Seminari 1

MATEMATIČKE METODE ZA INFORMATIČARE

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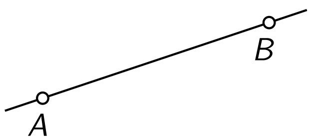
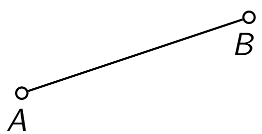
Oznake

- AB ⟿ pravac kroz točke A i B

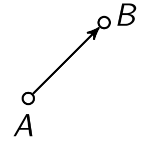


- $\overline{AB}$ ⟿ dužina čiji su krajevi točke A i B

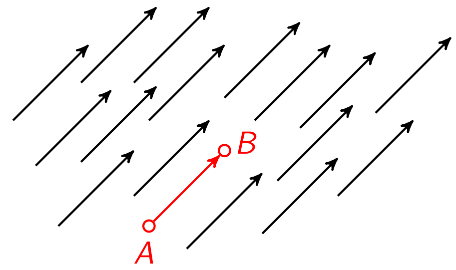


- $|AB|$ ⟿ duljina dužine $\overline{AB}$ (nenegativni realni broj)

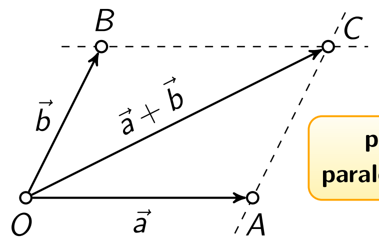
- $\overrightarrow{AB}$ ⟿ orijentirana dužina čiji početak je točka A, a kraj točka B
- $[\overrightarrow{AB}]$ ⟿ vektor čiji reprezentant je orijentirana dužina $\overrightarrow{AB}$



orijentirana dužina $\overrightarrow{AB}$

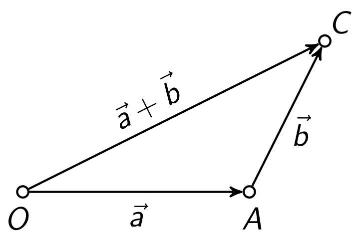


vektor $[\overrightarrow{AB}]$



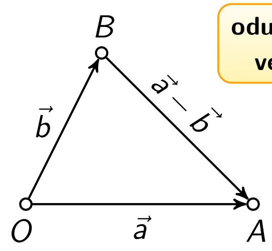
pravilo paralelograma

$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$



$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

pravilo trokuta



oduzimanje vektora

$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{BA}$

$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{BO} = \overrightarrow{BO} + \overrightarrow{OA} = \overrightarrow{BA}$

komutativnost zbrajanja vektora

Zadatak 1

Točke A, B, C i D su redom vrhovi jednakokračnog trapeza kojemu je duljina kraka jednaka d i kojemu su duljine osnovica $|AB| = 2d$ i $|CD| = d$.

a) *Nacrtajte vektor $\vec{v} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{CB}$.*

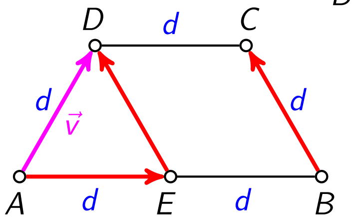
b) *Kolika je norma (duljina) vektora $\vec{v}$?*

Rješenje

$EB \parallel DC, \quad |EB| = |DC| \implies EBCD$ je paralelogram

$\overrightarrow{BC} = \overrightarrow{ED}$

a)



$|AD| = |BC| = d$

$|AB| = 2d, \quad |CD| = d$

$|AE| = |BE| = d$

$$\vec{v} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{CB} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AE} + \overrightarrow{BC} = \overrightarrow{AE} + \overrightarrow{ED} = \overrightarrow{AD}$$

b) $|\vec{v}| = d$

Zadatak 2

Točke A, B, C i D su redom vrhovi pravokutnika kojemu su duljine susjednih stranica $|AB| = a$ i $|AD| = b$.

a) *Nacrtajte vektor $\overrightarrow{AE} = \overrightarrow{AB} - \overrightarrow{BC}$.*

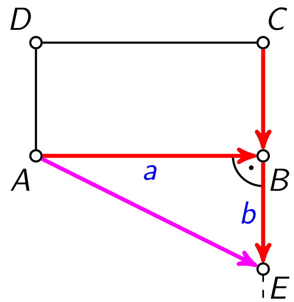
b) *Kolika je norma vektora $\overrightarrow{AE}$?*

c) *Nacrtajte vektor $\overrightarrow{AF} = \overrightarrow{AB} - \overrightarrow{AC}$.*

d) *Kolika je norma vektora $\overrightarrow{AF}$?*

Rješenje

a)



$|AB| = a$

$|AD| = b$

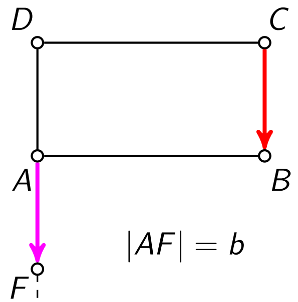
$|BE| = b$

$\overrightarrow{CB} = \overrightarrow{BE}$

$$\overrightarrow{AE} = \overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{CB} =$$
$$= \overrightarrow{AB} + \overrightarrow{BE}$$

b) $\left|\overrightarrow{AE}\right| = \sqrt{a^2 + b^2}$

c)



$|AF| = b$

$$\overrightarrow{AF} = \overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$$

d) $\left|\overrightarrow{AF}\right| = b$

Definicija kolinearnih vektora

Za dva vektora iz V^3 kažemo da su kolinearni ako imaju isti smjer. Po dogovoru je nulvektor kolinearan sa svakim vektorom iz V^3.

Iznimno važna propozicija

Neka su $\vec{a}, \vec{b} \in V^3$ različiti od nulvektora. Tada vrijedi:

$$\vec{a} \text{ i } \vec{b} \text{ imaju isti smjer} \iff \exists \lambda \in \mathbb{R},\ \vec{b} = \lambda \vec{a}.$$

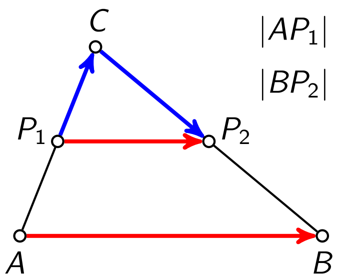
U tom je slučaju $\lambda \neq 0$ i vrijedi $\vec{a} = \frac{1}{\lambda}\vec{b}$. Nadalje, takav λ je jedinstven.

Zadatak 3

Dokažite da je srednjica trokuta paralelna s nasuprotnom stranicom trokuta i da je njezina duljina jednaka polovici duljine te stranice.

Rješenje

Srednjica trokuta je dužina koja spaja polovišta dviju stranica trokuta.



$|AP_1| = |CP_1|$

$|BP_2| = |CP_2|$

$\overrightarrow{P_1P_2} = \frac{1}{2}\overrightarrow{AB}$

$P_1P_2 \parallel AB$

$\left|\overrightarrow{P_1P_2}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|$

$|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$

$|P_1P_2| = \frac{1}{2}|AB|$

$$\overrightarrow{P_1P_2} = \overrightarrow{P_1C} + \overrightarrow{CP_2} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} = \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{CB}\right) = \frac{1}{2}\overrightarrow{AB}$$

Zadatak 4

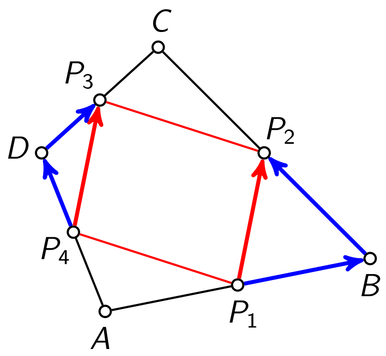
Dokažite da su polovišta stranica bilo kojeg četverokuta vrhovi paralelograma.

Rješenje

$$\implies \overrightarrow{P_1P_2} = \overrightarrow{P_4P_3} \implies P_1P_2P_3P_4 \text{ je paralelogram}$$

$|AP_1| = |BP_1| \qquad |CP_3| = |DP_3|$

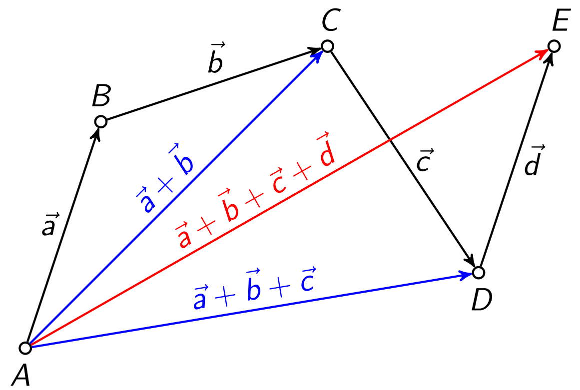
$|BP_2| = |CP_2| \qquad |AP_4| = |DP_4|$



$$\overrightarrow{P_1P_2} = \overrightarrow{P_1B} + \overrightarrow{BP_2} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}$$

$$\overrightarrow{P_4P_3} = \overrightarrow{P_4D} + \overrightarrow{DP_3} = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{DC}\right) = \frac{1}{2}\overrightarrow{AC}$$

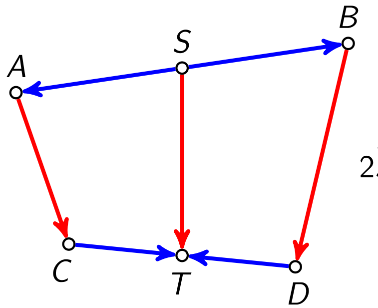
Zbrajanje više vektora



Zadatak 5

Neka su $\overline{AB}$ i $\overline{CD}$ bilo koje dužine, a točke S i T redom polovišta tih dužina. Dokažite da je $\overrightarrow{AC} + \overrightarrow{BD} = 2\overrightarrow{ST}$.

Rješenje



$$\overrightarrow{ST} = \overrightarrow{SA} + \overrightarrow{AC} + \overrightarrow{CT}$$
$$\overrightarrow{ST} = \overrightarrow{SB} + \overrightarrow{BD} + \overrightarrow{DT} \quad / +$$

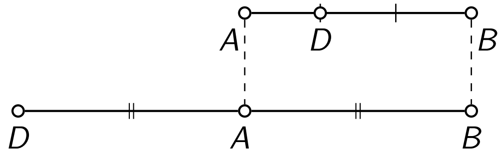
$$2\overrightarrow{ST} = \overrightarrow{AC} + \overrightarrow{BD} + \underbrace{\overrightarrow{SA} + \overrightarrow{SB}}_{=\vec{0}} + \underbrace{\overrightarrow{CT} + \overrightarrow{DT}}_{=\vec{0}}$$

$$2\overrightarrow{ST} = \overrightarrow{AC} + \overrightarrow{BD}$$

$|AS| = |BS| \rightsquigarrow \overrightarrow{SB} = -\overrightarrow{SA}$

$|CT| = |DT| \rightsquigarrow \overrightarrow{DT} = -\overrightarrow{CT}$

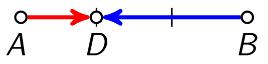
Dijeljenje dužine u zadanom omjeru



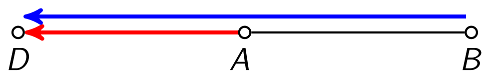
$\dfrac{|AD|}{|BD|} = \dfrac{1}{2}$ iznutra

$\dfrac{|AD|}{|BD|} = \dfrac{1}{2}$ izvana

1. način

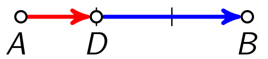


$\overrightarrow{AD} = -\frac{1}{2}\overrightarrow{BD}$

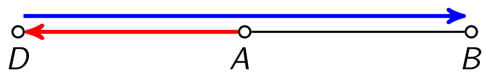


$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{BD}$

2. način



$\overrightarrow{AD} = \frac{1}{2}\overrightarrow{DB}$



$\overrightarrow{AD} = -\frac{1}{2}\overrightarrow{DB}$

1. način $\rightsquigarrow$ *koristit ćemo ovaj pristup*

Kažemo da točka D dijeli dužinu $\overline{AB}$ u omjeru λ ako je $\overrightarrow{AD} = \lambda\overrightarrow{BD}$.

Ako je $\lambda < 0$, točka D se nalazi unutar dužine $\overline{AB}$.

Ako je $\lambda > 0$, točka D se nalazi izvan dužine $\overline{AB}$.

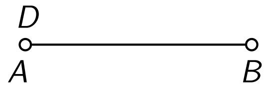
2. način

Kažemo da točka D dijeli dužinu $\overline{AB}$ u omjeru λ ako je $\overrightarrow{AD} = \lambda\overrightarrow{DB}$.

Ako je $\lambda > 0$, točka D se nalazi unutar dužine $\overline{AB}$.

Ako je $\lambda < 0$, točka D se nalazi izvan dužine $\overline{AB}$.

$\lambda = 0$



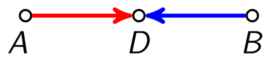
$\overrightarrow{AD} = \lambda\overrightarrow{BD}$

$\overrightarrow{AD} = 0 \cdot \overrightarrow{BD}$

$\overrightarrow{AD} = \vec{0}$

$D = A$

polovište

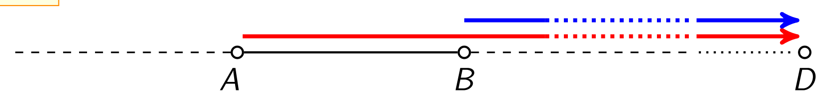


$\overrightarrow{AD} = \lambda\overrightarrow{BD}$

$\overrightarrow{AD} = -\overrightarrow{BD}$

$\boxed{\lambda = -1}$

$\lambda = 1$



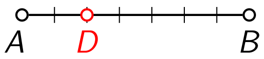
$\overrightarrow{AD} = \lambda\overrightarrow{BD}$

$\overrightarrow{AD} = \overrightarrow{BD}$

Točka D je beskonačno daleka točka na pravcu AB.

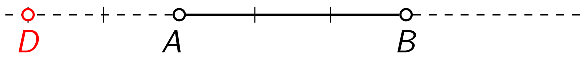
Primjer

$$\overrightarrow{AD} = -\frac{2}{5}\overrightarrow{BD}$$



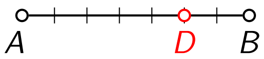
$$\frac{|AD|}{|BD|} = \frac{2}{5} \qquad 5+2=7$$

$$\overrightarrow{AD} = \frac{2}{5}\overrightarrow{BD}$$



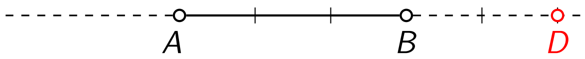
$$\frac{|AD|}{|BD|} = \frac{2}{5} \qquad 5-2=3$$

$$\overrightarrow{AD} = -\frac{5}{2}\overrightarrow{BD}$$



$$\frac{|AD|}{|BD|} = \frac{5}{2} \qquad 5+2=7$$

$$\overrightarrow{AD} = \frac{5}{2}\overrightarrow{BD}$$



$$\frac{|AD|}{|BD|} = \frac{5}{2} \qquad 5-2=3$$