

Neodređeni integral – 1. dio

MATEMATIKA 2

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Zadatak 2

Riješite neodređeni integral $\int \frac{(x-3)^2}{x^5} dx$.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Rješenje

$$\begin{aligned} \int \frac{(x-3)^2}{x^5} dx &= \int \frac{x^2 - 6x + 9}{x^5} dx = \int \left(\frac{1}{x^3} - \frac{6}{x^4} + \frac{9}{x^5} \right) dx = \\ &= \int x^{-3} dx - 6 \int x^{-4} dx + 9 \int x^{-5} dx = \\ &= \frac{x^{-2}}{-2} - 6 \cdot \frac{x^{-3}}{-3} + 9 \cdot \frac{x^{-4}}{-4} + C = \\ &= -\frac{1}{2x^2} + \frac{2}{x^3} - \frac{9}{4x^4} + C, \quad C \in \mathbb{R} \end{aligned}$$

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Zadatak 1

Riješite neodređeni integral $\int \frac{dx}{\sqrt[4]{x^3}}$.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Rješenje

$$\begin{aligned} \int \frac{dx}{\sqrt[4]{x^3}} &= \int x^{-\frac{3}{4}} dx = \frac{x^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + C = \\ &= \frac{x^{\frac{1}{4}}}{\frac{1}{4}} + C = 4\sqrt[4]{x} + C, \quad C \in \mathbb{R} \end{aligned}$$

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Zadatak 3

Riješite neodređeni integral $\int (5e^x - 3 \sin x) dx$.

Rješenje

$$\begin{aligned} \int (5e^x - 3 \sin x) dx &= 5 \int e^x dx - 3 \int \sin x dx = \\ &= 5e^x - 3 \cdot (-\cos x) + C = \\ &= 5e^x + 3 \cos x + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

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Zadatak 4

Riješite neodređeni integral $\int 3^x e^x dx$.

Rješenje

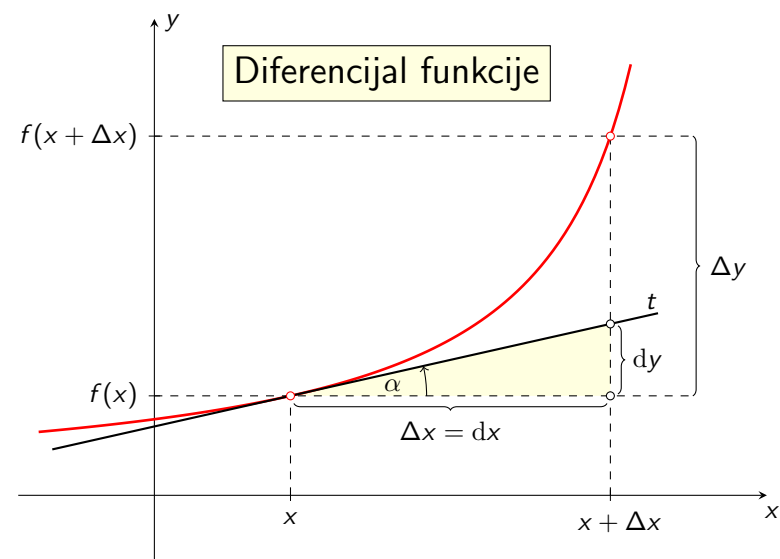
$$\int 3^x e^x dx = \int (3e)^x dx = \frac{(3e)^x}{\ln(3e)} + C, \quad C \in \mathbb{R}$$

$$(ab)^n = a^n b^n$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

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$$y = f(x), \quad f'(x) = \operatorname{tg} \alpha = \frac{dy}{dx}, \quad dy = f'(x) dx$$

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Zadatak 5

Riješite neodređeni integral $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$.

Rješenje

$$\begin{aligned} \int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} dx = \\ &= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx = \int \frac{dx}{\sin^2 x} - \int \frac{dx}{\cos^2 x} = \\ &= -\operatorname{ctg} x - \operatorname{tg} x + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

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Zadatak 6

Riješite neodređeni integral $\int (3 - 2x)^8 dx$.

Rješenje

$$\begin{aligned} \int (3 - 2x)^8 dx &= \left[\begin{array}{l} 3 - 2x = t \\ -2 dx = dt \end{array} \right] = \int t^8 \cdot \frac{dt}{-2} = \\ &= -\frac{1}{2} \int t^8 dt = -\frac{1}{2} \cdot \frac{t^9}{9} + C = -\frac{1}{18} t^9 + C = \\ &= -\frac{1}{18} (3 - 2x)^9 + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\begin{array}{l} t = f(x) \\ dt = f'(x) dx \end{array}$$

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Zadatak 7

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Riješite neodređeni integral $\int \sqrt[4]{(x-2)^3} dx$.

Rješenje

$$\begin{aligned} \int \sqrt[4]{(x-2)^3} dx &= \int (x-2)^{\frac{3}{4}} dx = \left[\begin{array}{l} x-2 = t/' \\ dx = dt \end{array} \right] = \\ &= \int t^{\frac{3}{4}} dt = \frac{t^{\frac{7}{4}}}{\frac{7}{4}} + C = \frac{4}{7} t^{\frac{7}{4}} + C = \\ &= \frac{4}{7} (x-2)^{\frac{7}{4}} + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

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Zadatak 9

Riješite neodređeni integral $\int \frac{x dx}{\sqrt{1-x^2}}$.

$$x dx = -\frac{dt}{2}$$

Rješenje

$$\begin{aligned} \int \frac{x dx}{\sqrt{1-x^2}} &= \left[\begin{array}{l} 1-x^2 = t/' \\ -2x dx = dt \end{array} \right] = \int \frac{-\frac{dt}{2}}{\sqrt{t}} = -\frac{1}{2} \int t^{-\frac{1}{2}} dt = \\ &= -\frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = -\sqrt{t} + C = -\sqrt{1-x^2} + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

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Zadatak 8

Riješite neodređeni integral $\int x \cdot 7^{x^2} dx$.

$$x dx = \frac{dt}{2}$$

Rješenje

$$\begin{aligned} \int x \cdot 7^{x^2} dx &= \left[\begin{array}{l} x^2 = t/' \\ 2x dx = dt \end{array} \right] = \int 7^t \cdot \frac{dt}{2} = \frac{1}{2} \int 7^t dt = \\ &= \frac{1}{2} \cdot \frac{7^t}{\ln 7} + C = \frac{7^t}{2 \ln 7} + C = \frac{7^{x^2}}{2 \ln 7} + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

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Zadatak 10

Riješite neodređeni integral $\int \frac{\sqrt[3]{1+\ln x}}{x} dx$.

Rješenje

$$\begin{aligned} \int \frac{\sqrt[3]{1+\ln x}}{x} dx &= \left[\begin{array}{l} 1+\ln x = t/' \\ \frac{1}{x} dx = dt \end{array} \right] = \int \sqrt[3]{t} dt = \int t^{\frac{1}{3}} dt = \\ &= \frac{t^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4} t^{\frac{4}{3}} + C = \frac{3}{4} (1+\ln x)^{\frac{4}{3}} + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

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Napomena

$$(\ln|x|)' = \frac{1}{x}, \quad x \neq 0$$

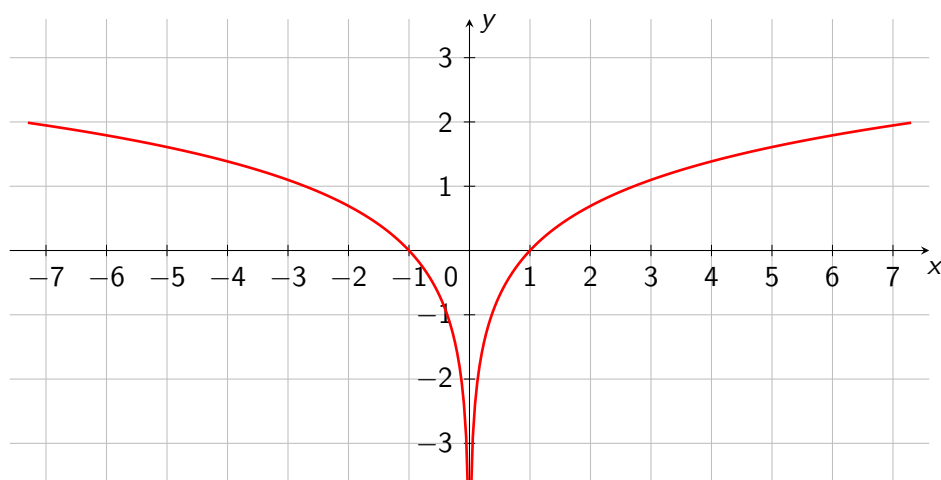
- Ako je $x > 0$, tada je $|x| = x$ pa znamo da vrijedi

$$(\ln x)' = \frac{1}{x}$$

- Ako je $x < 0$, tada je $|x| = -x$ pa korištenjem pravila za derivaciju složene funkcije ponovo dobivamo

$$(\ln(-x))' = \frac{1}{-x} \cdot (-x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

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Graf funkcije $f(x) = \ln|x|$ 

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Zadatak 11

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

Riješite neodređeni integral $\int \frac{dx}{3-2x}$.

Rješenje

$$\int \frac{dx}{3-2x} = \left[\begin{array}{l} 3-2x = t / ' \\ -2 dx = dt \end{array} \right] = \int \frac{\frac{dt}{-2}}{t} = -\frac{1}{2} \int \frac{dt}{t} =$$

$$= -\frac{1}{2} \ln|t| + C = -\frac{1}{2} \ln|3-2x| + C, \quad C \in \mathbb{R}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

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Zadatak 12

Riješite neodređeni integral $\int \frac{1-3x}{3+2x} dx$.

Rješenje

$$P_1(x) = P_2(x)Q(x) + R(x)$$

$$(-3x+1) : (2x+3) = -\frac{3}{2}$$

$$\frac{3x + \frac{9}{2}}{2x+3} = Q(x)$$

$$\frac{11}{2} = R(x)$$

$$\int \frac{1-3x}{3+2x} dx = \int \frac{-3x+1}{2x+3} dx = \int \left(-\frac{3}{2} + \frac{\frac{11}{2}}{2x+3} \right) dx =$$

$$= -\frac{3}{2} \int dx + \frac{11}{2} \int \frac{dx}{2x+3} = -\frac{3}{2}x + \frac{11}{2} \cdot \frac{1}{2} \ln|2x+3| + C =$$

$$= -\frac{3}{2}x + \frac{11}{4} \ln|2x+3| + C, \quad C \in \mathbb{R}$$

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

$$\frac{P_1(x)}{P_2(x)} = Q(x) + \frac{R(x)}{P_2(x)}$$

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Zadatak 13

Riješite neodređeni integral $\int \frac{x^2 + 5x - 4}{5x + 3} dx$.

Rješenje

$$(x^2 + 5x - 4) : (5x + 3) = \frac{1}{5}x + \frac{22}{25} \leftarrow Q(x)$$

$$\begin{array}{r} -x^2 - \frac{3}{5}x \\ \hline \end{array}$$

$$P_1(x) = P_2(x)Q(x) + R(x)$$

$$\frac{22}{5}x - 4$$

$$\frac{P_1(x)}{P_2(x)} = Q(x) + \frac{R(x)}{P_2(x)}$$

$$-\frac{22}{5}x - \frac{66}{25}$$

$$R(x) \rightarrow -\frac{166}{25}$$

$$\frac{x^2 + 5x - 4}{5x + 3} = \frac{1}{5}x + \frac{22}{25} + \frac{-\frac{166}{25}}{5x + 3}$$

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Zadatak 14

Riješite neodređeni integral $\int \frac{1 - \sin x}{x + \cos x} dx$.

Rješenje

$$\int \frac{1 - \sin x}{x + \cos x} dx = \int \frac{(x + \cos x)'}{x + \cos x} dx = \ln |x + \cos x| + C, \quad C \in \mathbb{R}$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

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$$\int \frac{x^2 + 5x - 4}{5x + 3} dx = \int \left(\frac{1}{5}x + \frac{22}{25} + \frac{-\frac{166}{25}}{5x + 3} \right) dx =$$

$$= \frac{1}{5} \int x dx + \frac{22}{25} \int dx - \frac{166}{25} \int \frac{dx}{5x + 3} =$$

$$= \frac{1}{5} \cdot \frac{x^2}{2} + \frac{22}{25}x - \frac{166}{25} \cdot \frac{1}{5} \ln |5x + 3| + C =$$

$$= \frac{1}{10}x^2 + \frac{22}{25}x - \frac{166}{125} \ln |5x + 3| + C, \quad C \in \mathbb{R}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{dx}{ax + b} = \frac{1}{a} \ln |ax + b| + C$$

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Zadatak 15

Riješite neodređeni integral $\int \frac{dx}{e^x + 2}$.

Rješenje

$$\int \frac{dx}{e^x + 2} = \frac{1}{2} \int \frac{(e^x + 2) - e^x}{e^x + 2} dx = \frac{1}{2} \int \left(1 - \frac{e^x}{e^x + 2} \right) dx =$$

$$= \frac{1}{2} \cdot \left(\int dx - \int \frac{e^x}{e^x + 2} dx \right) = \frac{1}{2}x - \frac{1}{2} \int \frac{(e^x + 2)'}{e^x + 2} dx =$$

$$= \frac{1}{2}x - \frac{1}{2} \ln(e^x + 2) + C, \quad C \in \mathbb{R}$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

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Zadatak 16

Riješite neodređeni integral $\int \operatorname{tg} x \, dx$.

$$\sin x \, dx = -dt$$

Rješenje

$$\begin{aligned} \int \operatorname{tg} x \, dx &= \int \frac{\sin x}{\cos x} \, dx = \left[\begin{array}{l} \cos x = t / ' \\ -\sin x \, dx = dt \end{array} \right] = \\ &= \int \frac{-dt}{t} = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C, \quad C \in \mathbb{R} \end{aligned}$$

$$\int \frac{dx}{x} = \ln|x| + C$$

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$$f(x) = \frac{\sin x}{\sqrt{1 + 2 \cos x}}$$

$$\int \frac{\sin x}{\sqrt{1 + 2 \cos x}} \, dx = -\sqrt{1 + 2 \cos x} + C$$

$$g(0) = 1$$

$$\rightarrow g(x) = -\sqrt{1 + 2 \cos x} + C$$

$$g(0) = -\sqrt{1 + 2 \cos 0} + C$$

$$1 = -\sqrt{1 + 2 \cdot 1} + C$$

$$1 = -\sqrt{3} + C$$

$$C = \sqrt{3} + 1$$

$$g(x) = -\sqrt{1 + 2 \cos x} + \sqrt{3} + 1$$

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Zadatak 17

Odredite primitivnu funkciju g funkcije $f(x) = \frac{\sin x}{\sqrt{1 + 2 \cos x}}$ za koju je $g(0) = 1$.

Rješenje

$$\sin x \, dx = -\frac{dt}{2}$$

$$\int \frac{\sin x}{\sqrt{1 + 2 \cos x}} \, dx = \left[\begin{array}{l} 1 + 2 \cos x = t / ' \\ -2 \sin x \, dx = dt \end{array} \right] = \int \frac{-\frac{dt}{2}}{\sqrt{t}} =$$

$$= -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \int t^{-\frac{1}{2}} \, dt = -\frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = -\sqrt{t} + C =$$

$$= -\sqrt{1 + 2 \cos x} + C, \quad C \in \mathbb{R}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

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