Derivacija funkcije – 1. dio

MATEMATIKA 2

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Zadatak 1

Odredite derivacije funkcija $f(x) = \sqrt[5]{x^2}$ *i* $g(x) = \left(\frac{2}{5}\right)^x$.

Rješenje

$f(x) = \sqrt[5]{x^2} \qquad f(x) = x^{\frac{2}{5}}$

$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

$f'(x) = \frac{2}{5}x^{\frac{2}{5}-1} = \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$

$(x^n)' = nx^{n-1}$

$g(x) = \left(\frac{2}{5}\right)^x \qquad g'(x) = \left(\frac{2}{5}\right)^x \ln\frac{2}{5}$

$(a^x)' = a^x \ln a$

$x^{-\frac{3}{5}} = \frac{1}{x^{\frac{3}{5}}} = \frac{1}{\sqrt[5]{x^3}}$

Zadatak 2

Odredite derivaciju funkcije $f(x) = \frac{3}{5}x^3 - \frac{7}{5}x^2 + \frac{9}{5}x + \frac{4}{5}$.

Rješenje

$$f'(x) = \left(\frac{3}{5}x^3 - \frac{7}{5}x^2 + \frac{9}{5}x + \frac{4}{5}\right)' =$$

$$= \left(\frac{3}{5}x^3\right)' - \left(\frac{7}{5}x^2\right)' + \left(\frac{9}{5}x\right)' + \left(\frac{4}{5}\right)' =$$

$$= \frac{3}{5}\cdot\left(x^3\right)' - \frac{7}{5}\cdot\left(x^2\right)' + \frac{9}{5}\cdot(x)' + 0 =$$

$$= \frac{3}{5}\cdot 3x^2 - \frac{7}{5}\cdot 2x + \frac{9}{5}\cdot 1 = \frac{9}{5}x^2 - \frac{14}{5}x + \frac{9}{5}$$

$(cu)'(x) = c\cdot u'(x)$ $\quad (c)' = 0$ $\quad (x^n)' = nx^{n-1}$

$(u+v)'(x) = u'(x) + v'(x)$ $\quad (u-v)'(x) = u'(x) - v'(x)$

Zadatak 3

Odredite derivaciju funkcije $y = \frac{4x^2}{3\sqrt[7]{x^4}}$.

$\frac{x^n}{x^m} = x^{n-m}$ $\quad \sqrt[n]{x^m} = x^{\frac{m}{n}}$

$(cu)'(x) = c\cdot u'(x)$

Rješenje

$(x^n)' = nx^{n-1}$

$$y = \frac{4x^2}{3\sqrt[7]{x^4}} = \frac{4x^2}{3x^{\frac{4}{7}}} = \frac{4}{3}x^{2-\frac{4}{7}}$$

$$y = \frac{4}{3}x^{\frac{10}{7}}$$

$$y' = \frac{4}{3}\cdot\left(x^{\frac{10}{7}}\right)' = \frac{4}{3}\cdot\frac{10}{7}x^{\frac{10}{7}-1}$$

$$y' = \frac{40}{21}x^{\frac{3}{7}} = \frac{40}{21}\sqrt[7]{x^3}$$

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$ $(x^n)' = nx^{n-1}$ $(e^x)' = e^x$

Zadatak 4

Odredite derivaciju funkcije $y = xe^x$ u točki 0.

Rješenje

$$y' = (x)' \cdot e^x + x \cdot (e^x)'$$

$$y' = 1 \cdot e^x + x \cdot e^x$$

$$y' = (1 + x)e^x$$

$$y'(0) = (1 + 0) \cdot e^0$$

$$y'(0) = 1 \cdot 1$$

$$y'(0) = 1$$

$(u - v)'(x) = u'(x) - v'(x)$

$(u + v)'(x) = u'(x) + v'(x)$

Zadatak 5

Odredite derivaciju funkcije $y = \frac{x^3 - 5}{x^3 + 5}$.

Rješenje

$$y' = \frac{(x^3 - 5)' \cdot (x^3 + 5) - (x^3 - 5) \cdot (x^3 + 5)'}{(x^3 + 5)^2} =$$

$$= \frac{3x^2 \cdot (x^3 + 5) - (x^3 - 5) \cdot 3x^2}{(x^3 + 5)^2} =$$

$$= \frac{3x^5 + 15x^2 - 3x^5 + 15x^2}{(x^3 + 5)^2} = \frac{30x^2}{(x^3 + 5)^2}$$

$\left(\frac{u}{v}\right)'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

$(x^n)' = nx^{n-1}$

$(c)' = 0$

Zadatak 6

Odredite derivaciju funkcije $y = \sqrt[3]{x} \log_2 x$.

$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

$\frac{x^m}{x^n} = x^{m-n}$

Rješenje

$$y' = \left(\sqrt[3]{x} \log_2 x\right)' = \left(x^{\frac{1}{3}} \log_2 x\right)' =$$

$$= \left(x^{\frac{1}{3}}\right)' \cdot \log_2 x + x^{\frac{1}{3}} \cdot (\log_2 x)' =$$

$$= \frac{1}{3} x^{-\frac{2}{3}} \log_2 x + x^{\frac{1}{3}} \cdot \frac{1}{x \ln 2} =$$

$$= \frac{1}{3} x^{-\frac{2}{3}} \log_2 x + \frac{x^{-\frac{2}{3}}}{\ln 2} =$$

$$= \left(\frac{1}{3} \log_2 x + \frac{1}{\ln 2}\right) x^{-\frac{2}{3}}$$

$(x^n)' = nx^{n-1}$

$(\log_a x)' = \frac{1}{x \ln a}$

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

$(u + v)'(x) = u'(x) + v'(x)$

Zadatak 7

Odredite derivaciju funkcije $y = \frac{\ln x + 1}{\ln x - 1}$.

Rješenje

$$y' = \frac{(\ln x + 1)' \cdot (\ln x - 1) - (\ln x + 1) \cdot (\ln x - 1)'}{(\ln x - 1)^2} =$$

$$= \frac{\frac{1}{x} \cdot (\ln x - 1) - (\ln x + 1) \cdot \frac{1}{x}}{(\ln x - 1)^2} =$$

$$= \frac{\frac{1}{x} \ln x - \frac{1}{x} - \frac{1}{x} \ln x - \frac{1}{x}}{(\ln x - 1)^2} = \frac{-\frac{2}{x}}{(\ln x - 1)^2} =$$

$$= \frac{-2}{x (\ln x - 1)^2}$$

$(c)' = 0$

$(\ln x)' = \frac{1}{x}$

$\left(\frac{u}{v}\right)'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

Zadatak 8

Odredite derivaciju funkcije $y = 10^x \log x + \ln 10$.

Rješenje

$$y' = \left(10^x \log x + \ln 10\right)' = \left(10^x \log x\right)' + \left(\ln 10\right)' =$$

$$= \left(10^x\right)' \cdot \log x + 10^x \cdot \left(\log x\right)' + 0 =$$

$$= 10^x \ln 10 \cdot \log x + 10^x \cdot \frac{1}{x \ln 10} =$$

$$= \left(\ln 10 \log x + \frac{1}{x \ln 10}\right) 10^x$$

$(c)' = 0$

$(a^x)' = a^x \ln a$

$(\log_a x)' = \frac{1}{x \ln a}$

$(u+v)'(x) = u'(x) + v'(x)$

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

Zadatak 9

Odredite derivaciju funkcije $y = e^{\sqrt{x}}$.

Rješenje

$$y' = e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$y' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$\left(e^{\text{nešto}}\right)' = e^{\text{nešto}} \cdot (\text{nešto})'$

$(e^x)' = e^x$

Zadatak 10

Odredite derivaciju funkcije $y = \left(x^2 + 3x - 5\right)^{20}$.

Rješenje

$$y' = 20\left(x^2 + 3x - 5\right)^{19} \cdot \left(x^2 + 3x - 5\right)' =$$

$$= 20\left(x^2 + 3x - 5\right)^{19} (2x + 3)$$

$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$

$(x^n)' = nx^{n-1}$

$(a^x)' = a^x \ln a$ $(x^n)' = nx^{n-1}$

Zadatak 11

Odredite derivaciju funkcije $y = \log_5\left(5^x - x^5\right)$.

Rješenje

$$y' = \frac{1}{\left(5^x - x^5\right)\ln 5} \cdot \left(5^x - x^5\right)' = \frac{5^x \ln 5 - 5x^4}{\left(5^x - x^5\right)\ln 5}$$

$\left(\log_a(\text{nešto})\right)' = \frac{1}{\text{nešto} \cdot \ln a} \cdot (\text{nešto})'$ $(\log_a x)' = \frac{1}{x \ln a}$

Zadatak 12

$\big((\text{nešto})^n\big)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$ $(x^n)' = nx^{n-1}$

Odredite derivacije funkcija $f(x) = \ln 5x$, $g(x) = \ln x^5$ *i* $h(x) = \ln^5 x$.

Rješenje

$f(x) = \ln 5x$

$f'(x) = \frac{1}{5x} \cdot (5x)'$

$f'(x) = \frac{1}{5x} \cdot 5$

$f'(x) = \frac{1}{x}$

$g(x) = \ln x^5$

$g'(x) = \frac{1}{x^5} \cdot (x^5)'$

$g'(x) = \frac{1}{x^5} \cdot 5x^4$

$g'(x) = \frac{5}{x}$

$h(x) = \ln^5 x$

$h(x) = (\ln x)^5$

$h'(x) = 5(\ln x)^4 \cdot (\ln x)'$

$h'(x) = 5(\ln x)^4 \cdot \frac{1}{x}$

$h'(x) = \frac{5}{x}\ln^4 x$

$\big(\ln(\text{nešto})\big)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$ $(\ln x)' = \frac{1}{x}$ $\ln^k x = (\ln x)^k$

Zadatak 13

Odredite derivaciju funkcije $y = \ln \frac{x-1}{x+1}$.

$(\ln x)' = \frac{1}{x}$

Rješenje

$\big(\ln(\text{nešto})\big)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$

$$y' = \frac{1}{\frac{x-1}{x+1}} \cdot \left(\frac{x-1}{x+1}\right)' =$$

$$= \frac{x+1}{x-1} \cdot \frac{(x-1)' \cdot (x+1) - (x-1) \cdot (x+1)'}{(x+1)^2} =$$

$$= \frac{x+1}{x-1} \cdot \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x-1)(x+1)} = \frac{2}{x^2-1}$$

$\left(\frac{u}{v}\right)'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

Zadatak 14

Odredite derivaciju funkcije $y = \sqrt{\sin(x^2-1)}$.

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Rješenje

$(\sin x)' = \cos x$

$$y' = \frac{1}{2\sqrt{\sin(x^2-1)}} \cdot \Big(\sin(x^2-1)\Big)' =$$

$$= \frac{1}{2\sqrt{\sin(x^2-1)}} \cdot \cos(x^2-1) \cdot (x^2-1)' =$$

$$= \frac{2x\cos(x^2-1)}{2\sqrt{\sin(x^2-1)}} = \frac{x\cos(x^2-1)}{\sqrt{\sin(x^2-1)}}$$

$\big(\sqrt{\text{nešto}}\big)' = \frac{1}{2\sqrt{\text{nešto}}} \cdot (\text{nešto})'$

$\big(\sin(\text{nešto})\big)' = \cos(\text{nešto}) \cdot (\text{nešto})'$

Zadatak 15

Odredite jednadžbu tangente na graf funkcije $y = \ln(5-4x)$ *u točki s apscisom* 1. *Odredite duljinu odsječka dobivene tangente između koordinatnih osi.*

Rješenje

- Jednadžba tangente na graf funkcije $y = f(x)$ u točki $T_0(x_0, y_0)$

$$t \ldots y - y_0 = k_t \cdot (x - x_0)$$

- Pritom je $y_0 = f(x_0)$ i $k_t = f'(x_0)$.

- Znamo da je $x_0 = 1$.

$y = \ln(5 - 4x)$

$$y_0 = \ln(5 - 4 \cdot 1) = \ln 1 = 0$$

Točka: $T_0(1, 0)$

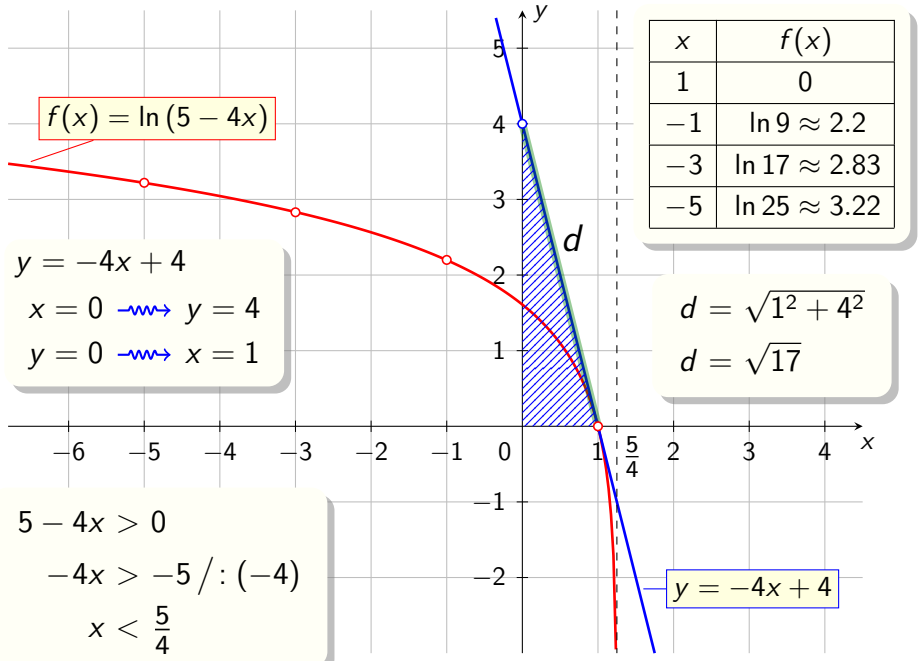
- Derivacija funkcije

$$y' = \frac{1}{5 - 4x} \cdot (5 - 4x)' = \frac{1}{5 - 4x} \cdot (-4) = \frac{-4}{5 - 4x}$$

- Koeficijent smjera tangente

$$k_t = y'(1) = \frac{-4}{5 - 4 \cdot 1} = -4$$

$\big(\ln(\text{nešto})\big)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$ $\quad (\ln x)' = \frac{1}{x}$



x	$f(x)$
1	0
-1	$\ln 9 \approx 2.2$
-3	$\ln 17 \approx 2.83$
-5	$\ln 25 \approx 3.22$

$y = -4x + 4$
$x = 0 \rightsquigarrow y = 4$
$y = 0 \rightsquigarrow x = 1$

$d = \sqrt{1^2 + 4^2}$
$d = \sqrt{17}$

$5 - 4x > 0$
$-4x > -5 \,/ : (-4)$
$x < \frac{5}{4}$

$x_0 = 1$ $\quad y_0 = 0$ $\quad k_t = -4$

- Jednadžba tangente

$$y - y_0 = k_t \cdot (x - x_0)$$
$$y - 0 = -4 \cdot (x - 1)$$
$$\boxed{y = -4x + 4}$$