $false$ and $base$. Without loss of generality, we may assume that $c(v_{true}) = true$, $c(v_{false}) = false$ and $c(v_{base}) = base$. Then for each variable $x_i$, $\{c(x_i), c(\bar{x}_i)\} = \{true, false\}$, so $c$ induces a truth assignment. Let us now prove that this truth assigment satisfies $\Phi$. Suppose for a contradiction, that a clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ is not satisfied. Then $c(\ell_1) = c(\ell_2) = c(\ell_3) = false$.

Hence $\{c(u_j), c(v_j)\} = \{base, true\}$, so $c(w_j) = false$. Now $c(t_j) = true$, because it is adajcent to $v_{false}$ and $v_{base}$, so $c(y_j) = base$. But $z_j$ is adjacent to $\ell_1$, $t_j$ and $y_j$ which are coloured $false$, $true$ and $base$ respectively. This contradicts the fact that $c$ is proper.

Reciprocally, suppose that $\Phi$ is satisfiable. Let $f$ be a truth assignment satisfying $\Phi$. Then $f$ defines a colouring of the $x_i$ and $\bar{x}_i$ with colours $true$ and $false$. This colouring can be extended properly by setting $f(v_{true}) = true$, $f(v_{false}) = false$ and $f(v_{base}) = base$. One can then check that this colouring can be extended properly to each $G_j$ using the colours three colours $true$, $false$, and $base$ (Exercise 8.2). □

**Corollary 8.4.** *For all $k \geq 3$, $k$-COLOURABILITY is $\mathcal{NP}$-complete.*

*Proof.* We can easily reduce 3-COLOURABILITY to $k$-COLOURABILITY. Let $G$ be a graph. Let $H$ be the graph obtained from the disjoint union of $G$ and the complete graph $K_{k-3}$ by adding all edges between $V(G)$ and $V(K_{k-3})$. Easily $G$ is 3-colourable if and only if $H$ is $k$-colourable. □

Furthermore, it is $\mathcal{NP}$-hard to approximate the chromatic number within $|V(G)|^{\varepsilon_0}$ for some positive constant $\varepsilon_0$ as shown by Lund and Yannakakis [19].

### 8.1.2 Lower bounds for $\chi(G)$

Clearly, the complete graph $K_n$ requires $n$ colours, so $\chi(K_n) = n$. Together with Proposition 8.1, it yields the following.

**Proposition 8.5.** $\chi(G) \geq \omega(G)$.

This bound can be tight, but it can also be very loose. Indeed, for any given integers $k \leq l$, there are graphs with clique number $k$ and chromatic number $l$. For example, the fact that a graph can be triangle-free ($\omega(G) \leq 2$) and yet have a large chromatic number has been established by a number of mathematicians including Descartes (alias Tutte) [7] (See Exercise 8.8), Kelly and Kelly [17] and Zykov [29]. We present here a proof of this fact due to Mycielski [21].

**Theorem 8.6.** *For every positive integer k, there exists a triangle-free k-chromatic graph.*

*Proof.* The *Mycielski graphs* $M_k$, $k \geq 1$ are defined inductively as follows. $M_1 = K_1$ and $M_2 = K_2$. For $k \geq 2$, let $V(M_k) = \{v_1, v_2, \dots, v_n\}$. The graph $M_{k+1}$ is defined by $V(M_{k+1}) = V(M_k) \cup \{w_1, w_2, \dots, w_n, z\}$ and $E(M_{k+1}) = E(M_k) \cup \{w_i v_j, v_i v_j \in E(M_k)\} \cup \{w_i z, 1 \leq i \leq n\}$. See Figure 8.2.

Let us show by induction on $k \geq 1$ that $M_{k+1}$ is triangle-free and $k$-chromatic, the result holding trivially for $k = 1$.

Let us first show that it is triangle-free. $W = \{w_1, w_2, \dots, w_n\}$ is a stable set of $M_{k+1}$ and $z$ is adjacent to no vertex of $M_k$. So $z$ is in no triangle. In addition, if there is a triangle $T$ in $M_{k+1}$, then two of the three vertices must belong to $M_k$ and the third vertex must belong to $W$, say $V(T) = \{w_i, v_j, v_k\}$. Since $w_i$ is adjacent to $v_j$ and $v_k$, by definition of $M_{k+1}$ it follows that $v_i$ is also adjacent to $v_j$ and $v_k$. Hence $v_i$, $v_j$ and $v_k$ induce a triangle in $M_k$, which is a contradiction. Thus, as claimed, $M_{k+1}$ is triangle-free.

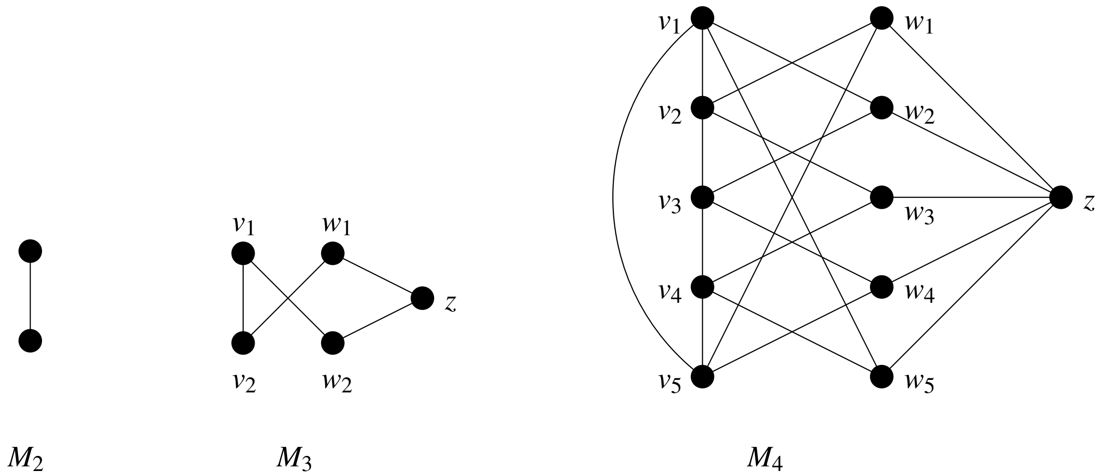



Figure 8.2: The graphs $M_2$, $M_3$ and $M_4$.

Next, we show that $\chi(M_{k+1}) = k+1$. Since $M_k$ is a subgraph of $M_{k+1}$, by Proposition 8.1, $\chi(M_{k+1}) \geq k$. Let a $k$-colouring of $H$ in $\{1,\dots,k\}$ be given. Assign to $w_i$ the same colour that is assigned to $v_i$ for $1 \leq i \leq n$ and assign $k+1$ to $z$. The obtained colouring is a proper $(k+1)$-colouring of $M_{k+1}$ and so $\chi(M_{k+1}) \leq k+1$. Suppose that $\chi(M_{k+1}) = k$. Then there is a $k$-colouring of $M_{k+1}$ with colours $\{1,\dots,k\}$. Without loss of generality, we may ssume that $z$ is coloured $k$. Then no vertex of $W$ is coloured $k$. For each vertex $v_i$ of $M_k$ coloured $k$, recolour it with the colour assigned to $w_i$. Because the neighbours of $v_i$ are also neighbours of $w_i$, this produces a proper colouring of $M_k$. Moreover this colouring uses $k-1$ colours. This is a contradiction, thus $\chi(M_{k+1}) \geq k+1$ and so $\chi(M_{k+1}) = k+1$. □

**Proposition 8.7.** $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.

*Proof.* Let $c$ be a proper colouring of $G$ with colours $1,2,\dots,\chi(G)$. For $1 \leq i \leq \chi(G)$, let $S_i$ be the stable set of vertices coloured $i$. Then $|S_i| \leq \alpha(G)$. So

$$|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G) \leq \chi(G).\alpha(G).$$

Hence $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. □

Again this bound can be very loose. For example, consider a graph $G$ with $n$ connected components all of which are isomorphic to $K_1$ except one which is isomorphic to $K_k$. Then $\chi(G) = k$ and $\frac{|V(G)|}{\alpha(G)} = \frac{n+k-1}{n}$ which is less than 2 for $n$ sufficiently large.

## 8.2 Chromatic number and maximum degree

Most upper bounds on the chromatic number come from algorithms that produce colourings. The most widespread one is the greedy algorithm. A *greedy colouring* relative to a vertex ordering $(v_1 < \cdots < v_n)$ of $V(G)$ is obtained by colouring the vertices in the order $v_1, \ldots, v_n$, assigning to $v_i$ the smallest-indexed colour not already used on its lower-indexed neighbourhood. In a vertex-ordering, each vertex has at most $\Delta(G)$ earlier neighbours, so the greedy colouring cannot be forced to use more than $\Delta(G)+1$ colours.

**Proposition 8.8.** $\chi(G) \leq \Delta(G)+1$.

The bound $\Delta(G)+1$ is the worst number of colours that a greedy colouring can have. However there is a vertex ordering whose associated colouring is optimal colouring. Indeed, if $c$ is an optimal colouring of $G$, then any ordering $\sigma_{opt} = (v_1 < \cdots < v_n)$ such that for any $i < j$, $c(v_i) \leq c(v_j)$ will be. But finding such an ordering among the $n!$ possible orderings is difficult because it is $\mathcal{NP}$-hard to determine the chromatic number of a graph.

The bound $\Delta(G)+1$ may be lowered by finding orderings yielding a greedy colouring with less than $\Delta(G)+1$ colours. A graph $G$ is *$k$-degenerate* if each of its subgraphs has a vertex of degree at most $k$. The *degeneracy* of $G$, denoted $\delta^*(G)$, is the smallest $k$ such that $G$ is $k$-degenerate. It is easy to see that a graph is $k$-degenerate if and only if there is an ordering $(v_1 < v_2 < \cdots < v_n)$ of the vertices such that for every $1 < i \leq n$, the vertex $v_i$ has at most $k$ neighbours in $\{v_1, \ldots, v_{i-1}\}$. Hence the greedy colouring relative to this ordering uses at most $\delta^*(G)+1$ colours.

**Proposition 8.9.**

$$\chi(G) \leq \delta^*(G)+1.$$

Note that finding an ordering as above (and thus the degeneracy of a graph) is easy. It suffices to recursively take a vertex $v_n$ of minimum degree in the graph and to put it at the end of the ordering $v_1, \ldots, v_{n-1}$ of $G - v_n$.

**Proposition 8.10.** *Let $G$ be a connected graph. Then $\delta^*(G) = \Delta(G)$ if and only if $G$ is regular.*

*Proof.* Assume first that $G$ is regular. Then for all ordering the last vertex has $\Delta(G)$ lower indexed neighbours. So $\delta^*(G) = \Delta(G)$.

Assume now that $G$ is not $\Delta$-regular. Let $v_n$ be a vertex of degree less than $\Delta$. Since $G$ is connected, one can grow a spanning tree of $G$ from $v_n$, assigning indices in decreasing order as we reach vertices. We obtain an ordering $v_1 < \cdots < v_n$ such that every vertex other than $v_n$ has a higher-indexed neighbour. Hence $\delta^*(G) < \Delta(G)$. □

This proposition and Proposition 8.8 implies the following.

**Corollary 8.11.** *Let $G$ be a connected graph. If $G$ is not regular, then $\chi(G) \leq \Delta(G)$.*

In view of this corollary, one may wonder which connected graphs $G$ satisfies $\chi(G) = \Delta(G)$. It is the case for complete graphs. On can also easily see that it is also the case for odd cycles. Brooks showed [6] that they are the only ones.

**Theorem 8.12** (BROOKS' THEOREM)**.** *Let $G$ be a connected graph. Then $\chi(G) \leq \Delta(G)$ unless $G$ is either a complete graph or an odd cycle.*

In order to prove this theorem, we need the following preliminary results.

**Proposition 8.13.** *Let $G$ be a connected graph which is not a complete graph. Then there exists three vertices $u$, $v$ and $w$ such that $uv \in E(G)$, $vw \in E(G)$ and $uw \notin E(G)$.*

*Proof.* Since $G$ is not complete, there exists two vertices $u$ and $u'$ which are not linked by an edge. Because $G$ is connected, there is a path between $u$ and $u'$. Let $P$ be a shortest $(u,u')$-path and let $v$ and $w$ be respectively the second and third vertices on $P$. Then $uv$ and $vw$ are edges of the paths and $uw$ is not an edge for otherwise it would shortcut $P$. □

**Lemma 8.14.** *Let $G$ be a 2-connected graph of minimum degree at least 3. If $G$ is not a complete graph, then there exists a vertex $x$ having two non-adjacent neighbours $v_1$ and $v_2$ such that $G - \{v_1, v_2\}$ is connected.*

*Proof.* If $G$ is 3-connected, then the results follows directly from Proposition 8.13.

So we may assume that $G$ is not 3-connected. Hence it as a separator of size 2, say $\{x,y\}$. Since $G$ is 2-connected, then $G - x$ is connected. Moreover $G - x$ has at least one separating vertex $y$ and this has two end-blocks $B_1$ and $B_2$. (See Exercise 5.27). Now, since $G$ is 2-connected, for $i = 1,2$, the separating vertex $y_i$ in $B_i$ is not a separating vertex of $G$ and thus $x$ is adjacent to a vertex $v_i$ of $B_i \setminus \{y_i\}$. Hence by Exercise 5.27 4), $G - \{x, v_1, v_2\}$ is connected. But $x$ has degree at least 3 in $G$, and so has a neighbour in $V(G) \setminus \{v_1, v_2\}$. Therefore, $G - \{v_1, v_2\}$ is connected. □

*Proof of Theorem 8.12.* If $G$ is not regular, Corollary 8.11 yields the result. So we may assume that $G$ is regular. In addition, we may assume that $\Delta = \Delta(G) \geq 3$, since $G$ is complete if $\Delta \leq 1$ and $G$ is a cycle when $\Delta = 2$, in which cases the result holds.

We shall find an ordering of the vertices so that the greedy colouring relative to it yields the desired bound.

Assume now that $G$ is $\Delta$-*regular*. If $G$ has a cut-vertex $x$. Let $C_1, C_2, \dots, C_p$ be the connected components of $G - x$. For $1 \leq i \leq p$ let $G_i$ be the graph induced by $V(C_i) \cup \{x\}$. Each of these graphs is not regular because $x$ has degree less than $\Delta$. So by Corollary 8.11, all the $G_i$, $1 \leq i \leq p$ have a proper $\Delta$-colouring. Free to permute the colours, one can assume that the colourings agree on $x$. Then the union of these colourings is a $\Delta$-colouring of $G$.

Hence we may assume that $G$ is 2-connected. In such a case, for $G$ is not complete, by Lemma 8.14 some vertex $v_n$ has two non-adjacent neighbours $v_1$ and $v_2$ such that $G - \{v_1, v_2\}$ is connected. Then indexing the vertices of a spanning tree of $G - \{v_1, v_2\}$ rooted in $v_n$ in a decreasing order, with $\{3, \dots, n\}$, we obtain an ordering $v_1, \dots, v_n$ such that every vertex other than $v_n$ has a higher-indexed neighbour. Now the greedy algorithm will assign colour 1 to both $v_1$ and $v_2$. So when colouring $v_n$ at most $\Delta - 1$ colours will be assigned to its neighbours. Hence the greedy colouring will use at most $\Delta$ colours. □

Note that the above proof is constructive and yields a polynomial-time algorithm for finding a $(\Delta(G) + 1)$-algorithm of a graph which is neither a complete graph nor an odd cycle.

Brooks' Theorem states that for $\Delta(G) > 2$, $\chi(G) = \Delta(G)+1$ if and only if $G$ contains a clique of size $\Delta(G)+1$. It is natural to ask whether this extends further. E. g. if $\chi(G) \geq \Delta+1-k$ does $G$ contain a large clique? One cannot expect a clique of size $\Delta+1-k$ if $k$ is large. Indeed consider the graph $H_{\Delta,p}$ formed by adding all the edges between a $(\Delta+1-5p)$-clique and $p$ disjoint 5-cycles. It is easy to see that $H_{\Delta,p}$ has maximum degree $\Delta$, chromatic number $\Delta+1-2p$ and clique number $\Delta+1-3p$. Reed [22] conjectured that if $\chi(G) \geq \Delta+1-k$ then $G$ contains a clique of size at least $\Delta+1-2k$.

**Conjecture 8.15** (REED'S CONJECTURE)**.** Let $G$ be a graph.
If $\chi(G) \geq \Delta(G)+1-k$, then $\omega(G) \geq \Delta(G)+1-2k$. In other words,

$$\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil .$$

Note that this value $2k$ is best possible. Indeed consider random graph $R$ on $n$ vertices with edge probability $(1-n^{-3/4})$. The expected number of cliques of size $i$ is

$$\begin{aligned} \binom{n}{i}\left(1-n^{-3/4}\right)^{\binom{i}{2}} &\leq 2^{i\log n}\left(1-n^{-3/4}\right)^{\frac{i^2}{4}} \\ &\leq 2^{i\log n}e^{-n^{-3/4}\frac{i^2}{4}}. \end{aligned}$$

For $i > n^{3/4}\log n$, this is $o(1)$ so (with high probability) $\omega(R) \leq n^{3/4}\log n$. Now the expected number of stable sets of size 3 is $\binom{n}{3} \times \left(n^{-3/4}\right)^3 = O(n^{3/4})$. Hence removing one vertex per such stable set, we obtain a graph $H$ with $n - O(n^{3/4})$ vertices and stability number $\alpha(H) = 2$. Hence its chromatic number is at least $n/2 - O(n^{3/4})$. Let $G$ be the graph obtained by connecting all the vertices of $H$ to a clique of size $\Delta - n$. Then $\Delta(G) = \Delta$, $\chi(G) = \Delta - n + \chi(H) \geq \Delta - n/2 - O(n^{3/4})$ and $\omega(G) \leq \Delta - n + \omega(G) \leq \Delta - n + n^{3/4}\log n$.

As an evidence for Conjecture 8.15, Reed [22] showed that there is an $\varepsilon > 0$ such that $\chi(G) \leq \varepsilon\omega(G) + (1-\varepsilon)(\Delta(G)+1)$. Johannson [16] settled Conjecture 8.15 for $\omega = 2$ and $\Delta$ sufficiently large. In fact, he proved that there is a constant $c$ such that if $\omega(G) = 2$ then $\chi(G) \leq c\frac{\Delta(G)}{\log\Delta(G)}$. Johannson's proof uses the probabilistic method and needs a careful probabilistic analysis. The interested reader is referred to Chapter 13 of [20]. However, one can easily improve the bound of Brooks' Theorem for triangle-free graphs.

**Proposition 8.16.** *Let $G$ be a triangle-free graph. Then* $\chi(G) \leq 3\left\lceil \frac{\Delta(G)+1}{4} \right\rceil$.

*Proof.* Set $k = \left\lceil \frac{\Delta(G)+1}{4} \right\rceil$. Let $(V_1, V_2, \ldots, V_k)$ be the partition of $V(G)$ in $k$ sets such that the number of internal edges (i.e. with two endvertices in a same part) is minimum. For all $i$, the graph $G_i$ induced by $V_i$ has maximum degree at most 3. Indeed, suppose that a vertex $x$ has 4 neighbours in the part it belongs to, say $V_1$. Then there is another part, say $V_2$, in which $x$ has at most 3 neighbours, otherwise $x$ would have at least $4k \geq \Delta(G)+1$ neighbours, which is impossible. Thus the partition $(V_1 - x, V_2 + x, \ldots, V_k)$ has less internal edges than $(V_1, V_2, \ldots, V_k)$ which contradicts the minimality of this.

Hence $\Delta(G_i) \leq 3$ and $\omega(G_i) \leq 2$ as it is a subgraph of $G$. Hence by Brooks' Theorem, $\chi(G_i) \leq 3$.

So colouring properly each $G_i$ with the colour set $\{3i-2, 3i-1, 3i\}$, we obtain a proper $3k$-colouring of $G$. □

**Remark 8.17.** Proposition 8.16 can be translated into a polynomial-time algorithm that colours a a triangle-free graph $G$ with at most $3\left\lceil\frac{\Delta(G)+1}{4}\right\rceil$. colours.

When $k = 1$ Conjecture 8.15 asserts that if $\chi(G) = \Delta(G)$ then $\omega(G) \geq \Delta - 1$. In fact, Reed [23] showed that when $\Delta$ is large if $\chi(G) = \Delta(G)$ then $\omega(G) = \Delta(G)$, thus settling a conjecture of Beutelspacher and Hering [4]. Borodin and Kostochka [5] conjectured that it is true for $\Delta \geq 9$; counterexamples are known for each $\Delta \leq 8$.

**Conjecture 8.18** (Borodin and Kostochka [5])**.** Let $G$ be a graph of maximum degree $\Delta \geq 9$. If $\chi(G) = \Delta$, then $\omega(G) = \Delta$.

When $k = 2$, one cannot expect all $(\Delta-1)$-chromatic graphs to have a clique of size $\Delta - 1$. Indeed $H_{\Delta,1}$ has chromatic number $\Delta - 1$ but clique number $\Delta - 2$. However, Farzad, Molloy and Reed [10] showed that for $\Delta$ sufficiently large if $\chi(G) \geq \Delta - 1$ then $G$ contains either a $(\Delta-1)$-clique or $H_{\Delta,1}$. They also proved similar results for $k = 3$ and $k = 4$; in these cases, $G$ must contain one of five or thirty eight graphs respectively.

Let $k_\Delta$ the maximum integer such that $(k+1)(k+2) \leq \Delta$. Thus, $k_\Delta \approx \sqrt{\Delta} - 2$. Molloy and Reed [20] showed that $k_\Delta$ is a threshold to Brooks-like theorems. Indeed if $k < k_\Delta$ then, if $\Delta$ is large enough, if $\chi(G) \geq \Delta - k + 1$ then $G$ must contain a graph $H$ that is close to a $(\Delta + 1 - k)$-clique, in that $H$ has small size ($|H| \leq \Delta + 1$) and cannot be $(\Delta - k)$-coloured. As a consequence, one can check polynomially if $\chi(G) \geq \Delta - k$ or not. On the opposite, if $k < k_\Delta$, then there are arbitrarily large $(\Delta + k - 1)$-critical graphs (i.e $(\Delta + 1 - k)$-chromatic graphs such that every proper subgraph is $(\Delta - k)$-colourable) with maximum degree $\Delta$. Furthermore, Embden-Weinert, Hougardy and Kreuter [8], proved that for any constant $\Delta$ and $\Delta - 3 \leq k < k_\Delta$, determining whether a graph of maximum degree $\Delta$ is $(\Delta - k)$-colourable is $\mathcal{NP}$-complete.

## 8.3 Colouring planar graphs

A graph is *embeddable* on a surface $\Sigma$ if its vertices can be mapped onto distincts points of $\Sigma$ and its edges onto simple curves of $\Sigma$ joining the points onto which its endvertices are mapped, so that two edge curves do not intersect except in their common extremity. A face of an embedding $\tilde{G}$ of a graph $G$ is a component of $\Sigma \setminus \tilde{G}$. We denote by $F(\tilde{G})$ the set of faces of $\tilde{G}$. A graph is *planar* if it can be embedded in the plane.

Let $\tilde{G}$ be an embedding of a planar graph $G$. Its numbers of vertices, faces and edges are related by Euler's Formula:

$$|V(\tilde{G})| + |F(\tilde{G})| - |E(\tilde{G})| = 1 + comp(G)$$

where $comp(G)$ is the number of connected components of $G$.

*Proof.* We prove Euler's Formula by induction on the number of edges of $G$.

If $G$ has no edges, then every vertex is a connected component and the graph has a unique face, the outer one.

Suppose now that $G$ is a planar graph on at least one edge and that the result holds for planar graphs with less edges. Let $e$ be an edge of $G$. Then two cases may occur.

Assume first that $e$ is a bridge (i.e. $G\setminus e$ has one more component than $G$). Then $e$ is incident to a unique face in $G$. So $G\setminus e$ has as many faces as $G$. By the induction hypothesis, $|V(\tilde{G}\setminus e)|+|F(\tilde{G}\setminus e)|-|E(\tilde{G}\setminus e)| = 1+comp(G\setminus e)$. So $|V(\tilde{G})|+|F(\tilde{G})|-(|E(\tilde{G})|-1) = 1+comp(G)+1$.

Assume now that $e$ is not a bridge. Then $G\setminus e$ has the same number of components as $G$. Then $e$ is incident to two faces in $G$. Removing $e$ transform these two faces into a single one (their union). So $G\setminus e$ has as many faces as $G$. By the induction hypothesis, $|V(\tilde{G}\setminus e)|+|F(\tilde{G}\setminus e)|-|E(\tilde{G}\setminus e)| = 2-comp(G\setminus e)$. So $|V(\tilde{G})|+(|F(\tilde{G})|)-1-(|E(\tilde{G})|-1) = 1+comp(G)$. □

**Corollary 8.19.** *If G is a planar graph, then*

$$|E(G)| \leq 3|V(G)|-6.$$

*Proof.* Let $\tilde{G}$ be an embedding of $G$. Every face de $\tilde{G}$ contains at least three edges and every edge is in at most two faces. Hence, considering the number $N$ of edge-face incidences, we have $2|E(G)| \geq 3|F(\tilde{G})|$. Putting this inequality into Euler's Formula we obtain $|V(G)| + 2|E(G)|/3 \geq |E(G)|+2$ so $3|V(G)|-6 \geq |E(G)|$. □

**Corollary 8.20.** *Every planar graph has a vertex of degree at most* 5.

*Proof.* Let $G$ be a planar graph. By Corollary 8.19, $\sum\{d(v) : v \in G\} = 2|E(G)| \leq 6|V(G)| - 12$. The minimum degree of $G$ is less or equal to the the average degree which is equal to $\frac{6|V(G)|-12}{|V(G)|} < 6$. Hence there is a vertex of degree less than 6. □

**Corollary 8.21.** *Every planar graph is* 6*-colourable.*

*Proof.* Let $G$ be a planar graph. Every subgraph of $G$ is planar and so has minimum degree at most 5 by Corollary 8.20. Hence $G$ is 5-degenerate. Thus, by Proposition 8.9, $\chi(G) \leq 6$. □

**Theorem 8.22.** *Every planar graph is* 5*-colourable.*

*Proof.* By induction on the number of vertices of $G$, the result holding trivially if $G$ has one vertex. By Corollary 8.20, there is a vertex $v$ of degree at most 5 in $G$. By the induction hypothesis, the graph $G-v$ is 5-colourable. Let $c$ be a proper 5-colouring of $G-v$. From $c$, we will construct a proper 5-colouring of $G$.

Assume first, that one of the colours, say $i$, is assigned to no neighbours of $v$. Then one can extend $c$ by setting $c(v) = i$. (Note that this is the case if $d(v) \leq 4$.)

Hence we may assume that $v$ has five neighbours coloured differently. Let $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$ be these neighbours in counter-clockwise order around $v$. Free to permute the colours, we may suppose that $c(v_i) = i$ for all $1 \leq i \leq 5$.

Let $C_{1,3}$ be the component of $v_1$ in the subgraph $G$ induced by the vertices coloured 1 or 3. If $v_3$ is not in $C_{1,3}$, then interchanging the colours 1 and 3 in $C_{1,3}$ and colouring $v$ with 1, we obtain a proper 5-colouring of $G$. If $v_3 \in C_{1,3}$, then there exists a path $P$ linking $v_1$ to $v_3$ in $C_1$. Together wiht $vv_1$ anf $vv_3$ it forms cycle $C$ which separates $v_2$ and $v_4$. Thus the component $C_{2,4}$ of $v_2$ in the subgraph of $G$ induced by the vertices coloured 2 and 4 does not contain $v_4$, otherwise an edge of the path joining $v_2$ to $v_4$ in$C_{2,4}$ would cross an edge of $C$. Hence on can interchange the colours 2 and 4 in $C_{2,4}$ and colour $v$ with 2 to obtain a proper 5-colouring of $G$. □

Theorem 8.22 is not best possible: the celebrated Four Colour Theorem by Appel and Haken [1, 2, 3] states that every planar graph is 4-colourable. A simpler proof was presented by Robertson, Sanders, Seymour and Thomas [24, 25]. However it still uses complicated reductions to a huge number of configurations (more than six hundreds) which need to be solved by computer assistance.

**Theorem 8.23** (Appel and Haken [1, 2, 3]). *Every planar graph is* 4*-colourable.*

Remark that the proof of Theorem 8.22 does not work for 4-colouring. Indded if we are in the configuration depicted in Figure 8.3, for every pair of colours $(i, j)$, a vertex coloured $i$ and a vertex coloured $j$ are in the same component in the subgraph induced by the vertices coloured $i$ and $j$.

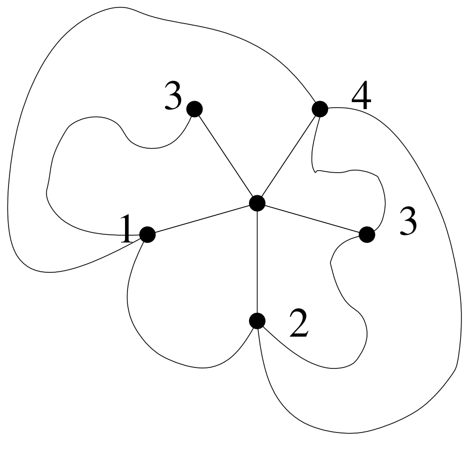



Figure 8.3: The probematic configuration. The curve from $i$ to $j$ represents a path whose vertices are alternately coloured $i$ and $j$.

It is $\mathcal{NP}$-Complete (see [11]) to decide if the chromatic number of a planar graph is 3 or 4, even if the maximum degree does not exceed 4.

## 8.4 Edge-colouring

An *edge-colouring* of $G$ is a mapping $f : E(G) \to S$. The element of $S$ are *colours*; the edges of one colour form a *colour class*. If $|S| = k$, then $f$ is a *k-edge-colouring*. An edge-colouring is *proper* if incident edges have different colours; that is, if each colour class is a matching. A graph is *k-edge-colourable* if it has a proper $k$-edge-colouring. The *chromatic index* or *edge-chromatic number* $\chi'(G)$ of a graph $G$ is the least $k$ such that $G$ is $k$-edge-colourable.

Since edges sharing an endvertex need different colours, $\chi'(G) \geq \Delta(G)$. Furthermore, if a subgraph $H$ of $G$ is odd, then a matching contains at most $\frac{|V(H)|-1}{2}$ edges. Hence at least $\frac{2|E(H)|}{|V(H)|-1}$ colours are needed to properly edge-colour $H$ and thus $G$. It follows that

$$\chi'(G) \geq \max\left\{\Delta(G), \max\left\{\frac{2|E(H)|}{|V(H)|-1} \mid H \text{ odd subgraph of } G\right\}\right\}. \tag{8.1}$$

Observe that for any $H$, $\frac{2|E(H)|}{|V(H)|-1} = \frac{\sum_{v\in V(H)} d_H(v)}{|V(H)|-1} \leq \frac{|V(H)| \times \Delta(H)}{|V(H)|-1} \leq \Delta(H)+1 \leq \Delta(G)+1$.

As an edge is incident to at most $2\Delta(G)-2$ other edges ($\Delta-1$ at each endvertex), colouring the edges greedily we use at most $2\Delta(G)-1$ colours. However, one needs less colours. Vizing [27] and Gupta [13] independently showed that $\chi'(G) \leq \Delta(G)+1$.

**Theorem 8.24** (Vizing [27], Gupta [13])**.** *If $G$ is a graph, then $\chi'(G) \leq \Delta(G)+1$.*

*Proof.* We prove the result by induction on $|E(G)|$. For $|E(G)| = 0$, it is trivial.

Suppose now that $|E(G)| \geq 1$ and that the assertion holds for graphs with fewer edges than $G$. Set $\Delta(G) = \Delta$.

Let $xy_0$ be an edge of $G$. By induction hpothesis, $G \setminus xy_0$ admits a $(\Delta+1)$-edge-colouring. As $y_0$ is incident to at most $\Delta-1$ edges in $G \setminus xy_0$, there exists a colour $c_1 \in \{1,2,\dots,\Delta+1\}$ *missing* at $y_0$, i.e. such that no edge incident to $y_0$ is coloured $c_1$. If $c_1$ is also missing at $x$, then colouring $xy_0$ with $c_1$, we obtain a $(\Delta+1)$-edge-colouring of $G$. So we may assume that there is an edge $xy_1$ coloured $c_1$.

Because $y_1$ is incident to at most $\Delta$ edges, a colour $c_2 \in \{1,2,\dots,\Delta+1\}$ is missing at $y_1$. If $c_2$ is missing at $x$ then recolouring $xy_1$ with $c_2$ and colouring $xy_0$ with $c_1$, we obtain a $(\Delta+1)$-edge-colouring of $G$. So we may assume that there is an edge $xy_2$ coloured $c_2$.

And so on, we construct a sequence $y_1, y_2, \dots$ of neighbours of $x$ and a sequence of colours $c_1, c_2, \dots$ such that: $xy_i$ is coloured $c_i$ and $c_{i+1}$ is missing at $y_i$. Since the degree of $x$ is bounded, there exists a smallest $l$ such that for an integer $k < l$,$c_{l+1} = c_k$.

Now, for $0 \leq i \leq k-1$, let us recolour the edge $xy_i$ with $c_{i+1}$.

There exists a colour $c_0 \in \{1,2,\dots,\Delta+1\}$ missing at $x$. In particular, $c_0 \neq c_k$. Let $P$ be the maximal path starting at $y_{k-1}$ with edges alternatively coloured $c_0$ and $c_k$. Let us interchange the colour $c_0$ and $c_k$ on $P + xy_{k-1}$. If $P$ does not contain $y_k$, we have a $(\Delta+1)$-edge-colouring of $G$. If $P$ contains (and thus ends in) $y_k$, recolouring the edge $xy_i$ with $c_{i+1}$ for $k \leq i \leq l$, we obtain a $(\Delta+1)$-edge-colouring of $G$. □

Hence $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$. A graph is said to be *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G)+1$. Holyer [14] showed that determining whether a graph is Class 1 or Class 2 is $\mathcal{NP}$-complete. While we will see many graphs of Class 1 and many graphs of Class 2, it turns out that it is much more likely that a graph is of Class 1. Erdős and Wilson [9] proved the following, where the set of graphs of order $n$ is denoted by $\mathcal{G}_n$ and the set of graphs of order $n$ and of Class 1 is denoted by $\mathcal{G}_n^1$.

**Theorem 8.25** (Erdős and Wilson [9])**.** *Almost every graph is of Class 1, that is,*

$$\lim_{n\to\infty} \frac{|\mathcal{G}_n^1|}{|\mathcal{G}_n|} = 1.$$

However there are classes of graphs for which we know if they are Class 1 or Class 2. For example, a regular graph of odd order, say $2n+1$, is Class 2 by Equation 8.1.

The following theorem of König [18] states that every bipartite graph is Class 1.

**Theorem 8.26** (König [18])**.** *Let $G$ be a bipartite graph. Then $\chi'(G) = \Delta(G)$.*

*Proof.* By induction of the number of edges, the result holding vacuously when $|E(G)| = 0$. Assume now that $|E(G)| \geq 1$. Set $\Delta = \Delta(G)$. Let $xy$ be an edge of $G$. By the induction hypothesis, $G \setminus xy$ admits a proper $\Delta$-edge-colouring.

In $G \setminus xy$, the vertices $x$ and $y$ are each incident to at most $\Delta - 1$ edges. So there exists $c_x$ and $c_y$ in $\{1, 2, \dots, \Delta\}$ such that $x$ (resp. $y$) is not incident to an edge coloured $c_x$ (resp. $c_y$). If $c_x = c_y$ then assigning this colour to $xy$ we obtain a $\Delta$-edge-colouring of $G$. Hence we may assume that $c_x \neq c_y$ and that the vertex $x$ is incident to an edge $e$ coloured $c_y$.

Extend this edge into a maximal path $P$ whose edges are coloured $c_y$ and $c_x$ alternatively. We claim that $y$ is not on $P$. Indeed if it would be on $P$, since $y$ is adajcent to no edge $c_y$, it would be an endvertex of $P$ and $P$ would terminate with an edge coloured $c_x$. Then $P \cup xy$ would be an odd cycle which is a contradicts Theorem 2.5. Hence one can invert the colours on $P$. By maximality of $P$, the edge-colouring is still proper. Then assigning $xy$ the colour $c_y$, we obtain a $\Delta$-edge-colouring of $G$. □

Planar graphs with sufficiently large maximum degree $\Delta$ are Class 1. Sanders and Zhao [26] showed that planar graphs with maximum degree $\Delta \geq 7$ are Class 1. Vizing edge-colouring conjecture [28] asserts that planar graphs of maximum degree 6 are also Class 1. This would be best possible as for any $\Delta \in \{2,3,4,5\}$, there are some planar graphs with maximum degree $\Delta$ which are Class 2 [28]. However, for $\Delta \in \{3,4,5\}$ the complexity of deciding if a planar graph with maximum degree $\Delta$ is $\Delta$-edge-colourable is still unknown.

Some planar cubic graphs are Class 2 as they have no perfect matching. An example is given Figure 8.4. However, by Petersen Theorem, cubic graphs with no perfect matching are bridgeless.

**Proposition 8.27.** *Every bridgeless cubic planar graph is* 3*-edge colourable.*

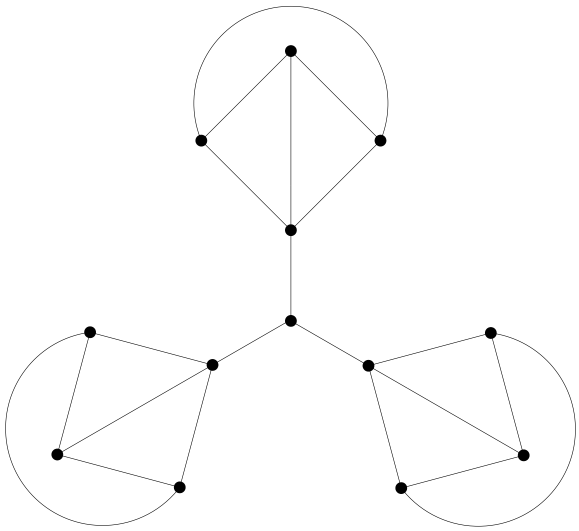
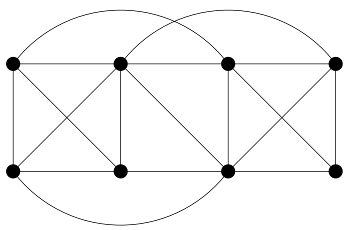


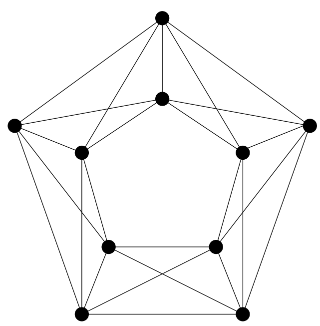
Figure 8.4: Planar cubic graph without perfect matching

*Proof.* Let $G$ be a bridgeless cubic planar graph and $\tilde{G}$ be one of its embedding. The *dual* of $\tilde{G}$ is the graph $G^*$ with vertex set $F(\tilde{G})$ such that two faces of $\tilde{G}$ are adjacent in $G^*$ if they are incident to a common edge. It is easy to see that $G^*$ is planar. So, by Theorem 8.23, $G^*$ is 4-colourable, so one can colour the faces of $G$ with $\{1,2,3,4\}$ such that two faces sharing an edge have different colours. For any $1 \leq i < j \leq 4$, let $E_{i,j}$ be the set of edges incident to a face coloured $i$ and one coloured $j$. Observe that since $G$ has no bridge, every edge is incident to two distinct faces. Then $E_{1,2} \cup E_{3,4}$, $E_{1,3} \cup E_{2,4}$ and $E_{1,4} \cup E_{2,3}$ are the three matchings corresponding to a proper 3-edge colouring of $G$. □

## 8.5 Exercises

**Exercise 8.1.** Find the chromatic number of the following graphs :





**Exercise 8.2.** Show that any partial colouring $f$ of the graph $G_j$ depicted Figure 8.1 with colours $\{true, false, base\}$ such that $f(v_{false}) = false$, $f(v_{base}) = base$, and $f(\ell_i) \neq false$ for some

$i \in \{1,2,3\}$, can be extended into a proper colouring of $G_j$ with colours $\{true, false, base\}$.

**Exercise 8.3.** Let $G$ be a $k$-regular bipartite graph. Show that for every proper 2-colouring of $G$, there are as many vertices coloured 1 as vertices coloured 2.

**Exercise 8.4.** Show that a graph $G = (V,E)$ is $2^k$-colourable if and only if $E$ may be partitionned into $k$ sets $E_1, \dots, E_k$ such that for every $1 \le i \le k$, $(V, E_i)$ is a bipartite graph.

**Exercise 8.5.** Show that if a $k$-chromatic graph $G$ admits a proper colouring for which every colour is assigned to at least 2 vertices then $G$ has a proper $k$-colouring with the same property.

**Exercise 8.6.** Let $c$ be a partial proper colouring of a graph $G$ with $\Delta(G) - k$ colours such that for every non-coloured vertex at least $k+1$ colours appears at least twice on its neighbourhood. Show that $c$ can be extended to a proper $(\Delta(G)-k)$-colouring of $G$.

**Exercise 8.7.** Let $G$ be a graph and $\overline{G}$ its complement.
1) Let $v$ be a vertex of $G$.

a) Show that $\chi(G) \le \chi(G-v)+1$.

b) Show that if $d_G(v) < \chi(G-v)$, then $\chi(G) = \chi(G-v)$.

c) Deduce that $\chi(G) + \chi(\overline{G}) \le |V(G)|+1$.

2) a) Show that $\chi(G) \times \chi(\overline{G}) \ge |V(G)|$.

b) Deduce that $\chi(G) + \chi(\overline{G}) \ge 2\sqrt{|V(G)|}$.

**Exercise 8.8.** Let $(G_i), i \ge n$, be the sequence of graphs defined as follows : $G_3$ is the cycle on 5 vertices. Suppose now that $G_k$ has $n_k$ vertices. Set $m_k = k(n_k - 1)$. Let $W$ be a set of $m_k$ vertices and for every subset $U$ of $W$ of cardinality $n_k$, let $G_U$ be a copy of $G_k$ such that $W$ and all $V(G_U)$ are pairwise disjoints. The graph $G_{k+1}$ is then obtained by adding for all $U \subset W$ of cardinality $n_k$ a perfect matching between $U$ and $V(G_U)$. Thus we have $|V(G_{k+1})| = n_{k+1} = \binom{m_k}{n_k} n_k + m_k$. Show that for all $k$, $G_k$ is triangle-free and $\chi(G_k) = k$.

**Exercise 8.9.** A *cograph* is a graph with no subgraph isomorphic to $P_4$ the path on 4 vertices. Show that for any ordering of the vertices $\sigma$, the greedy algorithm produces an optimal proper colouring. (*Hint:* Suppose that the greedy algorithm uses $k$ colours according to the ordering $(v_1 < \cdots < v_n)$ and let $i$ be the least integer such that there is a clique formed by $k-i+1$ vertices assigned coloured from $i$ to $k$. Show that $i = 1$.)

**Exercise 8.10.** Let $G_1$ and $G_2$ be two disjoint graphs, $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. The *Hajós sum* $G = (G_1, x_1y_1) + (G_2, x_2y_2)$ is the graph obtained from $G_1 \cup G_2$ by identifying $x_1$ and $x_2$, deleting $x_1y_1$ and $x_2y_2$, and adding $y_1y_2$.
1) Show that $\chi(G) \ge \min\{\chi(G_1), \chi(G_2)\}$.
2) Show that $\chi(G) \ge \max\{\chi(G_1), \chi(G_2)\} - 1$. Give an example for which $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} - 1$.

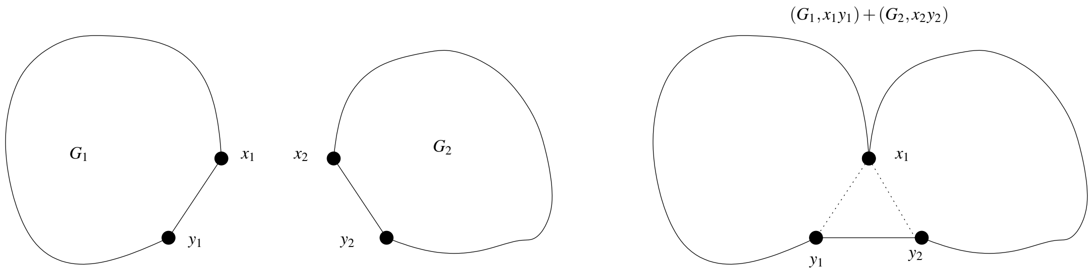



Figure 8.5: Hajós sum

**Exercise 8.11.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. The *disjoint union* of $G_1$ and $G_2$ is the graph $G_1 + G_2$, defined by $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2$.
The *join* of $G_1$ and $G_2$ is the graph $G_1 \oplus G_2$ defined by $V(G_1 \oplus G_2) = V_1 \cup V_2$ and $E(G_1 \oplus G_2) = E_1 \cup E_2 \cup \{v_1v_2 \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}$.

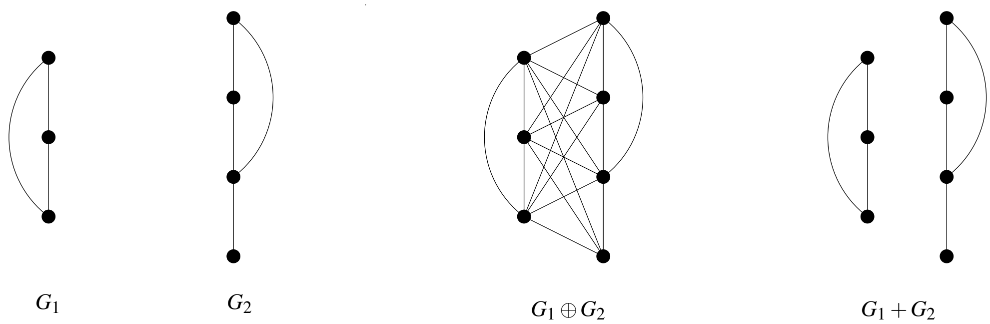



Figure 8.6: Two graphs, their join and their disjoint union

1) Show that $\chi(G_1 + G_2) = \max\{\chi(G_1), \chi(G_2)\}$ and $\chi(G_1 \oplus G_2) = \chi(G_1) + \chi(G_2)$.
2) The class of *cographs* is defined inductively as follows:

- the graph with one vertex $K_1$ is a cograph;
- the disjoint union of two cographs is a cograph;
- the join of two cographs is a cograph.

Prove that if $G$ is a cograph then $\chi(G) = \omega(G)$.

**Exercise 8.12.** Show that, in every $k$-chromatic graph, there are at least $k$ vertices of degree at least $k - 1$.

**Exercise 8.13.** Show that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n - 1$.

**Exercise 8.14.** Let $G$ be an $r$-regular bipartite graph and $E_0$ a set of $r-1$ edges. Show that $G \setminus E_0$ has a perfect matching.

**Exercise 8.15.** The *cartesian product* of two graphs $G$ and $H$ is the graph $G \times H$ defined by

$$\begin{aligned} V(G\Box H) &= V(G) \times V(H) \\ E(G\Box H) &= \{(a,x)(b,y) \mid a = b \text{ and } xy \in E(H) \text{ or } ab \in E(G) \text{ and } x = y\}. \end{aligned}$$

(a) Show that $\chi'(G\Box K_2) = \Delta(G\Box K_2)$.

(b) Deduce that if $H$ is non-empty (it has at least one edge) and $\chi'(H) = \Delta(H)$ then $\chi'(G\Box H) = \Delta(G\Box H)$.

**Exercise 8.16.**

1) Show the following lemma due to Isaacs [15]: *Let $G$ be a cubic 3-edge-coloured graph and $W \subseteq V(G)$ a set of vertices of $G$. Let $E_W$ be the set of edges of $G$ which connect $W$ to the remainder of the graph. If the number of edges of colour $i$ in $E_W$ is $k_i$ ($i = 1,2,3$), then* $k_1 \equiv k_2 \equiv k_3 \pmod 2$.

2) Show that in any 3-edge-colouring of the graph $I$, depicted Figure 8.7, one of the pairs of connecting edges marked $\{a,b\}$, or $\{c,d\}$ must have equal colours and the edges of the remaining pair and $e$ must have distinct colours.

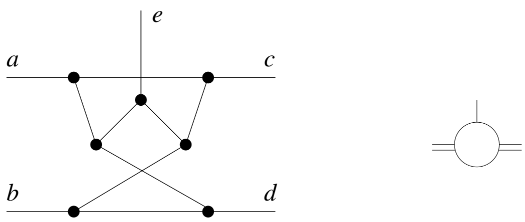



Figure 8.7: The graph $I$ and its symbolic representation

3) A pair of edges is said to be *of type $T$* if they have the same colour, and *of type $F$* if they have distinct colours. Show in any 3-edge-colouring of the graph $H$ depicted in Figure 8.8, the four pairs of connecting edges are of the same type.

4) Show that a 3-edge-colouring of the six connecting edges of the graph $C$ depicted in Figure 8.9, can be extended into a 3-edge-colouring of $C$, if and only if, the three pairs of connecting edges are not all of type $F$.

5) Conclude that deciding whether a graph is 3-edge colourable is an NP-complete problem.

6) Show also that deciding whether a 3-regular graph is 3-edge colourable is an NP-complete problem.

(Holyer [14])

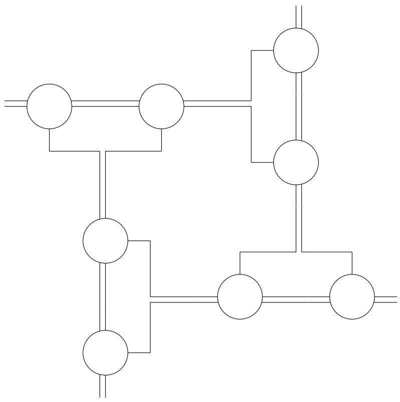


Figure 8.8: The graph *H*

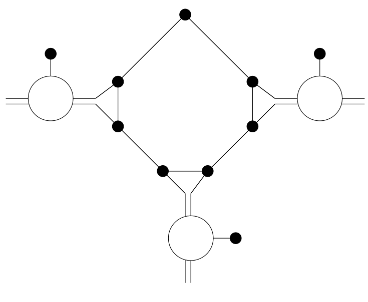


Figure 8.9: The graph *C*

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