Realne funkcije realne varijable – 1. dio

MATEMATIKA 2

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Zadatak 1

Odredite domene i nultočke sljedećih funkcija:

a) $f(x) = \sqrt[4]{\dfrac{x-3}{x+2} - 2} - 1$

b) $g(x) = (2 + x - x^2)^{\frac{1}{5}}$

c) $h(x) = \log\left(10^{x-1} - 5\right)$

d) $k(x) = \sqrt{\log_{\frac{1}{2}}(x+2)}$

$$f(x) = \sqrt[4]{\frac{x-3}{x+2} - 2} - 1$$

Rješenje

a) domena

$x + 2 \neq 0$ — uključeno u ovom uvjetu

$$\frac{x-3}{x+2} - 2 \geqslant 0$$

$$\frac{x - 3 - 2(x+2)}{x+2} \geqslant 0$$

$$\frac{-x-7}{x+2} \geqslant 0$$

$-x - 7 = 0 \qquad x + 2 = 0$

$x = -7 \qquad x = -2$

	$-\infty$		-7		-2		$+\infty$
$-x-7$		$+$	0	$-$	$\|$	$-$	
$x+2$		$-$	$\|$	$-$	0	$+$	
$\frac{-x-7}{x+2}$		$-$	0	$+$	$\|$	$-$	

RJEŠENJE: $x \in [-7, -2\rangle$

$D_f = [-7, -2\rangle$

nultočke

$$f(x) = \sqrt[4]{\frac{x-3}{x+2} - 2} - 1$$

$$\sqrt[4]{\frac{x-3}{x+2} - 2} - 1 = 0$$

$$\sqrt[4]{\frac{x-3}{x+2} - 2} = 1 \Big/ {}^4$$

$$\frac{x-3}{x+2} - 2 = 1$$

$$\frac{x-3}{x+2} = 3 \Big/ \cdot (x+2)$$

$$x - 3 = 3x + 6$$

$$-2x = 9$$

$$x = -\frac{9}{2}$$

$D_f = [-7, -2\rangle$

jest nultočka jer pripada domeni

b) domena

$g(x) = \sqrt[5]{2+x-x^2}$

$D_g = \mathbb{R}$ ⇜ neparni korijen je definiran za sve realne brojeve

$g(x) = (2+x-x^2)^{\frac{1}{5}}$

nultočke

$$\sqrt[5]{2+x-x^2} = 0 \,/^5$$
$$-x^2+x+2=0$$
$$x_{1,2} = \frac{-1 \pm \sqrt{1^2-4\cdot(-1)\cdot 2}}{2\cdot(-1)}$$
$$x_{1,2} = \frac{-1\pm 3}{-2}$$
$$x_1 = -1, \quad x_2 = 2$$

$ax^2+bx+c=0$

$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

c) domena

$10^{x-1}-5>0$

$10^{x-1}>5$

$10^{x-1}>10^{\log 5}$

$x-1>\log 5$

$x>1+\log 5$

$D_h = \langle 1+\log 5, +\infty\rangle$

$a^{\log_a x} = x$ $\quad \log = \log_{10}$ $\quad h(x) = \log\left(10^{x-1}-5\right)$

Ako je $a>1$: $a^x > a^y \Leftrightarrow x>y$

Ako je $0<a<1$: $a^x > a^y \Leftrightarrow x<y$

$\log_a x = b \rightsquigarrow x = a^b$

$a^x = b \rightsquigarrow x = \log_a b$

nultočke

$\log\left(10^{x-1}-5\right)=0$

$10^{x-1}-5=10^0$

$10^{x-1}=6$

$x-1=\log 6$

$x=1+\log 6$

jest nultočka jer pripada domeni

Ako je $a>1$: $\log_a x > \log_a y \Leftrightarrow x>y$

$\log_a a^x = x$ $\quad k(x) = \sqrt{\log_{\frac{1}{2}}(x+2)}$

d) domena

- $x+2>0$ ⇜ zbog $\log_{\frac{1}{2}}$
- $\log_{\frac{1}{2}}(x+2)\geqslant 0$ ⇜ zbog $\sqrt{\ }$

$x+2>0$

$x>-2$

$\log_{\frac{1}{2}}(x+2)\geqslant 0$

$\log_{\frac{1}{2}}(x+2)\geqslant \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^0$

$x+2\leqslant\left(\frac{1}{2}\right)^0$

$x+2\leqslant 1$

$x\leqslant -1$

presjek rješenja: $x\in\langle -2,-1]$

$D_k = \langle -2,-1]$

Ako je $0<a<1$: $\log_a x > \log_a y \Leftrightarrow x<y$

$k(x) = \sqrt{\log_{\frac{1}{2}}(x+2)}$

$D_k = \langle -2,-1]$

nultočke

$\sqrt{\log_{\frac{1}{2}}(x+2)} = 0 \,/^2$

$\log_{\frac{1}{2}}(x+2)=0$

$x+2=\left(\frac{1}{2}\right)^0$

$x+2=1$

$x=-1$

jest nultočka jer pripada domeni

$\log_a x = b \rightsquigarrow x = a^b$

Zadatak 2

Odredite nultočke funkcija

$f(x) = 2^{5-x} + 50$ *i* $g(x) = 2^{5-x} - 50$.

$$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$$

Rješenje

nultočke od f

$2^{5-x} + 50 = 0$

$2^{5-x} = -50$

$5 - x = \log_2(-50)$

funkcija f nema nultočki

nultočke od g

$2^{5-x} - 50 = 0$

$2^{5-x} = 50$

$5 - x = \log_2 50$

$-x = -5 + \log_2 50 \,/\cdot (-1)$

egzaktna vrijednost nultočke

$x = 5 - \log_2 50$

$x = 5 - \frac{\log 50}{\log 2}$

aproksimacija nultočke na 5 decimala

$x \approx -0.64386$

$a^x = b \rightsquigarrow x = \log_a b$

Nultočke funkcije g

1. način

$2^{5-x} - 50 = 0$

$2^{5-x} = 50 \,/\, \log_2$

$5 - x = \log_2 50$

$-x = -5 + \log_2 50 \,/\cdot (-1)$

$x = 5 - \log_2 50$

$x = 5 - \frac{\log 50}{\log 2}$

$x \approx -0.64386$

2. način

$2^{5-x} - 50 = 0$

$2^{5-x} = 50 \,/\, \log$

$\log 2^{5-x} = \log 50$

$(5 - x)\log 2 = \log 50 \,/: \log 2$

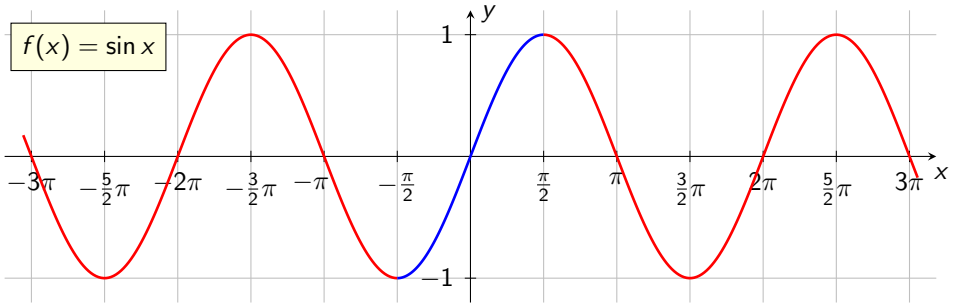
$5 - x = \frac{\log 50}{\log 2}$

$-x = -5 + \frac{\log 50}{\log 2} \,/\cdot (-1)$

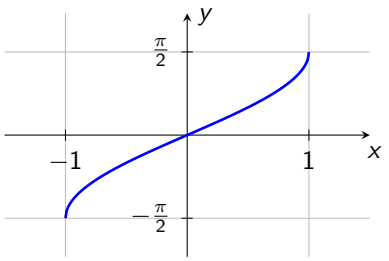
$x = 5 - \frac{\log 50}{\log 2}$

$a^x = b \rightsquigarrow x = \log_a b$

$\log_a x^k = k \cdot \log_a x$

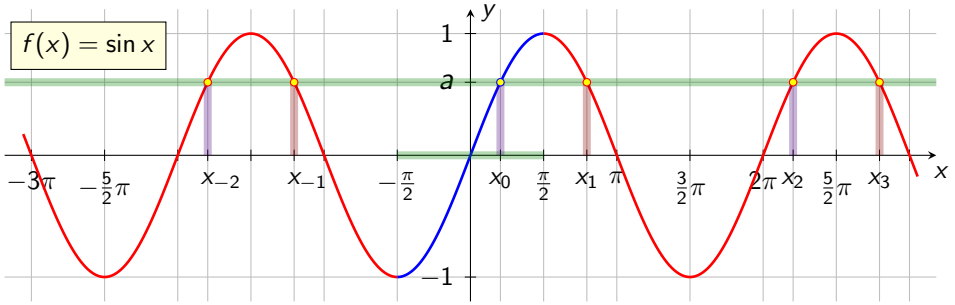


$f^{-1}(x) = \arcsin x$



$\sin x = 0 \;\Leftrightarrow\; x = k\pi,\ k \in \mathbb{Z}$

$\arcsin x = 0 \;\Leftrightarrow\; x = 0$



Rješenja jednadžbe $\sin x = a$ za $|a| \leqslant 1$

- $x_k^{(1)} = \arcsin a + 2k\pi,\ k \in \mathbb{Z}$
- $x_k^{(2)} = \pi - \arcsin a + 2k\pi,\ k \in \mathbb{Z}$

$x_0 = \arcsin a$

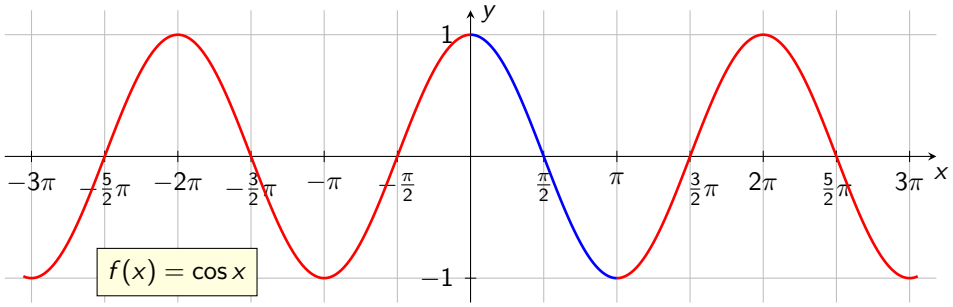
$x_1 = \pi - \arcsin a$

$x_k^{(1)} = x_{2k} = x_0 + 2k\pi$

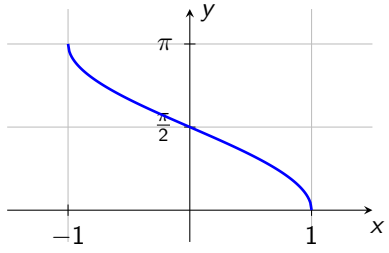
$x_k^{(2)} = x_{2k+1} = x_1 + 2k\pi$

Možemo sva rješenja zapisati pomoću jedne formule

$$x_k = (-1)^k \arcsin a + k\pi,\ k \in \mathbb{Z}$$

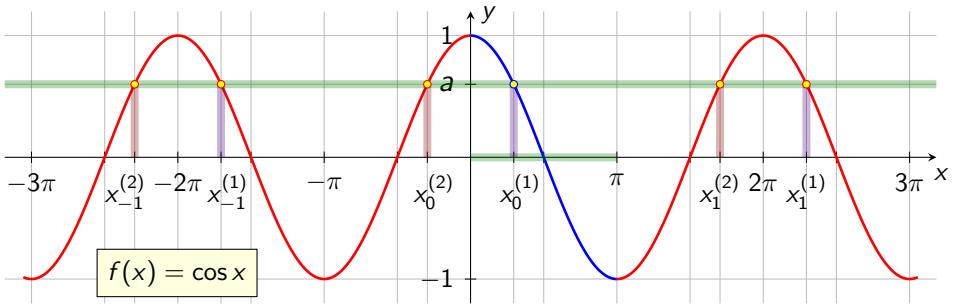


$f^{-1}(x) = \arccos x$



$$\cos x = 0 \;\Leftrightarrow\; x = \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$$

$$\arccos x = 0 \;\Leftrightarrow\; x = 1$$



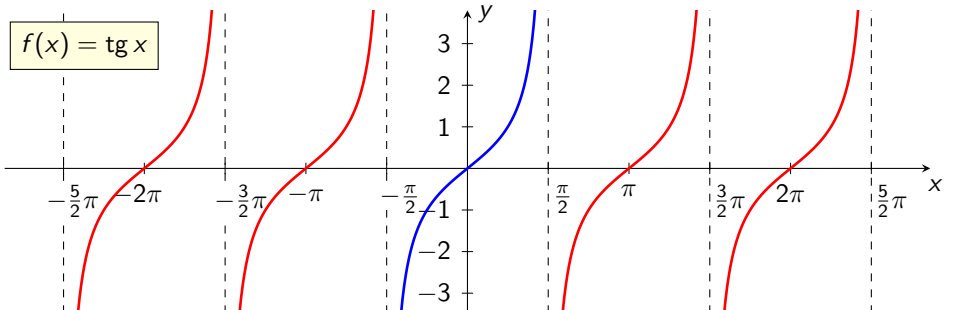
Rješenja jednadžbe $\cos x = a$ za $|a| \leqslant 1$

- $x_k^{(1)} = \arccos a + 2k\pi,\ k \in \mathbb{Z}$
- $x_k^{(2)} = -\arccos a + 2k\pi,\ k \in \mathbb{Z}$

$$x_0^{(1)} = \arccos a$$
$$x_0^{(2)} = -\arccos a$$
$$x_k^{(1)} = x_0^{(1)} + 2k\pi$$
$$x_k^{(2)} = x_0^{(2)} + 2k\pi$$

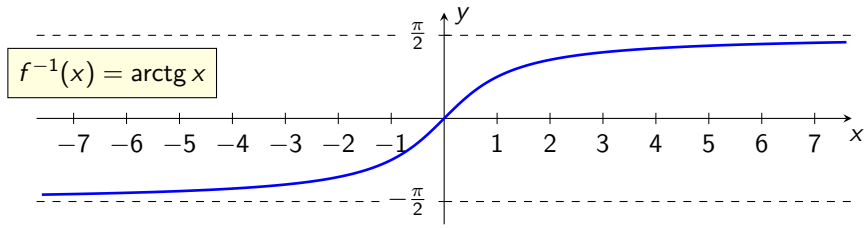
Bez indeksiranja možemo sva rješenja kratko zapisati

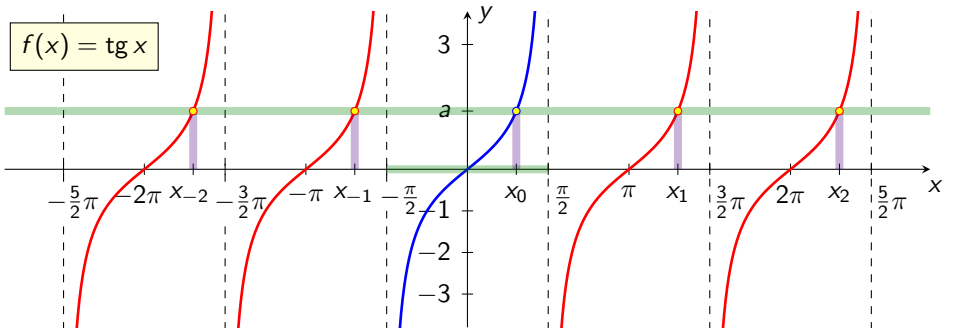
$$x = \pm\arccos a + 2k\pi,\ k \in \mathbb{Z}$$



$$\operatorname{tg} x = 0 \;\Leftrightarrow\; x = k\pi,\ k \in \mathbb{Z}$$

$$\operatorname{arctg} x = 0 \;\Leftrightarrow\; x = 0$$

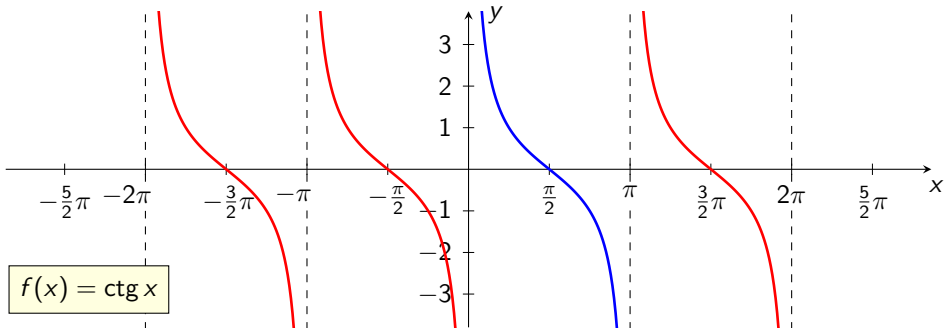




Rješenja jednadžbe $\operatorname{tg} x = a$

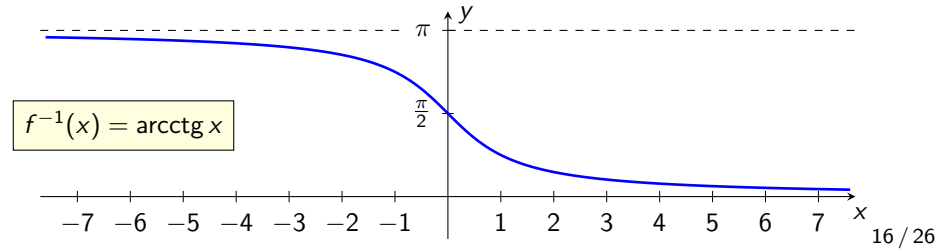
- $x_0 = \operatorname{arctg} a$
- $x_k = x_0 + k\pi,\ k \in \mathbb{Z}$

$$x_k = \operatorname{arctg} a + k\pi,\ k \in \mathbb{Z}$$



$\operatorname{ctg} x = 0 \Leftrightarrow x = \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$

$(\forall x \in \mathbb{R})\,(\operatorname{arcctg} x \neq 0)$



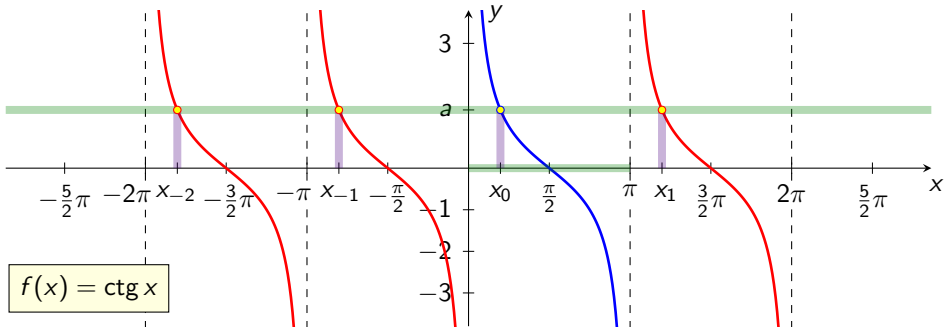
Zadatak 3

Odredite domenu i nultočke sljedećih funkcija:

a) $h(x) = \operatorname{ctg}(\pi x + 2)$

b) $f(x) = \sqrt{\sin 3x + \frac{1}{2}}$

c) $g(x) = \dfrac{\arccos(x^2 - 3)}{x - 2}$



Rješenja jednadžbe $\operatorname{ctg} x = a$

- $x_0 = \operatorname{arcctg} a$
- $x_k = x_0 + k\pi,\ k \in \mathbb{Z}$

$x_k = \operatorname{arcctg} a + k\pi,\ k \in \mathbb{Z}$

Rješenje

$h(x) = \operatorname{ctg}(\pi x + 2)$

a) domena

$\pi x + 2 \neq k\pi,\ k \in \mathbb{Z}$

$\pi x \neq k\pi - 2 \,/ : \pi$

$x \neq \frac{k\pi - 2}{\pi}$

$x \neq k - \frac{2}{\pi},\ k \in \mathbb{Z}$

$D_h = \mathbb{R} \setminus \{k - \frac{2}{\pi} : k \in \mathbb{Z}\}$

ekvivalentni zapis

$D_h = \bigcup_{k \in \mathbb{Z}} \left\langle k - \frac{2}{\pi},\ k + 1 - \frac{2}{\pi} \right\rangle$

nultočke

$\operatorname{ctg}(\pi x + 2) = 0$

$\pi x + 2 = \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$

$\pi x = \frac{2k+1}{2}\pi - 2 \,/ : \pi$

$x = \frac{2k+1}{2} - \frac{2}{\pi},\ k \in \mathbb{Z}$

jesu nultočke jer pripadaju domeni

b) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \qquad \pi - \left(-\frac{\pi}{6}\right) = \frac{7}{6}\pi$

$f(x) = \sqrt{\sin 3x + \frac{1}{2}}$

Domena

$\sin 3x + \frac{1}{2} \geqslant 0$

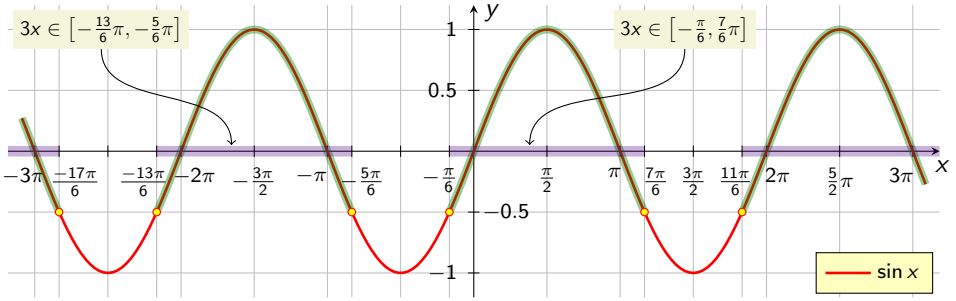
$\sin 3x \geqslant -\frac{1}{2}$

$3x \in \bigcup_{k\in\mathbb{Z}} \left[-\frac{\pi}{6} + 2k\pi,\ \frac{7}{6}\pi + 2k\pi\right] \Big/ : 3$

$x \in \bigcup_{k\in\mathbb{Z}} \left[-\frac{\pi}{18} + \frac{2}{3}k\pi,\ \frac{7}{18}\pi + \frac{2}{3}k\pi\right]$

$x \in \bigcup_{k\in\mathbb{Z}} \left[\frac{12k-1}{18}\pi,\ \frac{12k+7}{18}\pi\right]$

$D_f = \bigcup_{k\in\mathbb{Z}} \left[\frac{12k-1}{18}\pi,\ \frac{12k+7}{18}\pi\right]$



$$x = \frac{6k + (-1)^{k+1}}{18}\pi,\ k \in \mathbb{Z}$$

$k = 2s$ za neki $s \in \mathbb{Z}$, k je paran

$k = 2s + 1$ za neki $s \in \mathbb{Z}$, k je neparan

$$x = \frac{6 \cdot 2s + (-1)^{2s+1}}{18}\pi \qquad x = \frac{6 \cdot (2s+1) + (-1)^{2s+2}}{18}\pi$$

$$x = \frac{12s + (-1)^{\text{neparan}}}{18}\pi \qquad x = \frac{12s + 6 + (-1)^{\text{paran}}}{18}\pi$$

$$x = \frac{12s - 1}{18}\pi \qquad x = \frac{12s + 7}{18}\pi$$

$$D_f = \bigcup_{k\in\mathbb{Z}} \left[\frac{12k-1}{18}\pi,\ \frac{12k+7}{18}\pi\right]$$

b) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \qquad \pi - \left(-\frac{\pi}{6}\right) = \frac{7}{6}\pi$

$f(x) = \sqrt{\sin 3x + \frac{1}{2}}$

Nultočke

$\sqrt{\sin 3x + \frac{1}{2}} = 0 \Big/^2$

$\sin 3x + \frac{1}{2} = 0$

$\sin 3x = -\frac{1}{2}$

$3x = (-1)^k \arcsin\left(-\frac{1}{2}\right) + k\pi \Big/ : 3$

$x = \frac{(-1)^k}{3} \cdot \frac{-\pi}{6} + \frac{k\pi}{3}$

$x = \frac{(-1)^k \cdot (-1) \cdot \pi}{18} + \frac{k\pi}{3}$

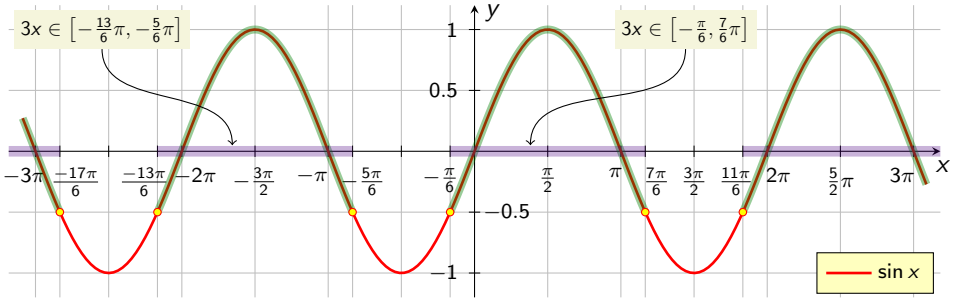
$x = \frac{(-1)^{k+1}}{18}\pi + \frac{k\pi}{3}$

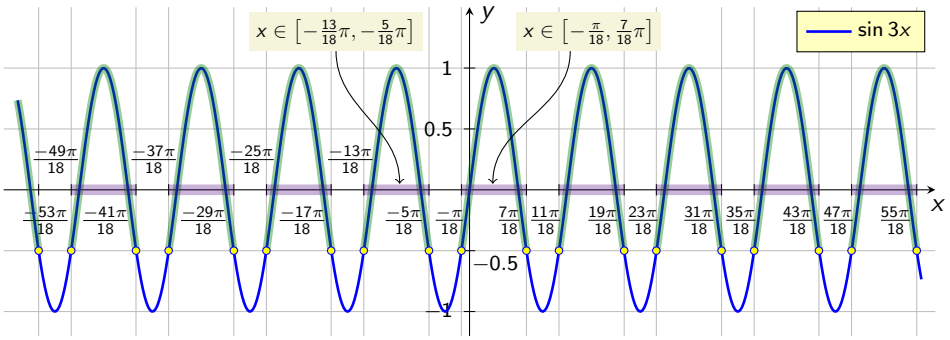
$x = \frac{6k + (-1)^{k+1}}{18}\pi,\ k \in \mathbb{Z}$

$D_f = \bigcup_{k\in\mathbb{Z}} \left[\frac{12k-1}{18}\pi,\ \frac{12k+7}{18}\pi\right]$

jesu nultočke jer pripadaju domeni

$\sin x = a \ \Leftrightarrow\ x = (-1)^k \arcsin a + k\pi,\ k \in \mathbb{Z}$





c) Domena

$g(x) = \dfrac{\arccos(x^2-3)}{x-2}$

✏ $x^2 - 3 \geqslant -1$

✏ $x^2 - 3 \leqslant 1$

✏ $x - 2 \neq 0$

domena funkcije arccos je segment $[-1, 1]$

zbog nazivnika

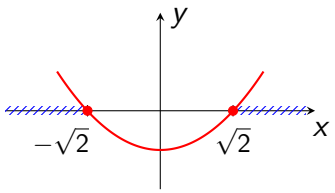
$x^2 - 3 \geqslant -1$

$x^2 - 3 + 1 \geqslant 0$

$x^2 - 2 \geqslant 0$

$x^2 - 2 = 0$

$x_1 = -\sqrt{2},\ x_2 = \sqrt{2}$



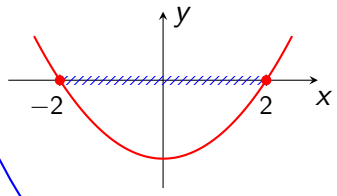
$x^2 - 3 \leqslant 1$

$x^2 - 3 - 1 \leqslant 0$

$x^2 - 4 \leqslant 0$

$x^2 - 4 = 0$

$x_1 = -2,\ x_2 = 2$



$x \neq 2$

presjek rješenja

$x \in \langle -\infty, -\sqrt{2}\,] \cup [\sqrt{2}, +\infty \rangle$

$x \in [-2, 2]$

c) Nultočke

$\arccos x = 0 \ \Leftrightarrow\ x = 1$

$g(x) = \dfrac{\arccos(x^2-3)}{x-2}$

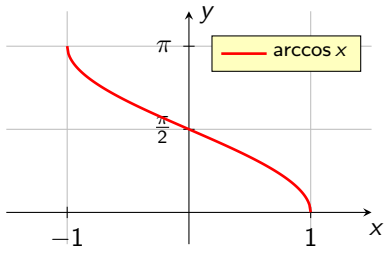
$\dfrac{\arccos(x^2-3)}{x-2} = 0$

$\arccos(x^2-3) = 0$

$x^2 - 3 = 1$

$x^2 = 4$

$x_1 = -2$ $\quad x_2 = 2$



jest nultočka jer pripada domeni

nije nultočka jer ne pripada domeni

$D_g = x \in [-2, -\sqrt{2}\,] \cup [\sqrt{2}, 2\rangle$

c) Domena

$g(x) = \dfrac{\arccos(x^2-3)}{x-2}$

$D_g = [-2, -\sqrt{2}\,] \cup [\sqrt{2}, 2\rangle$

✏ $x^2 - 3 \geqslant -1 \rightsquigarrow x \in \langle -\infty, -\sqrt{2}\,] \cup [\sqrt{2}, +\infty \rangle$

✏ $x^2 - 3 \leqslant 1 \rightsquigarrow x \in [-2, 2]$

✏ $x - 2 \neq 0 \rightsquigarrow x \neq 2$

presjek rješenja

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