Seminari 11

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

Primjenom Euklidovog algoritma ispitajte može li se skratiti razlomak

$$\frac{x^4+x^3-x^2+x-2}{x^4+x^3-3x^2-x+2}.$$

Ukoliko se može, skratite ga.

Rješenje $\dfrac{x^4+x^3-x^2+x-2}{x^4+x^3-3x^2-x+2}$

$$\begin{array}{l}
\big(x^4+x^3-x^2+x-2\big) : \big(x^4+x^3-3x^2-x+2\big) = 1 \longleftarrow \boxed{Q_1} \\
\underline{-x^4-x^3+3x^2+x-2} \\
\qquad 2x^2+2x-4 \longleftarrow \boxed{R_1}
\end{array}$$

$$\begin{array}{l}
\big(x^4+x^3-3x^2-x+2\big) : \big(2x^2+2x-4\big) = \frac{1}{2}x^2-\frac{1}{2} \longleftarrow \boxed{Q_2} \\
\underline{-x^4-x^3+2x^2} \\
\qquad -x^2-x+2 \\
\qquad \underline{x^2+x-2} \\
\qquad\qquad 0 \longleftarrow \boxed{R_2}
\end{array}$$

$$M\big(x^4+x^3-x^2+x-2,\ x^4+x^3-3x^2-x+2\big) =$$
$$= n(2x^2+2x-4) = x^2+x-2$$

$$\frac{x^4+x^3-x^2+x-2}{x^4+x^3-3x^2-x+2} = \frac{(x^2+x-2)(x^2+1)}{(x^2+x-2)(x^2-1)} = \frac{x^2+1}{x^2-1}$$

$$\begin{array}{l}
\big(x^4+x^3-x^2+x-2\big) : \big(x^2+x-2\big) = x^2+1 \\
\underline{-x^4-x^3+2x^2} \\
\qquad x^2+x-2 \\
\qquad \underline{-x^2-x+2} \\
\qquad\qquad 0
\end{array}$$

$$\begin{array}{l}
\big(x^4+x^3-3x^2-x+2\big) : \big(x^2+x-2\big) = x^2-1 \\
\underline{-x^4-x^3+2x^2} \\
\qquad -x^2-x+2 \\
\qquad \underline{x^2+x-2} \\
\qquad\qquad 0
\end{array}$$

$$M\big(x^4+x^3-x^2+x-2,\ x^4+x^3-3x^2-x+2\big) = x^2+x-2$$

Zadatak 2

Zadani su polinomi

$$f(x) = 2x^4 - x^3 + x^2 + 3x + 1 \quad i \quad g(x) = 2x^3 - 3x^2 + 2x + 2.$$

Odredite polinome $\tilde{f}$ i $\tilde{g}$ takve da je $f\tilde{f} + g\tilde{g} = M(f, g)$.

Rješenje

$$\begin{array}{l} (2x^4 - x^3 + x^2 + 3x + 1) : (2x^3 - 3x^2 + 2x + 2) = x + 1 \longleftarrow \boxed{Q_1} \\ \underline{-2x^4 + 3x^3 - 2x^2 - 2x} \\ \qquad 2x^3 - x^2 + x + 1 \\ \qquad \underline{-2x^3 + 3x^2 - 2x - 2} \\ \qquad\qquad 2x^2 - x - 1 \longleftarrow \boxed{R_1} \end{array}$$

$$M(f, g) = n(2x + 1) = x + \frac{1}{2}$$

$$\uparrow \boxed{\tfrac{1}{2}R_2}$$

$$\begin{array}{l} (2x^3 - 3x^2 + 2x + 2) : (2x^2 - x - 1) = x - 1 \longleftarrow \boxed{Q_2} \\ \underline{-2x^3 + x^2 + x} \\ \qquad -2x^2 + 3x + 2 \\ \qquad \underline{2x^2 - x - 1} \\ \qquad\qquad 2x + 1 \longleftarrow \boxed{R_2} \end{array}$$

$$\begin{array}{l} (2x^2 - x - 1) : (2x + 1) = x - 1 \\ \underline{-2x^2 - x} \\ \qquad -2x - 1 \\ \qquad \underline{2x + 1} \\ \qquad\qquad 0 \longleftarrow \boxed{R_3} \end{array}$$

$$\uparrow \boxed{Q_3}$$

$$f(x) = 2x^4 - x^3 + x^2 + 3x + 1$$
$$g(x) = 2x^3 - 3x^2 + 2x + 2$$

$$R_1 = f - gQ_1 \leftsquigarrow f = gQ_1 + R_1$$

$$g = R_1Q_2 + R_2$$

$$R_1 = R_2Q_3$$

$$Q_2(x) = x - 1$$

$$Q_1(x) = x + 1$$

$$g = R_1Q_2 + R_2$$

$$g - R_1Q_2 = R_2 \quad \Big/ \cdot \frac{1}{2}$$

$$\frac{1}{2}g - \frac{1}{2}R_1Q_2 = \frac{1}{2}R_2$$

$$\frac{1}{2}g - \frac{1}{2}R_1 \cdot (x - 1) = M(f, g)$$

$$\frac{1}{2}g - \frac{1}{2}(x - 1) \cdot (f - gQ_1) = M(f, g)$$

$$\frac{1}{2}g - \frac{1}{2}(x - 1) \cdot (f - g \cdot (x + 1)) = M(f, g)$$

$$\frac{1}{2}g - \frac{1}{2}(x - 1)f + \frac{1}{2}(x^2 - 1)g = M(f, g)$$

$$\left(-\frac{1}{2}x + \frac{1}{2}\right) \cdot f(x) + \frac{1}{2}x^2 \cdot g(x) = M(f, g)$$

$$\tilde{f}(x) = -\frac{1}{2}x + \frac{1}{2} \qquad \tilde{g}(x) = \frac{1}{2}x^2$$

$$\boxed{f\tilde{f} + g\tilde{g} = M(f, g)}$$

$$M(f, g) = \frac{1}{2}R_2$$

Zadatak 3

Riješite jednadžbu

$$x^4 - 4x^3 + 6x^2 - 4x + 5 = 0$$

ako je poznato jedno njezino rješenje $x_1 = 2 - i$.

Kompleksne nultočke polinoma

Neka je $P \in \mathbb{R}[x]$. Ako je $z_0 \in \mathbb{C}$ nultočka polinoma P, tada je i $\bar{z}_0$ također nultočka polinoma P.

Rješenje $\boxed{x_1 = 2 - i, \quad x_2 = 2 + i}$ $\qquad x^4 - 4x^3 + 6x^2 - 4x + 5 = 0$

	1	−4	6	−4	5
$2-i$	1	$-2-i$	1	$-2-i$	0
$2+i$	1	0	1	0	

$i^2 = -1$

$$(x-(2-i))\cdot(x-(2+i))\cdot(x^2+0\cdot x+1) = 0$$

$$x^2+1=0 \rightsquigarrow x^2=-1 \rightsquigarrow \boxed{x_3 = i,\ x_4 = -i}$$

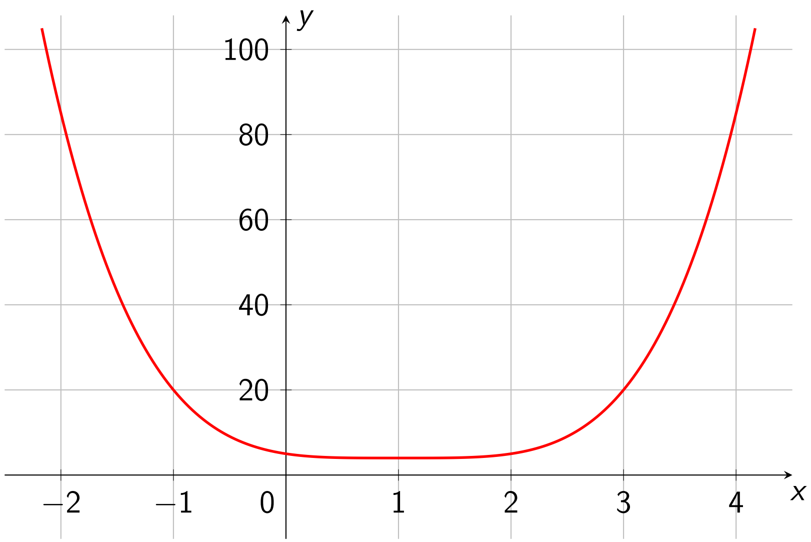
$$(2-i)(-2-i) = -4-2i+2i+i^2 = -4-1 = -5$$

2. način

$$\left(x^4-4x^3+6x^2-4x+5\right) : \left(x^2-4x+5\right) = x^2+1$$

$$\begin{array}{r} -x^4+4x^3-5x^2 \\ \hline x^2 - 4x + 5 \\ -x^2 + 4x - 5 \\ \hline 0 \end{array}$$

$$\begin{aligned} &(x-(2-i))\cdot(x-(2+i)) = \\ &= x^2-(2+i)x-(2-i)x+4-i^2 = \\ &= x^2-4x+5 \end{aligned}$$



$f(x) = x^4 - 4x^3 + 6x^2 - 4x + 5$

Zadatak 4

Odredite sva rješenja jednadžbe

$$x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$$

ako je poznato da ima barem jedno cjelobrojno kompleksno rješenje.

Cjelobrojne kompleksne nultočke polinoma

Ako je $\alpha + \beta i$ cjelobrojna kompleksna nultočka polinoma

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

s cjelobrojnim koeficijentima, onda je $\alpha^2 + \beta^2$ djelitelj slobodnog člana.

Rješenje $\qquad x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$

pozitivni djelitelji od 25: $1, 5, 25$

$\alpha^2 + \beta^2 \rightsquigarrow \alpha + \beta i$

$$1 = 0^2 + 1^2 = 0^2 + (-1)^2$$

$$\begin{aligned} 5 &= 1^2+2^2 = 1^2+(-2)^2 = (-1)^2+2^2 = (-1)^2+(-2)^2 \\ &= 2^2+1^2 = 2^2+(-1)^2 = (-2)^2+1^2 = (-2)^2+(-1)^2 \end{aligned}$$

$$\begin{aligned} 25 &= 0^2+5^2 = 0^2+(-5)^2 \\ &= 3^2+4^2 = 3^2+(-4)^2 = (-3)^2+4^2 = (-3)^2+(-4)^2 \\ &= 4^2+3^2 = 4^2+(-3)^2 = (-4)^2+3^2 = (-4)^2+(-3)^2 \end{aligned}$$

$i,\ -i,\ 1+2i,\ 1-2i,\ -1+2i,\ -1-2i,\ 2+i,\ 2-i,\ -2+i,\ -2-i,\ 5i,\ -5i,\ 3+4i,\ 3-4i,\ -3+4i,\ -3-4i,\ 4+3i,\ 4-3i,\ -4+3i,\ -4-3i$

$x_1 = 1 + 2i, \quad x_2 = 1 - 2i$ $\qquad x^4 - 6x^3 + 18x^2 - 30x + 25 = 0$

	1	−6	18	−30	25
$1+2i$	1	$-5+2i$	$9-8i$	$-5+10i$	0
$1-2i$	1	-4	5	0	

$i^2 = -1$ $\qquad \big(x - (1+2i)\big)\cdot\big(x-(1-2i)\big)\cdot\big(x^2-4x+5\big) = 0$

$$x^2 - 4x + 5 = 0 \qquad x_{3,4} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$(1+2i)(-5+2i) = -5 + 2i - 10i + 4i^2 = -9 - 8i$

$(1+2i)(9-8i) = 9 - 8i + 18i - 16i^2 = 25 + 10i$

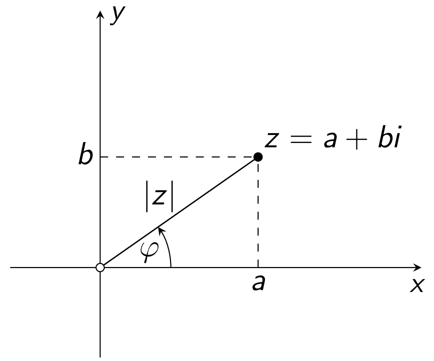
$(1+2i)(-5+10i) = -5 + 10i - 10i + 20i^2 = -25$

$x_3 = 2 + i$
$x_4 = 2 - i$

$i,\ -i,\ 1+2i,\ 1-2i,\ -1+2i,\ -1-2i,\ 2+i,\ 2-i,\ -2+i,$
$-2-i,\ 5i,\ -5i,\ 3+4i,\ 3-4i,\ -3+4i,\ -3-4i,\ 4+3i,$
$4-3i,\ -4+3i,\ -4-3i$

$f(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$

Trigonometrijski zapis kompleksnog broja



$a = r\cos\varphi$

$b = r\sin\varphi$

$\arg z = \varphi \in [0, 2\pi\rangle$

$\operatorname{tg}\varphi = \dfrac{b}{a}$

$r = |z| = \sqrt{a^2 + b^2}$

$$z = r(\cos\varphi + i\sin\varphi)$$

$$z^n = r^n\big(\cos(n\varphi) + i\sin(n\varphi)\big), \quad n \in \mathbb{N}$$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

Zadatak 5

U skupu kompleksnih brojeva riješite jednadžbu $z^6 + 3z^4 + z^2 + 3 = 0$.

Rješenje

$$z^6 + 3z^4 + z^2 + 3 = 0$$

$$z^2 = t$$

$$t^3 + 3t^2 + t + 3 = 0$$

$$1, -1, 3, -3$$

	1	3	1	3
−3	1	0	1	0

$$\big(t - (-3)\big)\big(t^2 + 0\cdot t + 1\big) = 0$$

$$(t+3)(t^2+1) = 0$$

$t_1 = -3 \qquad t^2 + 1 = 0$

$t_2 = i \qquad t_3 = -i$

$z^2 = t, \quad t_1 = -3, \quad t_2 = i, \quad t_3 = -i$ $\qquad z^6 + 3z^4 + z^2 + 3 = 0$

$z^2 = -3$

$z = \sqrt{-3}$

$\boxed{z_1 = \sqrt{3}i}$ $\quad \boxed{z_2 = -\sqrt{3}i}$

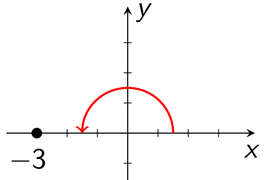
$$\left(\sqrt{-3}\right)_k = \sqrt{3} \cdot \left(\cos \frac{\pi + 2k\pi}{2} + i \sin \frac{\pi + 2k\pi}{2}\right)$$

$$\left(\sqrt{-3}\right)_0 = \sqrt{3} \cdot \left(\cos \tfrac{\pi}{2} + i \sin \tfrac{\pi}{2}\right)$$

$$\left(\sqrt{-3}\right)_0 = \sqrt{3}i$$

$$\left(\sqrt{-3}\right)_1 = \sqrt{3} \cdot \left(\cos \tfrac{3}{2}\pi + i \sin \tfrac{3}{2}\pi\right)$$

$$\left(\sqrt{-3}\right)_1 = -\sqrt{3}i$$



$r = 3 \qquad \varphi = \pi$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos \frac{\varphi+2k\pi}{n} + i \sin \frac{\varphi+2k\pi}{n}\right)$$

$z^2 = t, \quad t_1 = -3, \quad t_2 = i, \quad t_3 = -i$ $\qquad z^6 + 3z^4 + z^2 + 3 = 0$

$z^2 = i$

$z = \sqrt{i}$

$\boxed{z_3 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$ $\quad \boxed{z_4 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}$

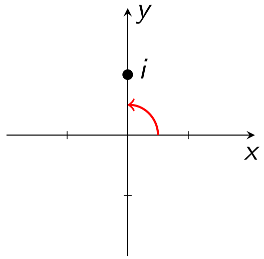
$$\left(\sqrt{i}\,\right)_k = \sqrt{1} \cdot \left(\cos \frac{\frac{\pi}{2} + 2k\pi}{2} + i \sin \frac{\frac{\pi}{2} + 2k\pi}{2}\right)$$

$$\left(\sqrt{i}\,\right)_0 = \cos \tfrac{\pi}{4} + i \sin \tfrac{\pi}{4}$$

$$\left(\sqrt{i}\,\right)_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\left(\sqrt{i}\,\right)_1 = \cos \tfrac{5}{4}\pi + i \sin \tfrac{5}{4}\pi$$

$$\left(\sqrt{i}\,\right)_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$



$r = 1 \qquad \varphi = \frac{\pi}{2}$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos \frac{\varphi+2k\pi}{n} + i \sin \frac{\varphi+2k\pi}{n}\right)$$

$z^2 = t, \quad t_1 = -3, \quad t_2 = i, \quad t_3 = -i$ $\qquad z^6 + 3z^4 + z^2 + 3 = 0$

$z^2 = -i$

$z = \sqrt{-i}$

$\boxed{z_5 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$ $\quad \boxed{z_6 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i}$

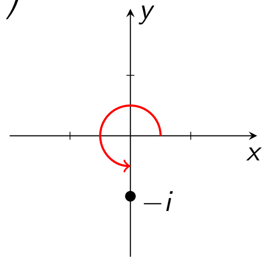
$$\left(\sqrt{-i}\,\right)_k = \sqrt{1} \cdot \left(\cos \frac{\frac{3}{2}\pi + 2k\pi}{2} + i \sin \frac{\frac{3}{2}\pi + 2k\pi}{2}\right)$$

$$\left(\sqrt{-i}\,\right)_0 = \cos \tfrac{3}{4}\pi + i \sin \tfrac{3}{4}\pi$$

$$\left(\sqrt{-i}\,\right)_0 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

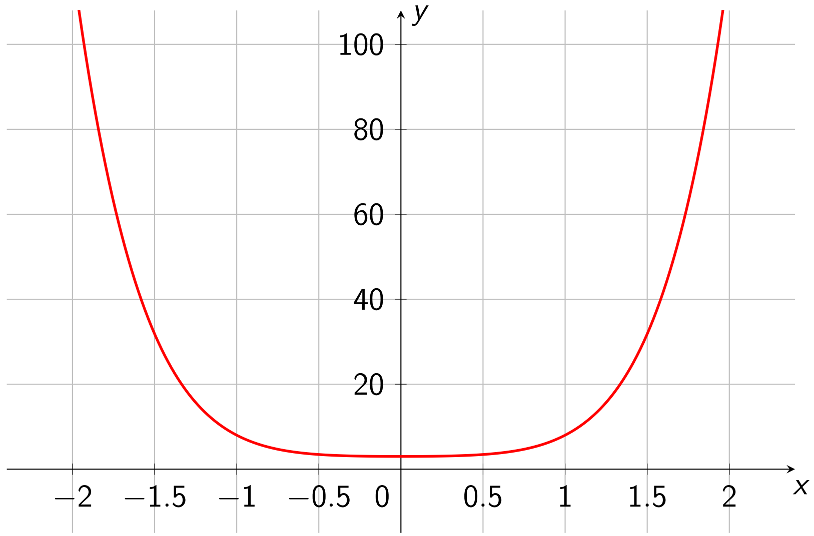
$$\left(\sqrt{-i}\,\right)_1 = \cos \tfrac{7}{4}\pi + i \sin \tfrac{7}{4}\pi$$

$$\left(\sqrt{-i}\,\right)_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$



$r = 1 \qquad \varphi = \frac{3}{2}\pi$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos \frac{\varphi+2k\pi}{n} + i \sin \frac{\varphi+2k\pi}{n}\right)$$



$f(x) = x^6 + 3x^4 + x^2 + 3$