Određeni integral

Matematika 2

Damir Horvat

FOI, Varaždin

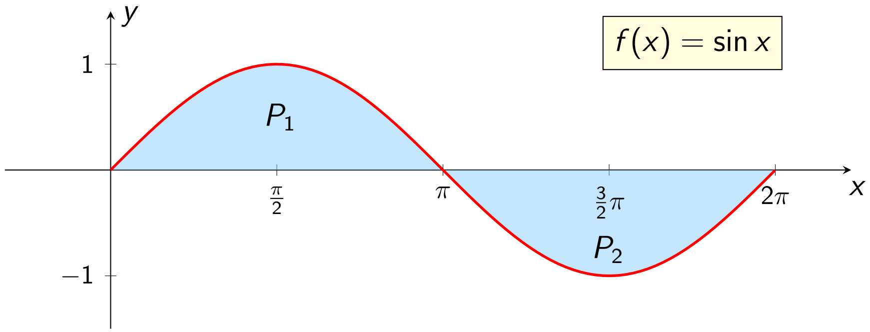
Newton-Leibnizova formula

Teorem

Ako je f neprekidna funkcija na otvorenom intervalu I i F bilo koja primitivna funkcija funkcije f na I, tada za svaki $[a,b] \subseteq I$ vrijedi

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b \qquad F'(x) = f(x),\ x \in [a,b]$$



Vrijednost integrala na segmentu $[0, 2\pi]$

$$\int_0^{2\pi} \sin x\,\mathrm{d}x = -\cos x\Big|_0^{2\pi} = -\cos 2\pi - \big(-\cos 0\big) =$$

$$= -1 - (-1) = -1 + 1 = 0$$

Površina između grafa funkcije i x-osi na segmentu $[0, 2\pi]$

$$P_1 = \int_0^{\pi} \sin x\,\mathrm{d}x = -\cos x\Big|_0^{\pi} = -\cos \pi - \big(-\cos 0\big) =$$

$$= -(-1) - (-1) = 1 + 1 = 2$$

$$P_2 = -\int_{\pi}^{2\pi} \sin x\,\mathrm{d}x = -\big(-\cos x\big)\Big|_{\pi}^{2\pi} = \cos x\Big|_{\pi}^{2\pi} =$$

$$= \cos 2\pi - \cos \pi = 1 - (-1) = 1 + 1 = 2$$

$$P = P_1 + P_2 = 2 + 2 = 4$$

Zadatak 1

Zadana je funkcija $f(x) = \dfrac{2}{x^2+1} - 1.$

a) *Izračunajte* $\int_0^{\sqrt{3}} f(x)\,\mathrm{d}x.$

b) *Izračunajte površinu koju graf funkcije* f *zatvara s* x*-osi na segmentu* $[0, \sqrt{3}]$.

Rješenje

a)

$$\int \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x = 2\int \frac{\mathrm{d}x}{x^2+1} - \int \mathrm{d}x =$$

$$= 2\,\mathrm{arctg}\,x - x + C, \quad C \in \mathbb{R}$$

$$\int_0^{\sqrt{3}} \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x = \left(2\,\mathrm{arctg}\,x - x\right)\Big|_0^{\sqrt{3}} =$$

$$= \left(2\,\mathrm{arctg}\,\sqrt{3} - \sqrt{3}\right) - \left(2\,\mathrm{arctg}\,0 - 0\right) =$$

$$= 2\cdot\frac{\pi}{3} - \sqrt{3} - 2\cdot 0 + 0 = \boxed{\frac{2}{3}\pi - \sqrt{3}} \approx 0.36234$$

egzaktna vrijednost

aproksimacija na pet decimala

b)

$$\lim_{x\to\pm\infty} f(x) = \frac{2}{+\infty} - 1 = 0 - 1 = -1$$

$$f(x) = \frac{2}{x^2+1} - 1$$

$\boxed{y = -1}$ ⟵ horizontalna asimptota

$$\frac{2}{x^2+1} - 1 = 0 \;\Big/ \cdot (x^2+1)$$

$$2 = x^2 + 1$$

$$x^2 = 1$$

$\boxed{x_1 = -1}$ $\boxed{x_2 = 1}$ nultočke

$$f'(x) = \frac{\overbrace{(2)'}^{=0} \cdot (x^2+1) - 2\cdot \overbrace{(x^2+1)'}^{=2x}}{(x^2+1)^2} - 0 = \frac{-4x}{(x^2+1)^2}$$

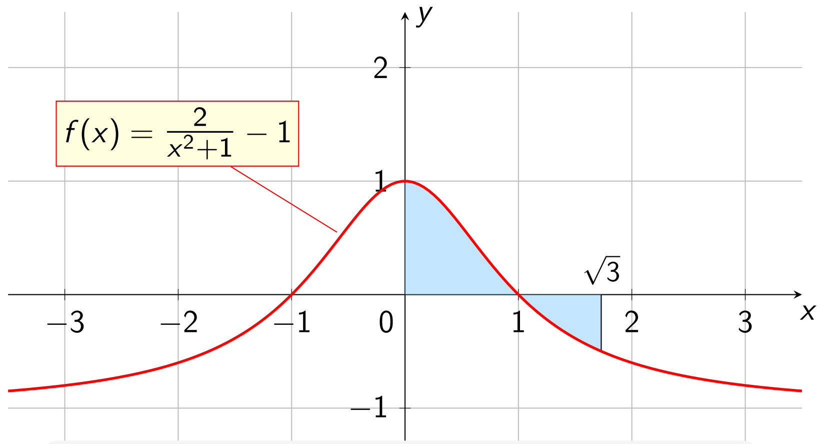
$$\frac{-4x}{(x^2+1)^2} = 0$$

$$-4x = 0$$

$\boxed{x = 0}$ ⟵ stacionarna točka

$f(0) = 1$ ⟵ globalni maksimum

	$-\infty$	0	$+\infty$
f'	$+$	$-$	
f	$\nearrow$	$\searrow$	



$$P = \int_0^1 \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x - \int_1^{\sqrt{3}} \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x$$

$$P = \int_0^1 \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x - \int_1^{\sqrt{3}} \left(\frac{2}{x^2+1} - 1\right) \mathrm{d}x =$$

$$= (2\operatorname{arctg} x - x)\Big|_0^1 - (2\operatorname{arctg} x - x)\Big|_1^{\sqrt{3}} =$$

$$= \Big[(2\operatorname{arctg} 1 - 1) - (2\operatorname{arctg} 0 - 0)\Big] -$$

$$- \Big[(2\operatorname{arctg}\sqrt{3} - \sqrt{3}) - (2\operatorname{arctg} 1 - 1)\Big] =$$

$$= \left(2\cdot\frac{\pi}{4} - 1\right) - (2\cdot 0 - 0) - \left(2\cdot\frac{\pi}{3} - \sqrt{3}\right) + \left(2\cdot\frac{\pi}{4} - 1\right) =$$

$$= \frac{\pi}{2} - 1 - 0 - \frac{2}{3}\pi + \sqrt{3} + \frac{\pi}{2} - 1 = \boxed{\frac{\pi}{3} + \sqrt{3} - 2} \approx 0.77925$$

egzaktna vrijednost

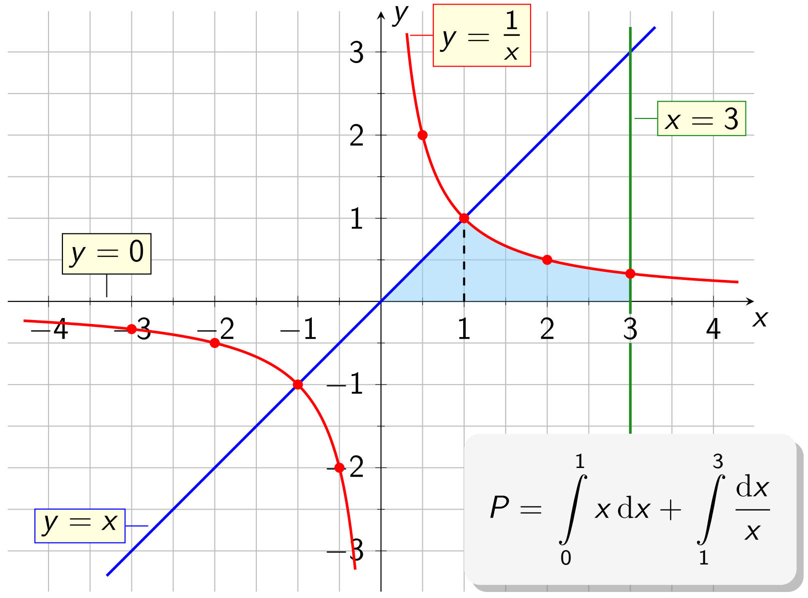
aproksimacija na pet decimala

Zadatak 2

Izračunajte površinu lika kojeg omeđuju krivulje

$$y = \frac{1}{x}, \quad y = x, \quad y = 0, \quad x = 3.$$

Rješenje



$$P = \int_0^1 x\mathrm{d}x + \int_1^3 \frac{\mathrm{d}x}{x} = \frac{x^2}{2}\Big|_0^1 + \ln|x|\Big|_1^3 =$$

$$= \left(\frac{1}{2} - 0\right) + (\ln 3 - \ln 1) = \frac{1}{2} + \ln 3$$

Zadatak 3

Izračunajte površinu lika omeđenog grafovima funkcija $f(x) = x^2$ i $g(x) = x + 2$.

$$ax^2 + bx + c = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Rješenje

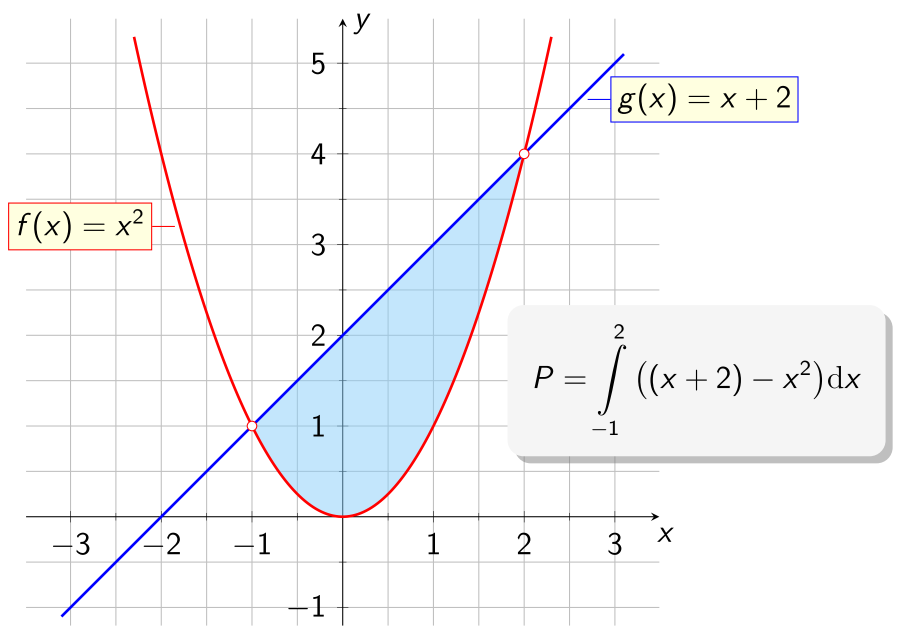
- Presjek pravca i parabole

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$$
$$x_{1,2} = \frac{1 \pm 3}{2}$$
$$x_1 = 2, \quad x_2 = -1$$
$$y_1 = 4, \quad y_2 = 1$$

točke presjeka

$T_1(2, 4)$

$T_2(-1, 1)$



$$P = \int_{-1}^{2} \left((x+2) - x^2\right) \mathrm{d}x = \int_{-1}^{2} \left(-x^2 + x + 2\right) \mathrm{d}x =$$
$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Big|_{-1}^{2} =$$
$$= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2 \cdot 2\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2 \cdot (-1)\right) =$$
$$= \left(-\frac{8}{3} + \frac{4}{2} + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) =$$
$$= -\frac{8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2 = \frac{9}{2}$$

Zadatak 4

Izračunajte površinu lika kojeg omeđuju krivulje

$$y = \frac{1}{x}, \quad y = 2^{x-1}, \quad y = 4.$$

Rješenje

- Presjek krivulja $y = \frac{1}{x}$ i $y = 4$

$$\frac{1}{x} = 4 \;\Big/ \cdot x$$
$$4x = 1$$
$$x = \frac{1}{4}$$
$$\left(\frac{1}{4}, 4\right)$$

- Presjek krivulja $y = 2^{x-1}$ i $y = 4$

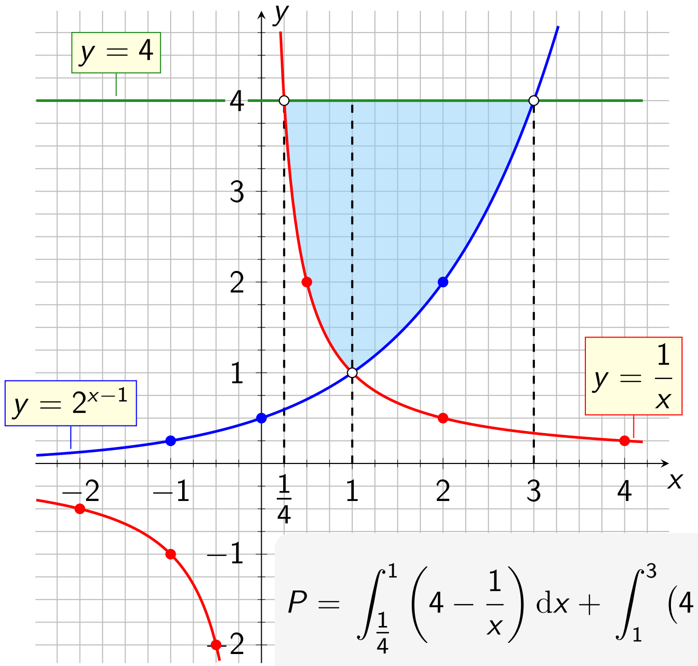
$$2^{x-1} = 4$$
$$x - 1 = \log_2 4$$
$$x = 2 + 1$$
$$x = 3$$
$$(3, 4)$$

- Presjek krivulja $y = 2^{x-1}$ i $y = \frac{1}{x}$

$$2^{x-1} = \frac{1}{x}$$

pogađamo rješenje

$$(1, 1)$$



$$P = \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) \mathrm{d}x + \int_{1}^{3} \left(4 - 2^{x-1}\right) \mathrm{d}x$$

$$\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C$$

$$P = \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) \mathrm{d}x + \int_{1}^{3} \left(4 - 2^{x-1}\right) \mathrm{d}x =$$

$$= \left(4x - \ln|x|\right)\Big|_{\frac{1}{4}}^{1} + \left(4x - \frac{2^{x-1}}{\ln 2}\right)\Big|_{1}^{3} =$$

$$= \left((4 - \ln 1) - \left(1 - \ln \frac{1}{4}\right)\right) + \left(\left(12 - \frac{4}{\ln 2}\right) - \left(4 - \frac{1}{\ln 2}\right)\right) =$$

$$= \left(3 + \ln \frac{1}{4}\right) + \left(8 - \frac{3}{\ln 2}\right) =$$

$$= 11 + \ln \frac{1}{4} - \frac{3}{\ln 2}$$

$P \approx 5.28562$

Zadatak 5

Pomoću određenog integrala dokažite da je površina kruga polumjera r jednaka $r^2\pi$.

Rješenje

- Jednadžba kružnice polumjera r sa središtem u ishodištu je

$$x^2 + y^2 = r^2.$$

- Kružnica nije graf niti jedne realne funkcije realne varijable. Međutim, *gornja polukružnica* jest graf funkcije

$$f(x) = \sqrt{r^2 - x^2}.$$

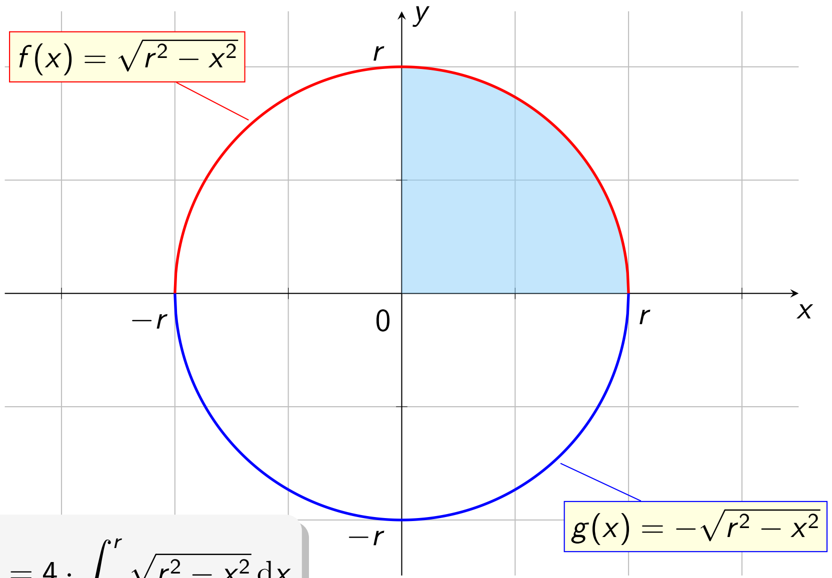
$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

+ gornja polukružnica

− donja polukružnica



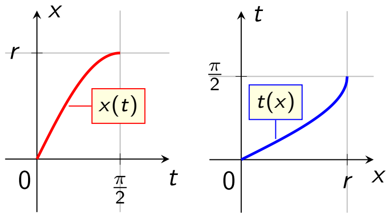
$$P = 4 \cdot \int_{0}^{r} \sqrt{r^2 - x^2}\,\mathrm{d}x$$

$$P = 4 \cdot \int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x =$$

$x : [0, \frac{\pi}{2}] \to [0, r],\ x(t) = r \sin t$

$t : [0, r] \to [0, \frac{\pi}{2}],\ t(x) = \arcsin \frac{x}{r}$

$$= \left[\begin{array}{ll} x = r \sin t \,/\,' & x = 0 \rightsquigarrow t = 0 \\ \mathrm{d}x = r\cos t\,\mathrm{d}t & x = r \rightsquigarrow t = \frac{\pi}{2} \end{array} \right] =$$



$$= 4 \cdot \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t} \cdot r\cos t\,\mathrm{d}t =$$

$$= 4 \cdot \int_0^{\frac{\pi}{2}} \sqrt{r^2(1 - \sin^2 t)} \cdot r\cos t\,\mathrm{d}t = 4 \cdot \int_0^{\frac{\pi}{2}} \sqrt{r^2\cos^2 t} \cdot r\cos t\,\mathrm{d}t =$$

$\sqrt{a^2} = |a|$

$$= 4 \cdot \int_0^{\frac{\pi}{2}} \underbrace{\sqrt{r^2}}_{=r} \cdot \sqrt{\cos^2 t} \cdot r\cos t\,\mathrm{d}t = 4 \cdot \int_0^{\frac{\pi}{2}} r \cdot \underbrace{|\cos t|}_{=\cos t} \cdot r\cos t\,\mathrm{d}t =$$

jer je $r > 0$ jer je $\cos t \geqslant 0$ za $t \in [0, \frac{\pi}{2}]$

$$= 4 \cdot \int_0^{\frac{\pi}{2}} r^2 \cdot \cos t \cdot \cos t\,\mathrm{d}t = 4r^2 \cdot \int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t$$

$$\int \cos^2 t\,\mathrm{d}t = \int \frac{\cos 2t + 1}{2}\,\mathrm{d}t = \int \left(\frac{1}{2}\cos 2t + \frac{1}{2}\right)\mathrm{d}t =$$

$$= \frac{1}{2}\int \cos 2t\,\mathrm{d}t + \frac{1}{2}\int \mathrm{d}t = \frac{1}{2} \cdot \frac{1}{2}\sin 2t + \frac{1}{2}t + C =$$

$$= \frac{1}{4}\sin 2t + \frac{1}{2}t + C, \quad C \in \mathbb{R}$$

$$P = 4r^2 \cdot \int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t = 4r^2 \left(\frac{1}{4}\sin 2t + \frac{1}{2}t\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= 4r^2\left(\frac{1}{4}\sin\left(2 \cdot \frac{\pi}{2}\right) + \frac{1}{2} \cdot \frac{\pi}{2}\right) - 4r^2\left(\frac{1}{4}\sin(2 \cdot 0) + \frac{1}{2} \cdot 0\right) =$$

$$= 4r^2\left(\frac{1}{4}\overbrace{\sin\pi}^{=0} + \frac{\pi}{4}\right) - 4r^2\left(\frac{1}{4}\overbrace{\sin 0}^{=0} + 0\right) = 4r^2 \cdot \frac{\pi}{4} - 4r^2 \cdot 0 = r^2\pi$$

Dobivanje decimala broja π pomoću integralne sume

- Pokazali smo da je

$$4 \cdot \int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x = r^2\pi.$$

- Ako uzmemo $r = 1$, dobivamo

$$4 \cdot \int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x = \pi. \qquad (\spadesuit)$$

- Integral $\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$ možemo aproksimirati pomoću integralne sume i na taj način dobiti određeni broj decimala broja π.

- Neka je

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$$

razdioba segmenta $[0, 1]$.

- Neka je $\Delta x_i = x_i - x_{i-1}$ i neka su $\xi_i \in [x_{i-1}, x_i]$ proizvoljno odabrani brojevi za $i = 1, 2, \ldots, n-1, n$.
- Integralna suma I_n funkcije $f(x) = \sqrt{1 - x^2}$ za danu razdiobu segmenta $[0, 1]$ i odabrane brojeve ξ_i je

$$I_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

- Specijalno, možemo uzeti $\xi_i = x_i,\ i = 1, 2, \ldots, n-1, n$.
- Možemo uzeti ekvidistantnu razdiobu segmenta $[0, 1]$.

$$\Delta x_i = \frac{1}{n}, \quad x_i = \frac{i}{n}, \quad i = 1, 2, \ldots, n-1, n$$

- U tom slučaju je

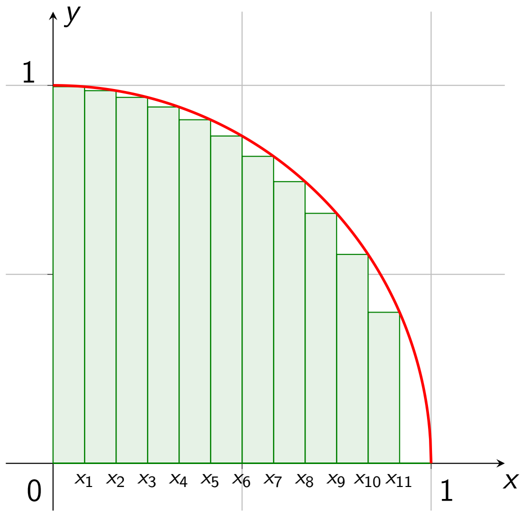
$$I_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right).$$

- U našem slučaju je $f(x) = \sqrt{1-x^2}$ pa slijedi

$$I_n = \frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}.$$

- Stoga za dovoljno veliki $n \in \mathbb{N}$ vrijedi

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}.$$



$$4\cdot\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \pi$$

$$\frac{4}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2} \approx \pi$$

konvergencija je spora

za dovoljno veliki $n \in \mathbb{N}$

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x \approx \frac{1}{n}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}$$

C++ kôd za integralnu sumu

```cpp
#include <iostream>
#include <vector>
#include <algorithm>
#include <numeric>
#include <cmath>
#include <iomanip>

// generator za podintegralnu funkciju, u ovom slucaju f(x)=sqrt(1-x^2)
class gen {
private:
  double x, dx;
public:
  gen(double x0, double pomak) : x(x0), dx(pomak) {}
  double operator()() {
    x += dx;
    return sqrt(1.0 - std::min(1.0, x * x));
  }
};

// racunanje vrijednosti integralne sume funkcije f(x)=sqrt(1-x^2) na segmentu [0,1]
double integrate(gen g, int n) {
  std::vector<double> fx(n);
  std::generate(fx.begin(), fx.end(), g);
  return (std::accumulate(fx.begin(), fx.end(), 0.0) / n);
}
```

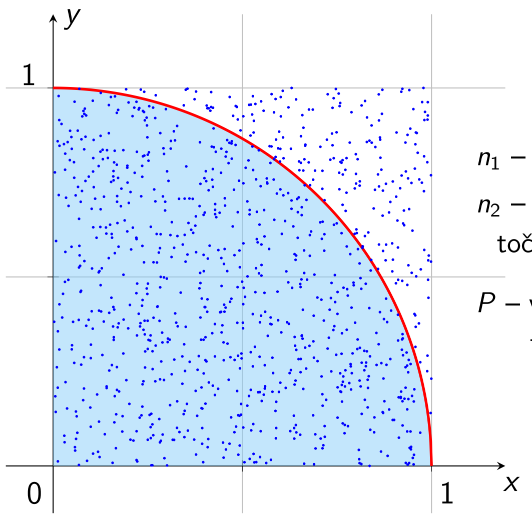

C++ kôd za integralnu sumu

```cpp
int main(void) {
  int n;
  std::cout << "\nNa koliko dijelova podijeliti segment [0,1]: ";
  std::cin >> n;

  gen g(0, 1.0/n);
  std::cout <<std::endl;
  std::cout << "--------------------------------------------------" << std::endl;
  std::cout << "Dobivanje decimala broja PI preko integralne sume" << std::endl;
  std::cout << "--------------------------------------------------" << std::endl;
  std::cout << std::setprecision(17) << 4 * integrate(g, n) << std::endl;
  std::cout << std::endl;

  return 0;
}
```

Monte Carlo integriranje



$$P = \frac{\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x}{\int_0^1 \mathrm{d}x} \approx \frac{n_1}{n_2}$$

n_1 – broj točaka unutar četvrtine kruga

n_2 – ukupni broj slučajno odabranih točaka unutar kvadrata $[0,1] \times [0,1]$

P – vjerojatnost da slučajno odabrana točka iz $[0,1] \times [0,1]$ leži unutar četvrtine kruga

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4}$$

$\pi \approx 4 \cdot \frac{n_1}{n_2}$ ⇝ za veliki broj ponavljanja slučajnog pokusa

C++ kôd za Monte Carlo integriranje

```cpp
#include <iostream>
#include <random>
#include <vector>
#include <tuple>
#include <ctime>
#include <cmath>
#include <iomanip>

typedef std::tuple<double, double> point;

std::ostream& operator<<(std::ostream& out, const point& pt) {
  out << "( " << std::get<0>(pt) << ", " <<  std::get<1>(pt) << ") ";
  return out;
}

std::default_random_engine e(time(nullptr));

point random_point() {
  std::uniform_real_distribution<double> u(0,1);
  point temp;
  std::get<0>(temp) = u(e);
  std::get<1>(temp) = u(e);
  return temp;
}
```


C++ kôd za Monte Carlo integriranje

```cpp
double mc_integral(double f(double), std::vector<point>::iterator first,
                   std::vector<point>::iterator last) {
  int total = 0;
  int below = 0;
  for (; first != last; ++first) {
    ++total;
    if (f(std::get<0>(*first)) > std::get<1>(*first))
      ++below;
  }
  return static_cast<double>(below) / total;
}

int main(void) {
  int data_size;
  std::cout << "Koliko slucajnih tocaka zelite generirati? ";
  std::cin >> data_size;
  std::vector<point> data(data_size);

  for (auto& element : data)
    element = random_point();

  std::cout << "PI (Monte Carlo) = " << std :: setprecision (17) <<
     4.0 * mc_integral([](double x){return sqrt(1 - x * x);}, data.begin(), data.end());
  std::cout << std::endl;

  return 0;
}
```