Seminari 12

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

Ferrarijevom metodom riješite jednadžbu $x^4 - x^3 + 9x^2 - 13x = -24$.

Rješenje

$$x^4 - x^3 + 9x^2 - 13x + 24 = 0$$

$$\underbrace{\left(x^2 - \tfrac{1}{2}x + y\right)^2} - \left[\left(\tfrac{1}{4} + 2y - 9\right)x^2 + (-y+13)x + (y^2 - 24)\right] = 0$$

$$x^4 + \underline{\tfrac{1}{4}x^2} + y^2 - x^3 + \underline{2x^2y} - xy$$

$b^2 - 4ac = 0$

$$\left(x^2 - \tfrac{1}{2}x + y\right)^2 - \left[\left(2y - \tfrac{35}{4}\right)x^2 + (-y+13)x + (y^2 - 24)\right] = 0$$

$$(-y+13)^2 - 4\left(2y - \tfrac{35}{4}\right)(y^2 - 24) = 0$$

$$(-y+13)^2 - (8y - 35)(y^2 - 24) = 0$$

$$y^2 - 26y + 169 - 8y^3 + 192y + 35y^2 - 840 = 0$$

$$\boxed{-8y^3 + 36y^2 + 166y - 671 = 0}$$

Ferrarijeva rezolventa

Eliminacija cjelobrojnih i racionalnih kandidata

Teorem

Ako je $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ polinom s cjelobrojnim koeficijentima i α njegova cjelobrojna nultočka, tada je za svaki $k \in \mathbb{Z}$ broj $f(k)$ djeljiv s $\alpha - k$.

Teorem

Ako je $M(p,q) = 1$ i $\alpha = \frac{p}{q}$ racionalna nultočka polinoma $f(x)$ s cjelobrojnim koeficijentima, tada je za svaki cijeli broj k broj $f(k)$ djeljiv s $p - kq$.

$q : 1, 2, 4, 8$

$p : 1, -1, 11, -11, 61, -61, 671, -671$

$$\underbrace{-8y^3 + 36y^2 + 166y - 671}_{f(y)} = 0$$

$$p - kq \xrightarrow{k=-1} p + q \qquad f(-1) = -793$$

$671 = 11 \cdot 61$

$793 = 13 \cdot 61$

$\frac{p}{q}$:	$\cancel{\frac{1}{1}}$	$\cancel{\frac{1}{2}}$	$\cancel{\frac{1}{4}}$	$\cancel{\frac{1}{8}}$	$\cancel{\frac{-1}{1}}$	$\frac{-1}{2}$	$\cancel{\frac{-1}{4}}$	$\cancel{\frac{-1}{8}}$	$\cancel{\frac{11}{1}}$	$\frac{11}{2}$	$\cancel{\frac{11}{4}}$
$p+q$:	2	3	5	9	0	1	3	7	12	13	15

$\frac{p}{q}$:	$\cancel{\frac{11}{8}}$	$\cancel{\frac{-11}{1}}$	$\cancel{\frac{-11}{2}}$	$\cancel{\frac{-11}{4}}$	$\cancel{\frac{-11}{8}}$	$\cancel{\frac{61}{1}}$	$\cancel{\frac{61}{2}}$	$\cancel{\frac{61}{4}}$	$\cancel{\frac{61}{8}}$	$\cancel{\frac{-61}{1}}$
$p+q$:	19	−10	−9	−7	−3	62	63	65	69	−60

$\frac{p}{q}$:	$\cancel{\frac{-61}{2}}$	$\cancel{\frac{-61}{4}}$	$\cancel{\frac{-61}{8}}$	$\cancel{\frac{671}{1}}$	$\cancel{\frac{671}{2}}$	$\cancel{\frac{671}{4}}$	$\cancel{\frac{671}{8}}$	$\cancel{\frac{-671}{1}}$
$p+q$:	−59	−57	−53	672	673	675	679	−670

$y = \frac{11}{2}$

$\frac{p}{q}$:	$\cancel{\frac{-671}{2}}$	$\cancel{\frac{-671}{4}}$	$\cancel{\frac{-671}{8}}$
$p+q$:	−669	−667	−663

	−8	36	166	−671
$\frac{11}{2}$	−8	−8	122	0

$$x^4 - x^3 + 9x^2 - 13x + 24 = 0$$

$$\left(x^2 - \tfrac{1}{2}x + y\right)^2 - \left[\left(2y - \tfrac{35}{4}\right)x^2 + (-y + 13)x + (y^2 - 24)\right] = 0$$

$y = \frac{11}{2}$

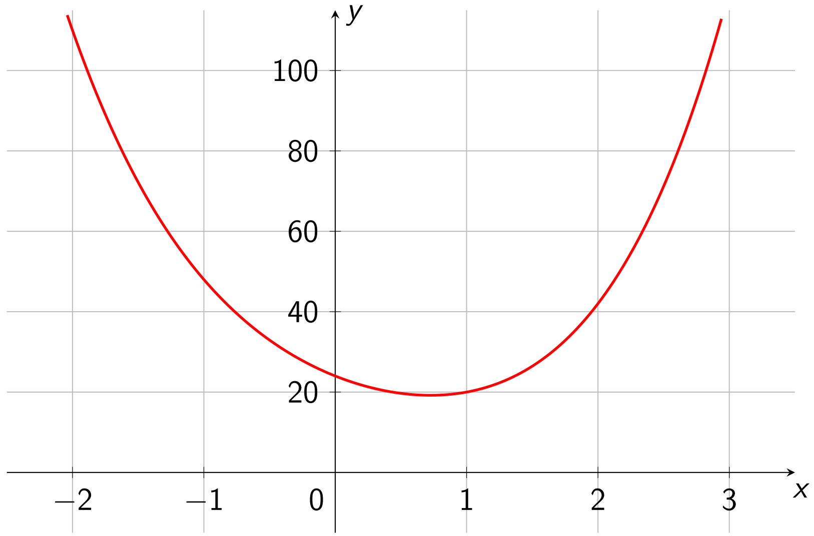
$$\left(x^2 - \tfrac{1}{2}x + \tfrac{11}{2}\right)^2 - \left[\tfrac{9}{4}x^2 + \tfrac{15}{2}x + \tfrac{25}{4}\right] = 0$$

$$\left(x^2 - \tfrac{1}{2}x + \tfrac{11}{2}\right)^2 - \left(\tfrac{3}{2}x + \tfrac{5}{2}\right)^2 = 0$$

$$\left[\left(x^2 - \tfrac{1}{2}x + \tfrac{11}{2}\right) + \left(\tfrac{3}{2}x + \tfrac{5}{2}\right)\right]\left[\left(x^2 - \tfrac{1}{2}x + \tfrac{11}{2}\right) - \left(\tfrac{3}{2}x + \tfrac{5}{2}\right)\right] = 0$$

$$\left(x^2 + x + 8\right)\left(x^2 - 2x + 3\right) = 0$$

$a^2 - b^2 = (a + b)(a - b)$



$f(x) = x^4 - x^3 + 9x^2 - 13x + 24$

$\sqrt{-31} = \sqrt{31}i$

$\sqrt{-8} = \sqrt{8}i = 2\sqrt{2}i$

$$x^4 - x^3 + 9x^2 - 13x + 24 = 0$$

$$\left(x^2 + x + 8\right)\left(x^2 - 2x + 3\right) = 0$$

$$x^2 + x + 8 = 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 32}}{2}$$

$$x_1 = -\frac{1}{2} + \frac{\sqrt{31}}{2}i$$

$$x_2 = -\frac{1}{2} - \frac{\sqrt{31}}{2}i$$

$$x^2 - 2x + 3 = 0$$

$$x_{3,4} = \frac{2 \pm \sqrt{4 - 12}}{2}$$

$$x_{3,4} = \frac{2 \pm 2\sqrt{2}i}{2}$$

$$x_3 = 1 + \sqrt{2}i$$

$$x_4 = 1 - \sqrt{2}i$$

Zadatak 2

Zadana je jednadžba $x^3 + 6x - 2 = 0$.

a) *Bez direktnog rješavanja jednadžbe komentirajte koliko ima realnih, a koliko pravih kompleksnih rješenja.*

b) *Pomoću Cardanove formule riješite zadanu jednadžbu.*

Rješenje

$x^3 + 6x - 2 = 0$

$p = 6, \quad q = -2$

a)

$$x^3 + px + q = 0$$

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

$$\Delta = \left(\frac{-2}{2}\right)^2 + \left(\frac{6}{3}\right)^3$$

$$\Delta = 1 + 8 = 9$$

$\Delta > 0$ ⇝ jednadžba ima jedno realno i dva konjugirano kompleksna rješenja

b)

$$u_0 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$u_0 = \sqrt[3]{-\frac{-2}{2} + \sqrt{9}}$$

$$u_0 = \sqrt[3]{1+3} \qquad \boxed{u_0 = \sqrt[3]{4}}$$

$$v_0 = -\frac{p}{3u_0}$$

$$v_0 = -\frac{6}{3\sqrt[3]{4}}$$

$$\boxed{v_0 = -\frac{2}{\sqrt[3]{4}}}$$

$x^3 + 6x - 2 = 0$

$p = 6, \quad q = -2$

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

$\Delta = 9$

$$\varepsilon = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$x_1 = u_0 + v_0 = \sqrt[3]{4} - \frac{2}{\sqrt[3]{4}} \qquad x_1 \approx 0.32748$$

$$x_2 = u_0\varepsilon + v_0\bar{\varepsilon} = \sqrt[3]{4}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) - \frac{2}{\sqrt[3]{4}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$x_2 = \left(\frac{1}{\sqrt[3]{4}} - \frac{1}{2}\sqrt[3]{4}\right) - \left(\frac{\sqrt{3}\cdot\sqrt[3]{4}}{2} + \frac{\sqrt{3}}{\sqrt[3]{4}}\right) i$$

$$x_2 \approx -0.16374 - 2.46585i$$

b)

$$u_0 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$u_0 = \sqrt[3]{-\frac{-2}{2} + \sqrt{9}}$$

$$u_0 = \sqrt[3]{1+3} \qquad \boxed{u_0 = \sqrt[3]{4}}$$

$$v_0 = -\frac{p}{3u_0}$$

$$v_0 = -\frac{6}{3\sqrt[3]{4}}$$

$$\boxed{v_0 = -\frac{2}{\sqrt[3]{4}}}$$

$x^3 + 6x - 2 = 0$

$p = 6, \quad q = -2$

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

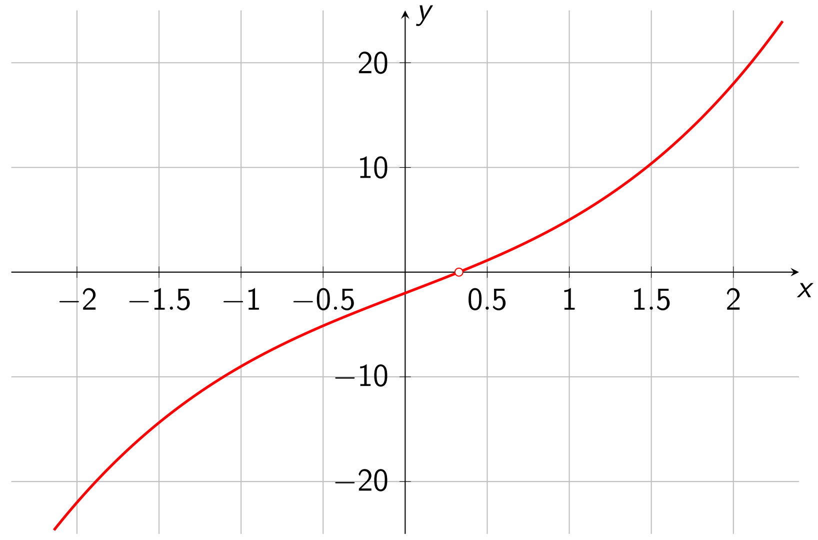
$\Delta = 9$

$$\varepsilon = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$x_3 = u_0\bar{\varepsilon} + v_0\varepsilon = \sqrt[3]{4}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - \frac{2}{\sqrt[3]{4}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$x_3 = \left(\frac{1}{\sqrt[3]{4}} - \frac{1}{2}\sqrt[3]{4}\right) + \left(\frac{\sqrt{3}\cdot\sqrt[3]{4}}{2} + \frac{\sqrt{3}}{\sqrt[3]{4}}\right) i$$

$$x_3 \approx -0.16374 + 2.46585i$$



$f(x) = x^3 + 6x - 2$

Zadatak 3

Riješite jednadžbu $5x^5 - 3x^4 + 2x^3 + 2x^2 - 3x + 5 = 0.$

Rješenje

simetrična jednadžba neparnog stupnja ⇝ jedno rješenje je $x_1 = -1$

	5	−3	2	2	−3	5
−1	5	−8	10	−8	5	0

$$(x+1)\left(5x^4 - 8x^3 + 10x^2 - 8x + 5\right) = 0$$

$$5x^4 - 8x^3 + 10x^2 - 8x + 5 = 0$$

simetrična jednadžba parnog stupnja

$$5x^4 - 8x^3 + 10x^2 - 8x + 5 = 0 \;/ : x^2$$

$$5x^2 - 8x + 10 - \frac{8}{x} + \frac{5}{x^2} = 0$$

$$5\left(x^2 + \frac{1}{x^2}\right) - 8\left(x + \frac{1}{x}\right) + 10 = 0$$

$$x + \frac{1}{x} = t \;/^2$$

$$x^2 + 2 + \frac{1}{x^2} = t^2 \rightsquigarrow x^2 + \frac{1}{x^2} = t^2 - 2$$

$$5\left(t^2 - 2\right) - 8t + 10 = 0$$

$$5t^2 - 8t = 0$$

$$t(5t - 8) = 0$$

$$t_1 = 0, \quad t_2 = \frac{8}{5}$$

$$x + \frac{1}{x} = 0 \;/ \cdot x$$

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x_2 = i, \quad x_3 = -i$$

$$x + \frac{1}{x} = \frac{8}{5} \;/ \cdot 5x$$

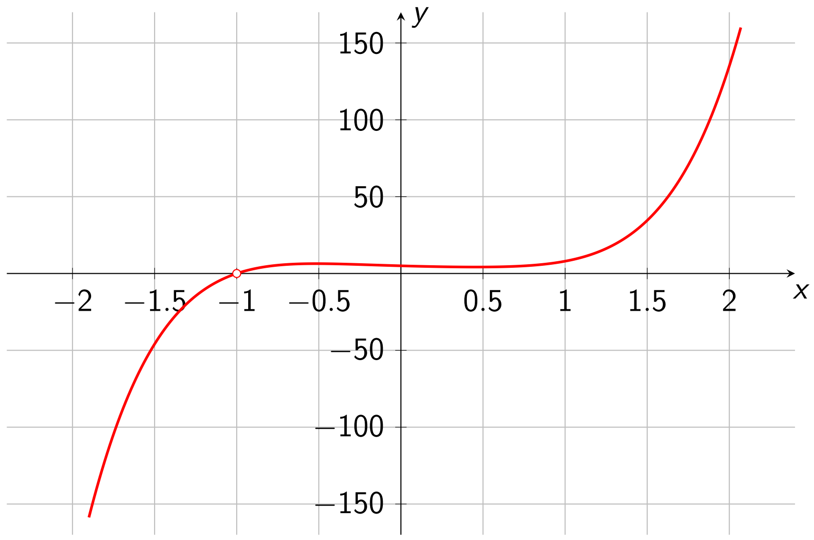
$$5x^2 - 8x + 5 = 0$$

$$x_{4,5} = \frac{8 \pm \sqrt{64 - 100}}{10}$$

$$x_{4,5} = \frac{8 \pm 6i}{10} = \frac{4 \pm 3i}{5}$$

$$x_4 = \frac{4}{5} + \frac{3}{5}i$$

$$x_5 = \frac{4}{5} - \frac{3}{5}i$$



$$f(x) = 5x^5 - 3x^4 + 2x^3 + 2x^2 - 3x + 5$$

Oznake

- Funkcija dvije varijable: $z = z(x, y)$
- Parcijalna derivacija po varijabli x

$$z_x \qquad z'_x \qquad \frac{\partial z}{\partial x}$$

- Parcijalna derivacija po varijabli y

$$z_y \qquad z'_y \qquad \frac{\partial z}{\partial y}$$

Parcijalne derivacije drugog reda – oznake

- Funkcija dvije varijable: $z = z(x, y)$

z_{xx}	z''_{xx}	$\dfrac{\partial^2 z}{\partial x^2}$
z_{xy}	z''_{xy}	$\dfrac{\partial^2 z}{\partial x \partial y}$
z_{yx}	z''_{yx}	$\dfrac{\partial^2 z}{\partial y \partial x}$
z_{yy}	z''_{yy}	$\dfrac{\partial^2 z}{\partial y^2}$

Zadatak 4

$(x^n)' = nx^{n-1}$

$(cu)'(x) = c \cdot u'(x)$

$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$

Odredite parcijalne derivacije sljedećih funkcija:

a) $f(x, y) = x^2 + y^2$

b) $g(x, y) = 3x^2 + xy + \sqrt{y}$

c) $z = \dfrac{y}{x}$ → $z = yx^{-1}$

Rješenje

a) $f_x = 2x + 0 = 2x$ $\quad$ $f_y = 0 + 2y = 2y$

b) $g_x = 6x + y + 0 = 6x + y$ $\quad$ $g_y = 0 + x + \dfrac{1}{2\sqrt{y}} = x + \dfrac{1}{2\sqrt{y}}$

c) $z_x = y \cdot \left(-x^{-2}\right) = -\dfrac{y}{x^2}$ $\quad$ $z_y = x^{-1} \cdot 1 = \dfrac{1}{x}$

Zadatak 5

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

Odredite parcijalne derivacije sljedećih funkcija:

a) $z = xe^y$ $\quad$ b) $z = ye^y + \sqrt{x}$ $\quad$ c) $u(x, y) = \dfrac{2x - y}{x + y}$

$\left(\dfrac{u}{v}\right)'(x) = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

Rješenje

$(cu)'(x) = c \cdot u'(x)$

a) $z_x = e^y \cdot 1 = e^y$ $\quad$ $z_y = x \cdot e^y = xe^y$

b) $z_x = 0 + \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$ $\quad$ $z_y = 1 \cdot e^y + y \cdot e^y + 0 = (1 + y)e^y$

$(e^x)' = e^x$

$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$

c) $u_x = \dfrac{2 \cdot (x + y) - (2x - y) \cdot 1}{(x + y)^2} = \dfrac{3y}{(x + y)^2}$

$u_y = \dfrac{-1 \cdot (x + y) - (2x - y) \cdot 1}{(x + y)^2} = \dfrac{-3x}{(x + y)^2}$

Zadatak 6

Odredite parcijalne derivacije sljedećih funkcija:

a) $z = 2^{\sin \frac{y}{x}}$

b) $z = x^y$

c) $f(x, y, z) = e^{2xz} - \ln(yz) + 1$

Rješenje

$(a^x)' = a^x \ln a$ $(a^{\text{nešto}})' = a^{\text{nešto}} \ln a \cdot (\text{nešto})'$

$(x^n)' = nx^{n-1}$ $(\sin x)' = \cos x$ $(\sin(\text{nešto}))' = \cos(\text{nešto}) \cdot (\text{nešto})'$

$z = 2^{\sin\frac{y}{x}}$

a) $$z_x = 2^{\sin\frac{y}{x}} \ln 2 \cdot \left(\sin\frac{y}{x}\right)'_x = 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x} \cdot \left(\frac{y}{x}\right)'_x =$$

$$= 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x} \cdot \frac{-y}{x^2} = -\frac{y}{x^2} \cdot 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x}$$

$$z_y = 2^{\sin\frac{y}{x}} \ln 2 \cdot \left(\sin\frac{y}{x}\right)'_y = 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x} \cdot \left(\frac{y}{x}\right)'_y =$$

$$= 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x} \cdot \frac{1}{x} = \frac{1}{x} \cdot 2^{\sin\frac{y}{x}} \ln 2 \cdot \cos\frac{y}{x}$$

$z = x^y$

b) $z_x = yx^{y-1}$ $z_y = x^y \ln x$

c)

$(e^{\text{nešto}})' = e^{\text{nešto}} \cdot (\text{nešto})'$ $(e^x)' = e^x$

$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$ $(\ln x)' = \frac{1}{x}$

$$f_x = e^{2xz} \cdot (2xz)'_x - 0 + 0 = e^{2xz} \cdot 2z = 2ze^{2xz}$$

$$f_y = 0 - \frac{1}{yz} \cdot (yz)'_y + 0 = -\frac{1}{yz} \cdot z = -\frac{1}{y}$$

$$f_z = e^{2xz} \cdot (2xz)'_z - \frac{1}{yz} \cdot (yz)'_z + 0 = e^{2xz} \cdot 2x - \frac{1}{yz} \cdot y = 2xe^{2xz} - \frac{1}{z}$$

$f(x, y, z) = e^{2xz} - \ln(yz) + 1$