Derivacija funkcije – 2. dio

MATEMATIKA 2

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Zadatak 1

Odredite jednadžbu normale na graf funkcije $y = \sqrt{8x^2 + 4}$ u točki T na grafu s apscisom 2. Odredite površinu trokuta kojeg normala u točki T zatvara s koordinatnim osima.

Rješenje

- Jednadžba normale na graf funkcije $y = f(x)$ u točki $T(x_0, y_0)$

$$n \ldots y - y_0 = k_n \cdot (x - x_0)$$

- Pritom je $y_0 = f(x_0)$, $\quad k_n = -\dfrac{1}{k_t}$, $\quad k_t = f'(x_0)$.

$y = \sqrt{8x^2 + 4}$

- Znamo da je $x_0 = 2$.

$$y_0 = \sqrt{8 \cdot 2^2 + 4} = \sqrt{36} = 6$$

Točka: $T(2, 6)$

- Derivacija funkcije

$$y' = \frac{1}{2\sqrt{8x^2 + 4}} \cdot \left(8x^2 + 4\right)' = \frac{1}{2\sqrt{8x^2 + 4}} \cdot 16x = \frac{8x}{\sqrt{8x^2 + 4}}$$

- Koeficijent smjera tangente

$$k_t = y'(2) = \frac{8 \cdot 2}{\sqrt{8 \cdot 2^2 + 4}} = \frac{16}{6} = \frac{8}{3}$$

$$\left(\sqrt{\text{nešto}}\,\right)' = \frac{1}{2\sqrt{\text{nešto}}} \cdot (\text{nešto})'$$

$$\left(\sqrt{x}\,\right)' = \frac{1}{2\sqrt{x}}$$

$k_t = \frac{8}{3}$

- Koeficijent smjera normale

$$k_n = -\frac{1}{k_t} = -\frac{1}{\frac{8}{3}} = -\frac{3}{8}$$

- Jednadžba normale

$x_0 = 2$

$y_0 = 6$

$k_n = -\frac{3}{8}$

$$y - y_0 = k_n \cdot (x - x_0)$$

$$y - 6 = -\frac{3}{8} \cdot (x - 2)$$

$$y - 6 = -\frac{3}{8}x + \frac{3}{4}$$

$$y = -\frac{3}{8}x + \frac{3}{4} + 6$$

$$\boxed{y = -\frac{3}{8}x + \frac{27}{4}}$$

Segmentni oblik jednadžbe normale

$$y=-\frac{3}{8}x+\frac{27}{4}$$

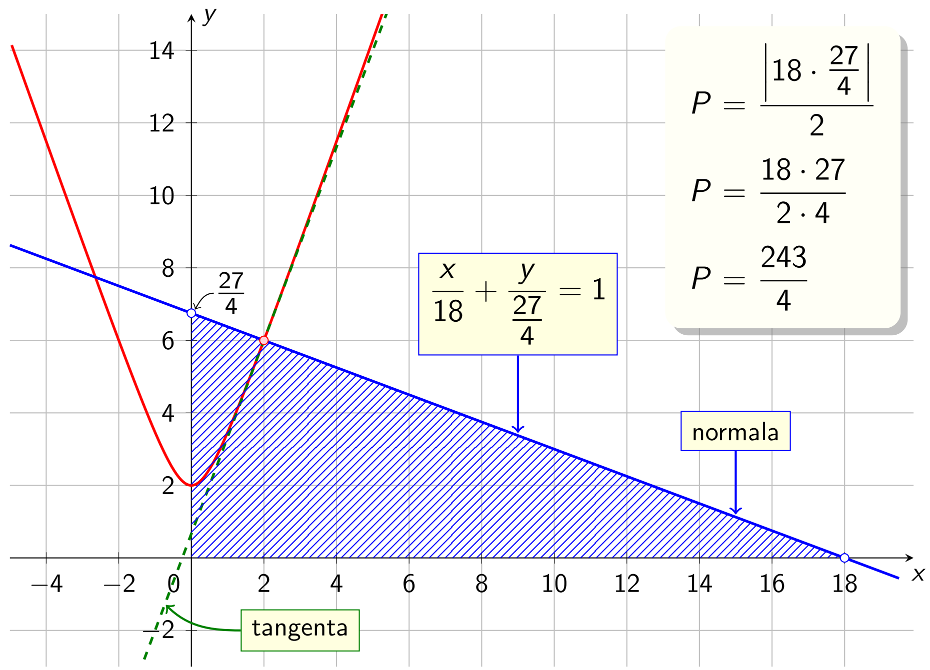
$$\frac{3}{8}x+y=\frac{27}{4}\ \Big/\cdot 8$$

$$3x+8y=54\ \Big/:54$$

$$\frac{3x}{54}+\frac{8y}{54}=1$$

$$\frac{x}{18}+\frac{4y}{27}=1$$

$$\frac{x}{18}+\frac{y}{\frac{27}{4}}=1$$



Zadatak 2

Odredite četvrtu derivaciju funkcije $f(x)=\ln(3x+1)$.

Rješenje

$$\left(\ln(\text{nešto})\right)'=\frac{1}{\text{nešto}}\cdot(\text{nešto})'$$

$$(\ln x)'=\frac{1}{x}$$

- Prva derivacija

$$f'(x)=\frac{1}{3x+1}\cdot(3x+1)'=\frac{1}{3x+1}\cdot 3=\frac{3}{3x+1}$$

$$f'(x)=3\cdot(3x+1)^{-1}$$

- Druga derivacija

$$f''(x)=3\cdot(-1)\cdot(3x+1)^{-2}\cdot(3x+1)'=-3\cdot(3x+1)^{-2}\cdot 3$$

$$f''(x)=-9\cdot(3x+1)^{-2}$$

$$\left((\text{nešto})^n\right)'=n(\text{nešto})^{n-1}\cdot(\text{nešto})'$$

$$(x^n)'=nx^{n-1}$$

$$f''(x)=-9\cdot(3x+1)^{-2}$$

- Treća derivacija

$$f'''(x)=-9\cdot(-2)\cdot(3x+1)^{-3}\cdot(3x+1)'=18\cdot(3x+1)^{-3}\cdot 3$$

$$f'''(x)=54\cdot(3x+1)^{-3}$$

- Četvrta derivacija

$$f^{(4)}(x)=54\cdot(-3)\cdot(3x+1)^{-4}\cdot(3x+1)'=-162\cdot(3x+1)^{-4}\cdot 3$$

$$f^{(4)}(x)=-486\cdot(3x+1)^{-4}$$

$$f^{(n)}(x)=\left(f^{(n-1)}(x)\right)'$$

$$\left((\text{nešto})^n\right)'=n(\text{nešto})^{n-1}\cdot(\text{nešto})'$$

$$(x^n)'=nx^{n-1}$$

Zadatak 3

Odredite derivaciju funkcije $y = y(x)$ *zadane implicitno s* $ye^y = e^{x+1}$.

Rješenje

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$ye^y = e^{x+1} \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$$y' \cdot e^y + y \cdot (e^y)' = e^{x+1} \cdot (x+1)'$$

$$y'e^y + y \cdot e^y y' = e^{x+1} \cdot 1$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$y'(e^y + ye^y) = e^{x+1}$$

$$\left(e^{\text{nešto}}\right)' = e^{\text{nešto}} \cdot (\text{nešto})'$$

$$y' = \frac{e^{x+1}}{e^y + ye^y}$$

$$(e^x)' = e^x$$

$$y' = \frac{e^{x+1}}{(1+y)e^y}$$

$$\boxed{y' = \frac{e^{x-y+1}}{1+y}}$$

$$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Zadatak 4

Odredite derivaciju funkcije $y = y(x)$ *zadane implicitno s*

$$y^2 = \cos 3x + \ln \frac{y}{x}.$$

Rješenje

$$(\ln x)' = \frac{1}{x}$$

$$\left(\frac{u}{v}\right)'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$$

$$y^2 = \cos 3x + \ln \frac{y}{x} \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$$(\cos x)' = -\sin x$$

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$2yy' = -\sin 3x \cdot (3x)' + \frac{1}{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'$$

$$(x^n)' = nx^{n-1}$$

$$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$$

$$2yy' = -3\sin 3x + \frac{x}{y} \cdot \frac{y'x - y \cdot 1}{x^2}$$

$$\left(\cos(\text{nešto})\right)' = -\sin(\text{nešto}) \cdot (\text{nešto})'$$

$$2yy' = -3\sin 3x + \frac{y'x - y}{xy} \Big/ \cdot xy$$

$$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$$

$$2xy^2y' = -3xy\sin 3x + y'x - y$$

$$2xy^2y' - xy' = -3xy\sin 3x - y$$

$$(2xy^2 - x)y' = -3xy\sin 3x - y$$

$$y' = \frac{-3xy\sin 3x - y}{2xy^2 - x} \cdot \frac{-1}{-1}$$

$$\boxed{y' = \frac{3xy\sin 3x + y}{x - 2xy^2}}$$

Zadatak 5

$$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$$\operatorname{tg}^2 x = (\operatorname{tg} x)^2$$

Odredite derivaciju funkcije $y = \left(x + \operatorname{tg}^2 x\right)^{\operatorname{ctg} x}$.

$$(x^n)' = nx^{n-1}$$

Rješenje

$$\log_a x^k = k \log_a x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$$

$$y = \left(x + \operatorname{tg}^2 x\right)^{\operatorname{ctg} x} \Big/ \ln$$

$$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$$

$$\ln y = \ln\left(x + \operatorname{tg}^2 x\right)^{\operatorname{ctg} x}$$

$$\ln y = \operatorname{ctg} x \cdot \ln\left(x + \operatorname{tg}^2 x\right) \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$$

$$\frac{1}{y} \cdot y' = (\operatorname{ctg} x)' \cdot \ln\left(x + \operatorname{tg}^2 x\right) + \operatorname{ctg} x \cdot \left(\ln\left(x + \operatorname{tg}^2 x\right)\right)'$$

$$\frac{y'}{y} = \frac{-1}{\sin^2 x} \cdot \ln\left(x + \operatorname{tg}^2 x\right) + \operatorname{ctg} x \cdot \frac{1}{x + \operatorname{tg}^2 x} \cdot \left(x + \operatorname{tg}^2 x\right)'$$

$$\frac{y'}{y} = -\frac{\ln\left(x + \operatorname{tg}^2 x\right)}{\sin^2 x} + \frac{\operatorname{ctg} x}{x + \operatorname{tg}^2 x} \cdot \left(1 + 2\operatorname{tg} x \cdot (\operatorname{tg} x)'\right)$$

$y = \left(x + \operatorname{tg}^2 x\right)^{\operatorname{ctg} x}$

$(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}$

$$\frac{y'}{y} = -\frac{\ln\left(x + \operatorname{tg}^2 x\right)}{\sin^2 x} + \frac{\operatorname{ctg} x}{x + \operatorname{tg}^2 x} \cdot \left(1 + 2\operatorname{tg} x \cdot (\operatorname{tg} x)'\right)$$

$$\frac{y'}{y} = -\frac{\ln\left(x + \operatorname{tg}^2 x\right)}{\sin^2 x} + \frac{\operatorname{ctg} x}{x + \operatorname{tg}^2 x} \cdot \left(1 + \frac{2\operatorname{tg} x}{\cos^2 x}\right) \Big/ \cdot y$$

$$y' = \left[\frac{\operatorname{ctg} x}{x + \operatorname{tg}^2 x} \cdot \left(1 + \frac{2\operatorname{tg} x}{\cos^2 x}\right) - \frac{\ln\left(x + \operatorname{tg}^2 x\right)}{\sin^2 x}\right] \cdot y$$

$$y' = \left[\frac{\operatorname{ctg} x}{x + \operatorname{tg}^2 x} \cdot \left(1 + \frac{2\operatorname{tg} x}{\cos^2 x}\right) - \frac{\ln\left(x + \operatorname{tg}^2 x\right)}{\sin^2 x}\right] \cdot \left(x + \operatorname{tg}^2 x\right)^{\operatorname{ctg} x}$$

Zadatak 6

Odredite derivaciju funkcije

$$y = \frac{\sqrt{x+2}}{\sqrt[3]{x+1} \cdot (x+3)^5}.$$

Rješenje

- Funkciju možemo derivirati direktno koristeći pravila za derivaciju kvocijenta, produkta i složene funkcije.
- Međutim, u ovom slučaju *logaritamska derivacija* znatno olakšava postupak deriviranja.

$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

$\log_a \dfrac{x}{y} = \log_a x - \log_a y$

$\log_a (xy) = \log_a x + \log_a y$

$\log_a x^k = k \log_a x$

$$y = \frac{\sqrt{x+2}}{\sqrt[3]{x+1} \cdot (x+3)^5}$$

$$y = \frac{(x+2)^{\frac{1}{2}}}{(x+1)^{\frac{1}{3}} \cdot (x+3)^5} \Big/ \ln$$

$$\ln y = \ln \frac{(x+2)^{\frac{1}{2}}}{(x+1)^{\frac{1}{3}} \cdot (x+3)^5}$$

$$\ln y = \ln (x+2)^{\frac{1}{2}} - \ln\left((x+1)^{\frac{1}{3}} \cdot (x+3)^5\right)$$

$$\ln y = \ln (x+2)^{\frac{1}{2}} - \ln (x+1)^{\frac{1}{3}} - \ln (x+3)^5$$

$$\ln y = \frac{1}{2}\ln(x+2) - \frac{1}{3}\ln(x+1) - 5\ln(x+3) \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$(\ln x)' = \dfrac{1}{x}$

$\big(\ln(\text{nešto})\big)' = \dfrac{1}{\text{nešto}} \cdot (\text{nešto})'$

$$\ln y = \frac{1}{2}\ln(x+2) - \frac{1}{3}\ln(x+1) - 5\ln(x+3) \Big/ \frac{\mathrm{d}}{\mathrm{d}x}$$

$$\frac{1}{y} \cdot y' = \frac{1}{2} \cdot \frac{1}{x+2} \cdot \underbrace{(x+2)'}_{=1} - \frac{1}{3} \cdot \frac{1}{x+1} \cdot \underbrace{(x+1)'}_{=1} - 5 \cdot \frac{1}{x+3} \cdot \underbrace{(x+3)'}_{=1}$$

$$\frac{y'}{y} = \frac{1}{2x+4} - \frac{1}{3x+3} - \frac{5}{x+3} \Big/ \cdot y$$

$$y' = \left(\frac{1}{2x+4} - \frac{1}{3x+3} - \frac{5}{x+3}\right) \cdot y$$

$y = \dfrac{\sqrt{x+2}}{\sqrt[3]{x+1} \cdot (x+3)^5}$

$$y' = \left(\frac{1}{2x+4} - \frac{1}{3x+3} - \frac{5}{x+3}\right) \cdot \frac{\sqrt{x+2}}{\sqrt[3]{x+1} \cdot (x+3)^5}$$

Zadatak 7

Funkcija $y = y(x)$ je zadana implicitno

Odredite jednadžbu tangente i normale na krivulju

$$\ln(xy) = x^3y^3 - 1$$

u točki $T(1,1)$. ($x_0 = 1$, $y_0 = 1$)

zadana točka pripada krivulji

$$\ln(1 \cdot 1) = 1^3 \cdot 1^3 - 1 \rightsquigarrow 0 = 0$$

Rješenje

- Jednadžba tangente na graf funkcije $y = f(x)$ u točki $T_0(x_0, y_0)$

$$t \ldots y - y_0 = k_t \cdot (x - x_0)$$

- Jednadžba normale na graf funkcije $y = f(x)$ u točki $T(x_0, y_0)$

$$n \ldots y - y_0 = k_n \cdot (x - x_0)$$

- Pritom je $y_0 = f(x_0)$, $k_t = f'(x_0)$, $k_n = -\dfrac{1}{k_t}$.

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$

$$\ln(xy) = x^3y^3 - 1 \Big/ \frac{d}{dx}$$

$$\frac{1}{xy} \cdot (xy)' = 3x^2 \cdot y^3 + x^3 \cdot 3y^2 \cdot y' - 0$$

$$\frac{1 \cdot y + xy'}{xy} = 3x^2y^3 + 3x^3y^2y' \Big/ \cdot xy$$

$$y + xy' = 3x^3y^4 + 3x^4y^3y'$$

$$xy' - 3x^4y^3y' = 3x^3y^4 - y$$

$$(x - 3x^4y^3)y' = 3x^3y^4 - y$$

$$\boxed{y' = \frac{3x^3y^4 - y}{x - 3x^4y^3}}$$

$$y'(x) = \frac{3x^3y(x)^4 - y(x)}{x - 3x^4y(x)^3}$$

$$k_t = y'(x_0) = \frac{3x_0^3y_0^4 - y_0}{x_0 - 3x_0^4y_0^3}$$

$$k_t = y'(1) = \frac{3 \cdot 1^3 \cdot 1^4 - 1}{1 - 3 \cdot 1^4 \cdot 1^3} = \frac{2}{-2}$$

$$\boxed{k_t = -1} \quad \boxed{k_n = 1}$$

$$k_n = -\frac{1}{k_t} = -\frac{1}{-1} = 1$$

$y_0 = y(x_0)$ $\quad T(1, 1)$ (x_0, y_0)

$\left(\ln(\text{nešto})\right)' = \dfrac{1}{\text{nešto}} \cdot (\text{nešto})'$ $\quad (\ln x)' = \dfrac{1}{x}$ $\quad (x^n)' = nx^{n-1}$

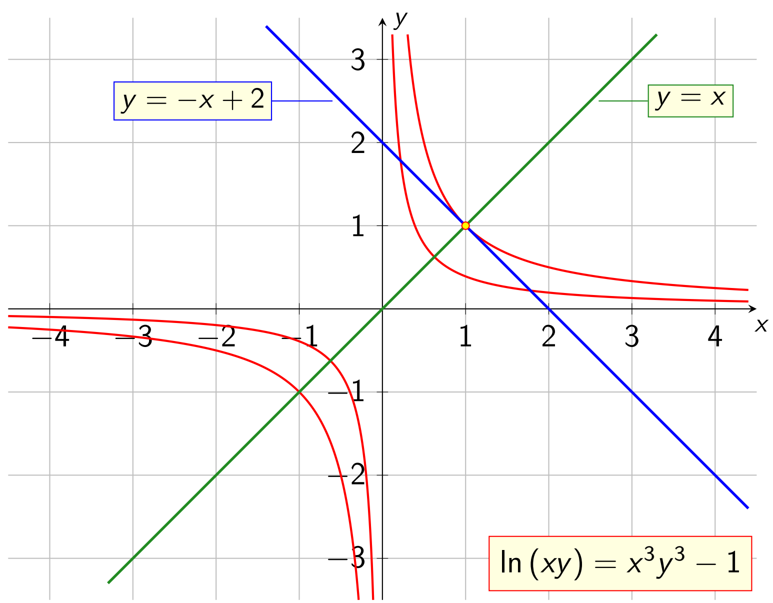
$x_0 = 1$ $\quad y_0 = 1$ $\quad k_t = -1$ $\quad k_n = 1$

- Jednadžba tangente

$$y - y_0 = k_t \cdot (x - x_0)$$
$$y - 1 = -1 \cdot (x - 1)$$
$$y - 1 = -x + 1$$
$$\boxed{t \ldots y = -x + 2}$$

- Jednadžba normale

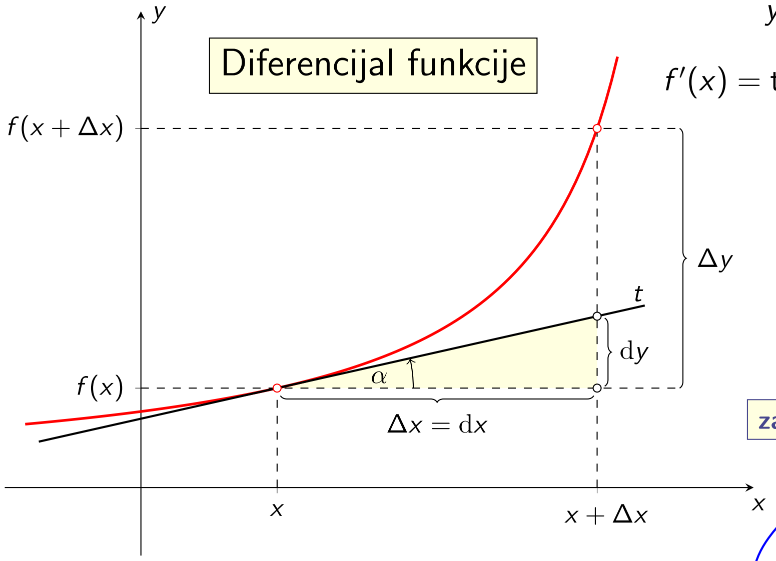
$$y - y_0 = k_n \cdot (x - x_0)$$
$$y - 1 = 1 \cdot (x - 1)$$
$$y - 1 = x - 1$$
$$\boxed{n \ldots y = x}$$



Diferencijal funkcije

$y = f(x)$

$f'(x) = \operatorname{tg}\alpha = \dfrac{\mathrm{d}y}{\mathrm{d}x}$



za male Δx

$$\Delta y = f(x+\Delta x) - f(x), \qquad \mathrm{d}y = f'(x)\,\mathrm{d}x, \qquad \Delta y \approx \mathrm{d}y$$

$$f(x+\Delta x) \approx f(x) + f'(x)\,\mathrm{d}x$$

Zadatak 8

$f(x_0 + \mathrm{d}x) \approx f(x_0) + f'(x_0)\,\mathrm{d}x$

Pomoću diferencijala približno izračunajte $\sqrt{6.26^3}$.

Rješenje

$(x^n)' = nx^{n-1}$

$\sqrt{6.23^3} \approx ???$

Domaća zadaća

- $f(x) = \sqrt{x^3} = x^{\frac{3}{2}}, \quad f'(x) = \dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{3}{2}\sqrt{x}$
- $x_0 = 6.25, \ \mathrm{d}x = 0.01, \ x_0 + \mathrm{d}x = 6.26$
- $f(x_0) = f(6.25) = \sqrt{6.25^3} = \sqrt{6.25}^{\,3} = 2.5^3 = 15.625$
- $f'(x_0) = f'(6.25) = \dfrac{3}{2}\sqrt{6.25} = 1.5 \cdot 2.5 = 3.75$

$$f(6.26) \approx f(6.25) + f'(6.25) \cdot 0.01$$
$$f(6.26) \approx 15.625 + 3.75 \cdot 0.01$$
$$f(6.26) \approx 15.6625$$

$\boxed{\sqrt{6.26^3} \approx 15.6625}$

$\sqrt{6.26^3} = 15.662514996002\cdots$