Neodređeni integral – 2. dio

Matematika 2

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Parcijalna integracija

$$\int {\color{red}u'(x)}v(x)\,{\color{red}\mathrm{d}x} = u(x)v(x) - \int u(x){\color{red}v'(x)\,\mathrm{d}x}$$

$\mathrm{d}u = u'(x)\,\mathrm{d}x$ $\qquad$ $\mathrm{d}v = v'(x)\,\mathrm{d}x$

$$\int v\,\mathrm{d}u = uv - \int u\,\mathrm{d}v$$

Zadatak 1

Riješite neodređeni integral $\int \ln x\,\mathrm{d}x$.

Rješenje

$$\int \ln x\,\mathrm{d}x = \int 1\cdot\ln x\,\mathrm{d}x = \int (x)'\ln x\,\mathrm{d}x =$$

$$= x\ln x - \int x\cdot(\ln x)'\,\mathrm{d}x = x\ln x - \int x\cdot\frac{1}{x}\,\mathrm{d}x =$$

$$= x\ln x - \int \mathrm{d}x = x\ln x - x + C, \quad C\in\mathbb{R}$$

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

Zadatak 2

Riješite neodređeni integral $\int x^4\ln 8x\,\mathrm{d}x$.

Rješenje

$$\int x^4\ln 8x\,\mathrm{d}x = \int\left(\frac{x^5}{5}\right)'\ln 8x\,\mathrm{d}x = \frac{x^5}{5}\ln 8x - \int\frac{x^5}{5}\big(\ln 8x\big)'\,\mathrm{d}x =$$

$$= \frac{x^5}{5}\ln 8x - \int\frac{x^5}{5}\cdot\frac{1}{8x}\cdot 8\,\mathrm{d}x = \frac{x^5}{5}\ln 8x - \frac{1}{5}\int x^4\,\mathrm{d}x =$$

$$= \frac{x^5}{5}\ln 8x - \frac{1}{5}\cdot\frac{x^5}{5} + C = \frac{x^5}{5}\ln 8x - \frac{1}{25}x^5 + C, \quad C\in\mathbb{R}$$

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

Zadatak 3

Riješite neodređeni integral $\int x\cos 3x\,\mathrm{d}x$.

Rješenje

$$\int x\cos 3x\,\mathrm{d}x = \int x\cdot\left(\frac{1}{3}\sin 3x\right)'\mathrm{d}x =$$

$$= x\cdot\frac{1}{3}\sin 3x - \int (x)'\cdot\frac{1}{3}\sin 3x\,\mathrm{d}x = \frac{x}{3}\sin 3x - \frac{1}{3}\int \sin 3x\,\mathrm{d}x =$$

$$= \frac{x}{3}\sin 3x - \frac{1}{3}\cdot\frac{-1}{3}\cos 3x + C = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + C,\quad C\in\mathbb{R}$$

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

$$\int \cos 3x\,\mathrm{d}x = \left[\begin{array}{c} 3x = t\ /\ ' \\ 3\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = \int \cos t\cdot\frac{\mathrm{d}t}{3} =$$

$$= \frac{1}{3}\int \cos t\,\mathrm{d}t = \frac{1}{3}\sin t + C = \frac{1}{3}\sin 3x + C,\quad C\in\mathbb{R}$$

$$\int \sin 3x\,\mathrm{d}x = \left[\begin{array}{c} 3x = t\ /\ ' \\ 3\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = \int \sin t\cdot\frac{\mathrm{d}t}{3} =$$

$$= \frac{1}{3}\int \sin t\,\mathrm{d}t = -\frac{1}{3}\cos t + C = -\frac{1}{3}\cos 3x + C,\quad C\in\mathbb{R}$$

Zadatak 4

Riješite neodređeni integral $\int (x^2+x)e^{5x}\,\mathrm{d}x$.

Rješenje

$$\int (x^2+x)e^{5x}\,\mathrm{d}x = \int (x^2+x)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

$$= (x^2+x)\cdot\frac{1}{5}e^{5x} - \int (x^2+x)'\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2 + \frac{1}{5}x\right)e^{5x} - \frac{1}{5}\int (2x+1)e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2 + \frac{1}{5}x\right)e^{5x} - \frac{1}{5}\int (2x+1)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

$$= \left(\frac{1}{5}x^2 + \frac{1}{5}x\right)e^{5x} - \frac{1}{5}\cdot\int (2x+1)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2 + \frac{1}{5}x\right)e^{5x} - \frac{1}{5}\cdot\left[(2x+1)\cdot\frac{1}{5}e^{5x} - \int (2x+1)'\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x\right] =$$

$$= \left(\frac{1}{5}x^2 + \frac{1}{5}x\right)e^{5x} - \left(\frac{2}{25}x + \frac{1}{25}\right)e^{5x} + \frac{1}{5}\int 2\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2 + \frac{3}{25}x - \frac{1}{25}\right)e^{5x} + \frac{2}{25}\int e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2 + \frac{3}{25}x - \frac{1}{25}\right)e^{5x} + \frac{2}{25}\cdot\frac{1}{5}e^{5x} + C =$$

$$= \left(\frac{1}{5}x^2 + \frac{3}{25}x - \frac{3}{125}\right)e^{5x} + C,\quad C\in\mathbb{R}$$

Zadatak 5

Riješite neodređeni integral $\int e^{2x}\sin 3x\,\mathrm{d}x$.

Rješenje

$$\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x$$

$$\int e^{2x}\sin 3x\,\mathrm{d}x = \int \left(\frac{1}{2}e^{2x}\right)' \cdot \sin 3x\,\mathrm{d}x =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \int \frac{1}{2}e^{2x}\cdot(\sin 3x)'\,\mathrm{d}x =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{1}{2}\int e^{2x}\cdot 3\cos 3x\,\mathrm{d}x =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,\mathrm{d}x =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int \left(\frac{1}{2}e^{2x}\right)' \cdot \cos 3x\,\mathrm{d}x =$$

$$\int e^{2x}\sin 3x\,\mathrm{d}x = \cdots = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} - \frac{9}{4}\int e^{2x}\sin 3x\,\mathrm{d}x$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int \left(\frac{1}{2}e^{2x}\right)' \cdot \cos 3x\,\mathrm{d}x =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\cdot\left[\frac{1}{2}e^{2x}\cos 3x - \int \frac{1}{2}e^{2x}\cdot(\cos 3x)'\,\mathrm{d}x\right] =$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + \frac{3}{2}\int \frac{1}{2}e^{2x}\cdot(-3\sin 3x)\,\mathrm{d}x =$$

$$= \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} - \frac{9}{4}\int e^{2x}\sin 3x\,\mathrm{d}x$$

početni integral

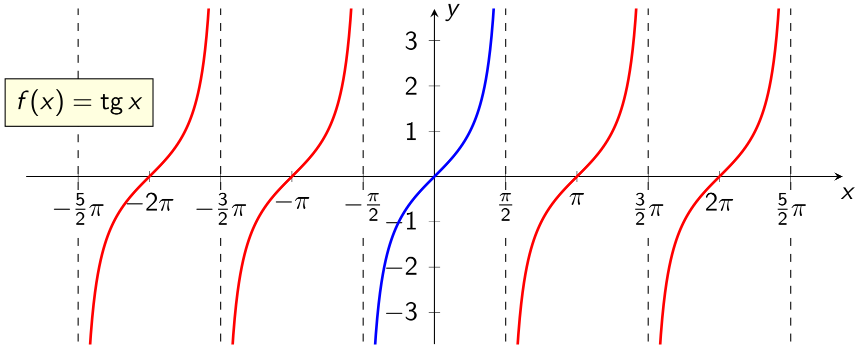
$$\int e^{2x}\sin 3x\,\mathrm{d}x = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} - \frac{9}{4}\int e^{2x}\sin 3x\,\mathrm{d}x$$

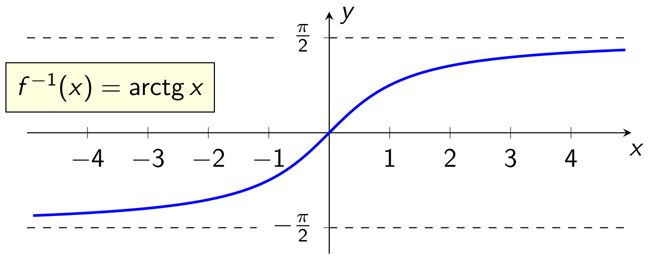
$$\int e^{2x}\sin 3x\,\mathrm{d}x + \frac{9}{4}\int e^{2x}\sin 3x\,\mathrm{d}x = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x}$$

$$\frac{13}{4}\int e^{2x}\sin 3x\,\mathrm{d}x = \left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} \Big/ \cdot \frac{4}{13}$$

$$\int e^{2x}\sin 3x\,\mathrm{d}x = \frac{4}{13}\cdot\left(\frac{1}{2}\sin 3x - \frac{3}{4}\cos 3x\right)e^{2x} + C$$

$$\int e^{2x}\sin 3x\,\mathrm{d}x = \left(\frac{2}{13}\sin 3x - \frac{3}{13}\cos 3x\right)e^{2x} + C, \quad C\in\mathbb{R}$$





Funkcija

$$f : \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle \to \mathbb{R}, \quad f(x) = \operatorname{tg} x$$

je bijekcija i ima inverznu funkciju

$$f^{-1} : \mathbb{R} \to \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle, \quad f^{-1}(x) = \operatorname{arctg} x.$$

Derivacija inverzne funkcije jednaka je

$$\big(\operatorname{arctg} x\big)' = \frac{1}{x^2+1}$$

odnosno

$$\int \frac{\mathrm{d}x}{x^2+1} = \operatorname{arctg} x + C, \quad C \in \mathbb{R}.$$

Zadatak 6

Riješite neodređeni integral $\displaystyle\int \frac{x^2}{x^6+1}\,\mathrm{d}x$.

$$x^3 = t \;/\;^2$$
$$x^6 = t^2$$

$$x^2\,\mathrm{d}x = \frac{\mathrm{d}t}{3}$$

Rješenje

$$\int \frac{x^2}{x^6+1}\,\mathrm{d}x = \left[\begin{array}{c} x^3 = t \;/\;' \\ 3x^2\,\mathrm{d}x = \mathrm{d}t \end{array}\right] = \int \frac{\frac{\mathrm{d}t}{3}}{t^2+1} = \frac{1}{3}\int \frac{\mathrm{d}t}{t^2+1} =$$

$$= \frac{1}{3}\operatorname{arctg} t + C = \frac{1}{3}\operatorname{arctg} x^3 + C, \quad C \in \mathbb{R}$$

$$\int \frac{\mathrm{d}x}{x^2+1} = \operatorname{arctg} x + C$$

Zadatak 7

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

Riješite neodređeni integral $\displaystyle\int \frac{\mathrm{d}x}{3x^2+5}$.

Rješenje

$$\int \frac{\mathrm{d}x}{3x^2+5} = \int \frac{\mathrm{d}x}{3\left(x^2+\frac{5}{3}\right)} = \frac{1}{3}\int \frac{\mathrm{d}x}{x^2+\sqrt{\frac{5}{3}}^{\,2}} =$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{\frac{5}{3}}}\operatorname{arctg}\frac{x}{\sqrt{\frac{5}{3}}} + C = \frac{\sqrt{3}}{3\sqrt{5}}\operatorname{arctg}\frac{\sqrt{3}x}{\sqrt{5}} + C =$$

$$= \frac{\sqrt{15}}{15}\operatorname{arctg}\frac{\sqrt{15}}{5}x + C, \quad C \in \mathbb{R}$$

$$\frac{\sqrt{3}}{3\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{3\cdot 5} = \frac{\sqrt{15}}{15} \qquad \frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$$

Zadatak 8

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

Riješite neodređeni integral $\displaystyle\int \frac{\mathrm{d}x}{x^2-3}$.

Rješenje

$$\frac{1}{x^2-3} = \frac{1}{\big(x-\sqrt{3}\big)\big(x+\sqrt{3}\big)} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}} =$$

$$= \frac{A\big(x+\sqrt{3}\big) + B\big(x-\sqrt{3}\big)}{\big(x-\sqrt{3}\big)\big(x+\sqrt{3}\big)}$$

$$1 = A\big(x+\sqrt{3}\big) + B\big(x-\sqrt{3}\big)$$

$x = \sqrt{3}$

$$1 = A\cdot 2\sqrt{3} + B\cdot 0$$

$$A = \frac{1}{2\sqrt{3}}$$

$x = -\sqrt{3}$

$$1 = A\cdot 0 + B\cdot\big(-2\sqrt{3}\big)$$

$$B = -\frac{1}{2\sqrt{3}}$$

$$\frac{1}{x^2-3} = \frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} + \frac{-\frac{1}{2\sqrt{3}}}{x+\sqrt{3}}$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \qquad \int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\mathrm{d}x}{x^2-3} = \int \left(\frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} + \frac{-\frac{1}{2\sqrt{3}}}{x+\sqrt{3}} \right) \mathrm{d}x =$$

$$= \frac{1}{2\sqrt{3}} \int \frac{\mathrm{d}x}{x-\sqrt{3}} - \frac{1}{2\sqrt{3}} \int \frac{\mathrm{d}x}{x+\sqrt{3}} =$$

$$= \frac{1}{2\sqrt{3}} \ln\left|x-\sqrt{3}\right| - \frac{1}{2\sqrt{3}} \ln\left|x+\sqrt{3}\right| + C =$$

$$= \frac{1}{2\sqrt{3}} \ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C, \quad C \in \mathbb{R}$$

Zadatak 9

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} + C$$

Riješite neodređeni integral $\int \frac{\mathrm{d}x}{3x^2+x+4}$.

Rješenje

$$\left(\frac{\frac{1}{3}}{2}\right)^2$$

$$3x^2+x+4 = 3\cdot\left(x^2+\frac{1}{3}x+\frac{4}{3}\right) =$$

$$= 3\cdot\left(x^2+\frac{1}{3}x+\frac{1}{36} - \frac{1}{36}+\frac{4}{3}\right) =$$

$$= 3\cdot\left(\left(x+\frac{1}{6}\right)^2 + \frac{47}{36}\right) =$$

$$= 3\cdot\left(\left(x+\frac{1}{6}\right)^2 + \left(\frac{\sqrt{47}}{6}\right)^2\right)$$

$$\int \frac{\mathrm{d}x}{3x^2+x+4} = \int \frac{\mathrm{d}x}{3\cdot\left(\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{47}}{6}\right)^2\right)} =$$

$$= \frac{1}{3}\int \frac{\mathrm{d}x}{\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{47}}{6}\right)^2} = \left[\begin{array}{c} x+\frac{1}{6} = t \;/\,' \\ \mathrm{d}x = \mathrm{d}t \end{array}\right] =$$

$$= \frac{1}{3}\int \frac{\mathrm{d}t}{t^2+\left(\frac{\sqrt{47}}{6}\right)^2} = \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{47}}{6}}\,\mathrm{arctg}\,\frac{t}{\frac{\sqrt{47}}{6}} + C =$$

$$= \frac{2}{\sqrt{47}}\,\mathrm{arctg}\,\frac{6t}{\sqrt{47}} + C = \frac{2}{\sqrt{47}}\,\mathrm{arctg}\,\frac{6\cdot\left(x+\frac{1}{6}\right)}{\sqrt{47}} + C =$$

$$= \frac{2}{\sqrt{47}}\,\mathrm{arctg}\,\frac{6x+1}{\sqrt{47}} + C, \quad C \in \mathbb{R}$$

Zadatak 10

Riješite neodređeni integral $\int \frac{\mathrm{d}x}{x^2+5x-4}$.

$$\left(\frac{5}{2}\right)^2$$

Rješenje

$$x^2+5x-4 = x^2+5x+\frac{25}{4} - \frac{25}{4} - 4 =$$

$$= \left(x+\frac{5}{2}\right)^2 - \frac{41}{4} =$$

$$= \left(x+\frac{5}{2}\right)^2 - \left(\frac{\sqrt{41}}{2}\right)^2$$

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\mathrm{d}x}{x^2+5x-4} = \int \frac{\mathrm{d}x}{\left(x+\frac{5}{2}\right)^2 - \left(\frac{\sqrt{41}}{2}\right)^2} = \begin{bmatrix} x+\frac{5}{2} = t \;/\,' \\ \mathrm{d}x = \mathrm{d}t \end{bmatrix} =$$

$$= \int \frac{\mathrm{d}t}{t^2 - \left(\frac{\sqrt{41}}{2}\right)^2} = \frac{1}{2\cdot\frac{\sqrt{41}}{2}} \ln\left|\frac{t-\frac{\sqrt{41}}{2}}{t+\frac{\sqrt{41}}{2}}\right| + C =$$

$$= \frac{1}{\sqrt{41}} \ln\left|\frac{2t-\sqrt{41}}{2t+\sqrt{41}}\right| + C = \frac{1}{\sqrt{41}} \ln\left|\frac{2\cdot\left(x+\frac{5}{2}\right)-\sqrt{41}}{2\cdot\left(x+\frac{5}{2}\right)+\sqrt{41}}\right| + C =$$

$$= \frac{1}{\sqrt{41}} \ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C, \quad C \in \mathbb{R}$$

Zadatak 11

Riješite neodređeni integral $\int \frac{5x+3}{x^2+5x-4}\,\mathrm{d}x$.

Rješenje

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$$

$$\int \frac{5x+3}{x^2+5x-4}\,\mathrm{d}x = \int \frac{\frac{5}{2}\cdot(2x+5) - \frac{19}{2}}{x^2+5x-4}\,\mathrm{d}x =$$

$$= \frac{5}{2}\int \frac{2x+5}{x^2+5x-4}\,\mathrm{d}x - \frac{19}{2}\boxed{\int \frac{\mathrm{d}x}{x^2+5x-4}} =$$

prethodni zadatak

$$= \frac{5}{2}\ln\left|x^2+5x-4\right| - \frac{19}{2}\cdot\frac{1}{\sqrt{41}}\ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C =$$

$$= \frac{5}{2}\ln\left|x^2+5x-4\right| - \frac{19}{2\sqrt{41}}\ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C, \quad C \in \mathbb{R}$$

Zadatak 12

Riješite neodređeni integral $\int \frac{4x^2+3x-20}{(x+2)^2(x-3)}\,\mathrm{d}x$.

Rješenje

$$\frac{4x^2+3x-20}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3} =$$

$$= \frac{A(x+2)(x-3) + B(x-3) + C(x+2)^2}{(x+2)^2(x-3)}$$

$$4x^2+3x-20 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$$

$$4x^2+3x-20 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$$

$x = -2$

$$4\cdot(-2)^2 + 3\cdot(-2) - 20 = A\cdot 0 + B\cdot(-5) + C\cdot 0$$
$$-10 = -5B$$
$$\boxed{B = 2}$$

$x = 3$

$$4\cdot 3^2 + 3\cdot 3 - 20 = A\cdot 0 + B\cdot 0 + C\cdot 25$$
$$25 = 25C$$
$$\boxed{C = 1}$$

$x = 0$

$$4\cdot 0 + 3\cdot 0 - 20 = A\cdot(0+2)\cdot(0-3) + B\cdot(0-3) + C\cdot(0+2)^2$$
$$-20 = -6A - 3B + 4C$$
$$-20 = -6A - 3\cdot 2 + 4\cdot 1$$
$$-20 = -6A - 2$$
$$6A = 18 \rightarrow \boxed{A = 3}$$

$$\frac{4x^2+3x-20}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$$

$$\frac{4x^2+3x-20}{(x+2)^2(x-3)} = \frac{3}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x-3}$$

$$\int \frac{4x^2+3x-20}{(x+2)^2(x-3)}\,\mathrm{d}x = \int \left(\frac{3}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x-3}\right)\mathrm{d}x =$$

$$= 3\int \frac{\mathrm{d}x}{x+2} + 2\int \frac{\mathrm{d}x}{(x+2)^2} + \int \frac{\mathrm{d}x}{x-3} =$$

$$= 3\ln|x+2| - \frac{2}{x+2} + \ln|x-3| + C, \quad C \in \mathbb{R}$$

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{\mathrm{d}x}{(x+2)^2} = \left[\begin{array}{c} x+2=t\ /' \\ \mathrm{d}x=\mathrm{d}t \end{array}\right] = \int \frac{\mathrm{d}t}{t^2} = \int t^{-2}\,\mathrm{d}t =$$

$$= \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C = -\frac{1}{x+2} + C, \quad C \in \mathbb{R}$$

Zadatak 13

Riješite neodređeni integral $\int \frac{x+1}{(x-1)^2(x^2+1)}\,\mathrm{d}x$.

Rješenje

$$\frac{x+1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} =$$

$$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$x+1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$x+1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$x = 1$

$$1+1 = A\cdot 0 + B\cdot 2 + (C+D)\cdot 0$$

$$2 = 2B$$

$$B = 1$$

$i^2 = -1$

$z_1 = x_1 + y_1 i, \quad z_2 = x_2 + y_2 i$

$z_1 = z_2 \Leftrightarrow (x_1 = x_2) \wedge (y_1 = y_2)$

$x = i$

$$i+1 = A\cdot(i-1)\cdot\underbrace{(i^2+1)}_{=0} + B\cdot\underbrace{(i^2+1)}_{=0} + (Ci+D)\cdot(i-1)^2$$

$$i+1 = (Ci+D)\cdot(i^2-2i+1)$$

$$i+1 = (Ci+D)\cdot(-2i)$$

$$1+i = 2C - 2Di$$

$2C = 1 \qquad -2D = 1$

$C = \frac{1}{2} \qquad D = -\frac{1}{2}$

$x = 0$

$$0+1 = A\cdot(0-1)\cdot(0^2+1) + B\cdot(0^2+1) + (C\cdot 0 + D)(0-1)^2$$

$$1 = -A + B + D$$

$$1 = -A + 1 - \frac{1}{2}$$

$$A = -\frac{1}{2}$$

$$\frac{x+1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{x+1}{(x-1)^2(x^2+1)} = \frac{-\frac{1}{2}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}$$

$$\int \frac{x+1}{(x-1)^2(x^2+1)}\,\mathrm{d}x = \int \left(\frac{-\frac{1}{2}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1} \right) \mathrm{d}x =$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}x}{x-1} + \int \frac{\mathrm{d}x}{(x-1)^2} + \int \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}\,\mathrm{d}x =$$

$$= -\frac{1}{2}\ln|x-1| - \frac{1}{x-1} + \frac{1}{4}\ln\left(x^2+1\right) - \frac{1}{2}\operatorname{arctg} x + C, \quad C \in \mathbb{R}$$

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{\mathrm{d}x}{(x-1)^2} = \left[\begin{array}{c} x-1 = t \,/\, ' \\ \mathrm{d}x = \mathrm{d}t \end{array} \right] = \int \frac{\mathrm{d}t}{t^2} = \int t^{-2}\,\mathrm{d}t =$$

$$= \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C = -\frac{1}{x-1} + C, \quad C \in \mathbb{R}$$

$$\int \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}\,\mathrm{d}x = \int \frac{\frac{1}{4}\cdot \boxed{2x} - \frac{1}{2}}{x^2+1}\,\mathrm{d}x = \qquad (x^2+1)'$$

$$= \frac{1}{4}\int \frac{2x}{x^2+1}\,\mathrm{d}x - \frac{1}{2}\int \frac{\mathrm{d}x}{x^2+1} = \frac{1}{4}\ln\left(x^2+1\right) - \frac{1}{2}\operatorname{arctg} x + C,$$

$$C \in \mathbb{R}$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C \qquad \int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$