Seminari 10

Matematičke metode za informatičare

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Zadatak 1

Odredite sliku, jezgru, rang i defekt linearnog operatora $B : \mathbb{R}^5 \to \mathbb{R}^3$ zadanog matricom

$$B = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \end{bmatrix}$$

u paru kanonskih baza. Je li B izomorfizam?

Rješenje

Kako je $\dim \mathbb{R}^5 \neq \dim \mathbb{R}^3$, zaključujemo da $\mathbb{R}^5$ i $\mathbb{R}^3$ nisu izomorfni vektorski prostori.

Stoga ne postoji niti jedan linearni operator $\mathbb{R}^5 \to \mathbb{R}^3$ koji je bijekcija.

Dakle, linearni operator B nije izomorfizam.

Ker B $\quad B : \mathbb{R}^5 \to \mathbb{R}^3$

$B(x) = \Theta_{\mathbb{R}^3} \rightsquigarrow BX = O$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_1	x_2	x_3	x_4	x_5		
1	2	3	1	0	0	+
(1)	2	5	2	1	0	$/\cdot(-1)$ $/\cdot(-1)$
1	2	5	2	1	0	+
0	0	−2	−1	−1	0	
1	2	5	2	1	0	
0	0	0	0	0	0	
0	0	−2	−1	(−1)	0	$/\cdot 1$
1	2	5	2	1	0	+
0	0	−2	−1	−1	0	
1	2	3	1	0	0	

$$\left.\begin{aligned} -2x_3 - x_4 - x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + x_4 &= 0 \end{aligned}\right\}$$

$$x_5 = -2x_3 - x_4$$
$$x_1 = -2x_2 - 3x_3 - x_4$$

$Y_{\mathcal{B}} = F_{(\mathcal{A},\mathcal{B})} X_{\mathcal{A}}$

Ker B $\quad B : \mathbb{R}^5 \to \mathbb{R}^3$

$$x_5 = -2x_3 - x_4$$
$$x_1 = -2x_2 - 3x_3 - x_4$$

$$\text{Ker}\, B = \{(-2x_2 - 3x_3 - x_4,\ x_2,\ x_3,\ x_4,\ -2x_3 - x_4) : x_2, x_3, x_4 \in \mathbb{R}\}$$

$$(-2x_2 - 3x_3 - x_4,\ x_2,\ x_3,\ x_4,\ -2x_3 - x_4) =$$
$$= x_2 \cdot (-2, 1, 0, 0, 0) + x_3 \cdot (-3, 0, 1, 0, -2) + x_4 \cdot (-1, 0, 0, 1, -1)$$

$$\mathcal{B}_{\text{Ker}\, B} = \{(-2, 1, 0, 0, 0),\ (-3, 0, 1, 0, -2),\ (-1, 0, 0, 1, -1)\}$$

$d(B) = 3 \longrightarrow B$ nije injekcija

$\operatorname{Im} B$

$$B : \mathbb{R}^5 \to \mathbb{R}^3 \qquad B = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \end{bmatrix}$$

$$r(B) + d(B) = \dim \mathbb{R}^5$$
$$r(B) + 3 = 5$$
$$r(B) = 2$$

$r(B) \neq \dim \mathbb{R}^3 \longrightarrow B$ nije surjekcija

$$\begin{bmatrix} \textcircled{1} & 2 & 3 & 1 & 0 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \end{bmatrix} / \cdot (-1) / \cdot (-1) \;\sim\; \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & \textcircled{2} & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} / \cdot (-1) \;\sim$$

$$\sim \begin{bmatrix} \textcircled{1} & 2 & 3 & 1 & 0 \\ 0 & 0 & \textcircled{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B}_{\operatorname{Im} B} = \{(1,1,1), (3,5,5)\}$$

Ako je $\dim U = \dim V$, je li linearni operator $f : U \to V$ izomorfizam?

Glavne minore

Neka je $A \in M_n(F)$ pri čemu je F polje.

- **Glavna podmatrica** reda r matrice A je svaka podmatrica $A_{i_1,i_2,\ldots,i_r}$ koja se sastoji od onih elemenata matrice A koji se nalaze na presjeku r redaka i r stupaca s istim indeksima $i_1, i_2, \ldots, i_r$.
- Glavnih podmatrica reda r matrice A ima ukupno $\binom{n}{r}$.
- **Glavna minora** $\Delta_{i_1,i_2,\ldots,i_r}$ reda r matrice A je determinanta pripadne glavne podmatrice, tj. $\Delta_{i_1,i_2,\ldots,i_r} = \det A_{i_1,i_2,\ldots,i_r}$.

Karakteristični polinom

- $k_A^{(1)}(\lambda) = \det(A - \lambda I)$
$$k_A^{(1)}(\lambda) = (-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n$$
- $k_A^{(2)}(\lambda) = \det(\lambda I - A)$
$$k_A^{(2)}(\lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n$$
- $c_r = (-1)^n a_r, \quad r = 1, 2, \ldots, n$
- $c_r = (-1)^r \sum\limits_{i_1 < i_2 < \cdots < i_r} \Delta_{i_1,i_2,\ldots,i_r}, \qquad \{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$
- $c_1 = -\operatorname{tr} A, \quad c_n = (-1)^n \det A$

Problem svojstvenih vrijednosti

- $V \leftsquigarrow$ konačnodimenzionalni vektorski prostor nad poljem F
- $\mathcal{B} \leftsquigarrow$ neka baza za vektorski prostor V

$$f : V \to V$$

$$f(x) = \lambda x \rightsquigarrow F_{\mathcal{B}} X_{\mathcal{B}} = \lambda X_{\mathcal{B}}$$

$$(F_{\mathcal{B}} - \lambda I) X_{\mathcal{B}} = O$$

$$S(\lambda) = \{x \in V : f(x) = \lambda x\}$$

Zadatak 2

Zadana je matrica $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$.

a) *Odredite svojstvene vrijednosti matrice A.*

b) *Odredite svojstvene potprostore matrice A.*

c) *Odredite minimalni polinom matrice A.*

d) *Izrazite A^{-1} pomoću potencija matrice A.*

Rješenje

a)

$$k_A(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3$$

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$k_A(\lambda) = \lambda^3 - 11\lambda^2 + 39\lambda - 45$$

$$c_1 = (-1)^1 \cdot (\Delta_1 + \Delta_2 + \Delta_3) = -(4 + 5 + 2) = -11$$

$$c_2 = (-1)^2 \cdot (\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}) = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 5 & -2 \\ 1 & 2 \end{vmatrix} =$$

$$= 18 + 9 + 12 = 39$$

$$c_3 = (-1)^3 \cdot \Delta_{1,2,3} = -1 \cdot \begin{vmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{vmatrix} = -1 \cdot 45 = -45$$

$$k_A(\lambda) = \lambda^3 - 11\lambda^2 + 39\lambda - 45$$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$1, -1, 3, -3, 5, -5, 9, -9, 15, -15, 45, -45$$

	1	−11	39	−45
3	1	−8	15	0

algebarska kratnost jednaka je 1

$$\sigma(A) = \{3, 5\}$$

algebarska kratnost jednaka je 2

$$(\lambda - 3)(\lambda^2 - 8\lambda + 15) = 0$$

$$\boxed{\lambda_1 = 3} \qquad \lambda^2 - 8\lambda + 15 = 0$$

$$\lambda_{2,3} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 15}}{2 \cdot 1}$$

$$k_A(\lambda) = (\lambda - 3)^2 \cdot (\lambda - 5)$$

$$\lambda_{2,3} = \frac{8 \pm 2}{2} \qquad \boxed{\lambda_2 = 5,\ \lambda_3 = 3}$$

b) $\mathcal{B}_{S(3)} = \{(1,0,1),\ (0,1,1)\}$ $\qquad \dim S(3) = 2$

geometrijska kratnost svojstvene vrijednosti $\lambda = 3$

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$(A - 3I)X = O$$

x_1	x_2	x_3		
1	1	−1	0	
2	2	−2	0	/ : 2
1	1	−1	0	
1	1	−1	0	
1	1	−1	0	
1	1	−1	0	
1	1	−1	0	

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 - x_3 = 0 \rightsquigarrow x_3 = x_1 + x_2$$

$$S(3) = \{(x_1, x_2, x_1 + x_2) : x_1, x_2 \in \mathbb{R}\}$$

$$(x_1, x_2, x_1 + x_2) = x_1 \cdot (1, 0, 1) + x_2 \cdot (0, 1, 1)$$

$$S(\lambda) = \{x \in V : f(x) = \lambda x\} \qquad (F_{\mathcal{B}} - \lambda I)X_{\mathcal{B}} = O$$

x_1	x_2	x_3		
-1	1	-1	0	
2	0	-2	0	$/:2$
1	1	-3	0	
-1	$\textcircled{1}$	-1	0	$/\cdot(-1)$
1	0	-1	0	
1	1	-3	0	$+$
-1	1	-1	0	
1	0	-1	0	
2	0	-2	0	
-1	1	-1	0	$+$
$\textcircled{1}$	0	-1	0	$/\cdot 1$
0	1	-2	0	
1	0	-1	0	

$$(A-5I)X = O \qquad A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{r} x_2 - 2x_3 = 0 \\ x_1 - x_3 = 0 \end{array}\right\} \begin{array}{l} \rightsquigarrow x_2 = 2x_3 \\ \rightsquigarrow x_1 = x_3 \end{array}$$

$$S(5) = \{(x_3, 2x_3, x_3) : x_3 \in \mathbb{R}\}$$

$$(x_3, 2x_3, x_3) = x_3 \cdot (1,2,1) \qquad \dim S(5) = 1$$

$$\mathcal{B}_{S(5)} = \{(1,2,1)\}$$

geometrijska kratnost svojstvene vrijednosti $\lambda = 5$

$$S(\lambda) = \{x \in V : f(x) = \lambda x\} \qquad (F_{\mathcal{B}} - \lambda I)X_{\mathcal{B}} = O$$

c) $k_A(\lambda) = \lambda^3 - 11\lambda^2 + 39\lambda - 45$

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$k_A(\lambda) = (\lambda-3)^2 \cdot (\lambda-5)$

$m_A(\lambda) = (\lambda-3)\cdot(\lambda-5)$ $\qquad$ $\boxed{m_A(\lambda) = \lambda^2 - 8\lambda + 15}$

$$(A-3I)\cdot(A-5I) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d)

$$k_A(A) = O$$

$$A^3 - 11A^2 + 39A - 45I = O$$

$$45I = A^3 - 11A^2 + 39A \;/\cdot A^{-1}$$

$$45A^{-1} = A^2 - 11A + 39I \;/:45$$

$$\boxed{A^{-1} = \frac{1}{45}A^2 - \frac{11}{45}A + \frac{39}{45}I}$$

$$m_A(A) = O$$

$$A^2 - 8A + 15I = O$$

$$15I = -A^2 + 8A \;/\cdot A^{-1}$$

$$15A^{-1} = -A + 8I \;/:15$$

$$\boxed{A^{-1} = -\frac{1}{15}A + \frac{8}{15}I}$$

Zadatak 3

Postoji li linearni operator $f : \mathbb{R}^3 \to \mathbb{R}^2$ za kojeg vrijedi

$$f(1,0,0) = (1,0), \quad f(0,1,0) = (1,3), \quad f(1,1,1) = (2,4)?$$

Ako postoji, odredite u tom slučaju $f(0,0,1)$ i njegovu matricu u paru kanonskih baza.

Rješenje

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f(1,0,0) = (1,0), \quad f(0,1,0) = (1,3), \quad f(1,1,1) = (2,4)$$

$\mathcal{B} = \{(1,0,0), (0,1,0), (1,1,1)\}$ je baza za $\mathbb{R}^3$.

$$\begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

Svaki linearni operator zadan je svojim djelovanjem na nekoj bazi. Postoji jedinstveni linearni operator f koji zadovoljava zadane uvjete.

$$f(0,0,1) = ?$$

$$(0,0,1) = \alpha_1 \cdot (1,0,0) + \alpha_2 \cdot (0,1,0) + \alpha_3 \cdot (1,1,1)$$

$$\left.\begin{array}{r} \alpha_1 + \alpha_3 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ \alpha_3 = 1 \end{array}\right\} \quad \rightsquigarrow \alpha_3 = 1, \quad \alpha_2 = -1, \quad \alpha_1 = -1$$

f je linearni operator

$$f(0,0,1) = f\big(-1 \cdot (1,0,0) + (-1)\cdot(0,1,0) + 1 \cdot (1,1,1)\big) =$$

$$= -1 \cdot f(1,0,0) + (-1)\cdot f(0,1,0) + 1 \cdot f(1,1,1) =$$

$$= -(1,0) - (1,3) + (2,4) = (0,1)$$

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f(1,0,0) = (1,0), \quad f(0,1,0) = (1,3), \quad f(1,1,1) = (2,4)$$

$$\mathcal{B} = \{(1,0,0),\ (0,1,0),\ (1,1,1)\} \qquad \mathcal{A}_{\text{kan}} = \{(1,0),\ (0,1)\}$$

$$F_{(\mathcal{B},\mathcal{A}_{\text{kan}})} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

$$Y_{\mathcal{A}_{\text{kan}}} = F_{(\mathcal{B},\mathcal{A}_{\text{kan}})} X_{\mathcal{B}}$$

$$f(0,0,1) = ?$$

$$Y_{\mathcal{A}_{\text{kan}}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad X_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} f(0,0,1) &= f\left(-1 \cdot (1,0,0) + (-1) \cdot (0,1,0) + 1 \cdot (1,1,1)\right) = \\ &= -1 \cdot f(1,0,0) + (-1) \cdot f(0,1,0) + 1 \cdot f(1,1,1) = \\ &= -(1,0) - (1,3) + (2,4) = (0,1) \end{aligned}$$

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f(1,0,0) = (1,0), \quad f(0,1,0) = (1,3), \quad f(1,1,1) = (2,4)$$

$$\mathcal{B} = \{(1,0,0),\ (0,1,0),\ (1,1,1)\} \qquad \mathcal{A}_{\text{kan}} = \{(1,0),\ (0,1)\}$$

$$\mathcal{B}_{\text{kan}} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

$$F_{(\mathcal{B},\mathcal{A}_{\text{kan}})} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{DZ} \quad S = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_{(\mathcal{B}_{\text{kan}},\mathcal{A}_{\text{kan}})} = T^{-1} F_{(\mathcal{B},\mathcal{A}_{\text{kan}})} S$$

$$\mathcal{B} \xrightarrow{S} \mathcal{B}_{\text{kan}} \quad (\mathcal{B}_{\text{kan}} \xrightarrow{S^{-1}} \mathcal{B}) \qquad \mathcal{A}_{\text{kan}} \xrightarrow{T} \mathcal{A}_{\text{kan}}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_{(\mathcal{B}_{\text{kan}},\mathcal{A}_{\text{kan}})} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$f(1,0,0) = (1,0), \quad f(0,1,0) = (1,3), \quad f(1,1,1) = (2,4)$$

$$\mathcal{B} = \{(1,0,0),\ (0,1,0),\ (1,1,1)\} \qquad \mathcal{A}_{\text{kan}} = \{(1,0),\ (0,1)\}$$

$$\mathcal{B}_{\text{kan}} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

$$F_{(\mathcal{B},\mathcal{A}_{\text{kan}})} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \qquad F_{(\mathcal{B}_{\text{kan}},\mathcal{A}_{\text{kan}})} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$f(0,0,1) = ?$$

$$Y_{\mathcal{A}_{\text{kan}}} = F_{(\mathcal{B}_{\text{kan}},\mathcal{A}_{\text{kan}})} X_{\mathcal{B}_{\text{kan}}}$$

$$X_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Y_{\mathcal{A}_{\text{kan}}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$