Seminari 13

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

Zadana je funkcija $f(x,y) = \ln\left(x + y^2\right)$.

a) *Prikažite grafički domenu funkcije* f.

b) *Odredite nivo-linije funkcije* f *i specijalno nacrtajte nivo-liniju za vrijednost* $z = \ln 5$.

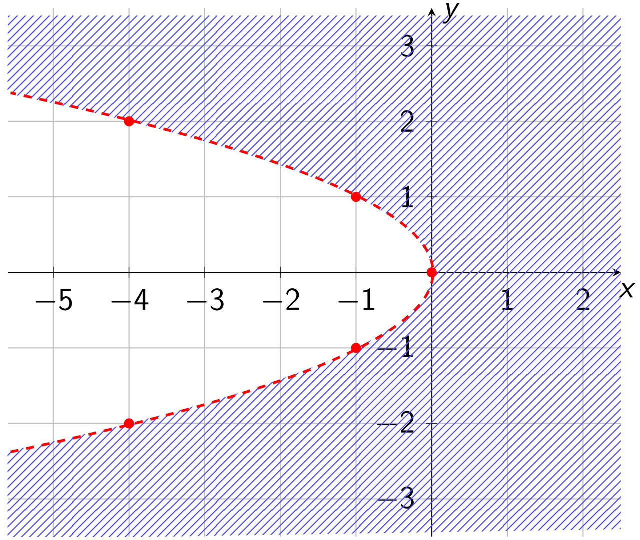
c) *Odredite nultočke funkcije* f.

d) *Odredite parcijalne derivacije funkcije* f.

e) *Odredite* $\dfrac{\partial^4 f}{\partial x^3 \partial y}$.

a) **Rješenje**

$f(x,y) = \ln\left(x + y^2\right)$



$$x + y^2 > 0$$

$$y^2 > -x$$

crtamo krivulju

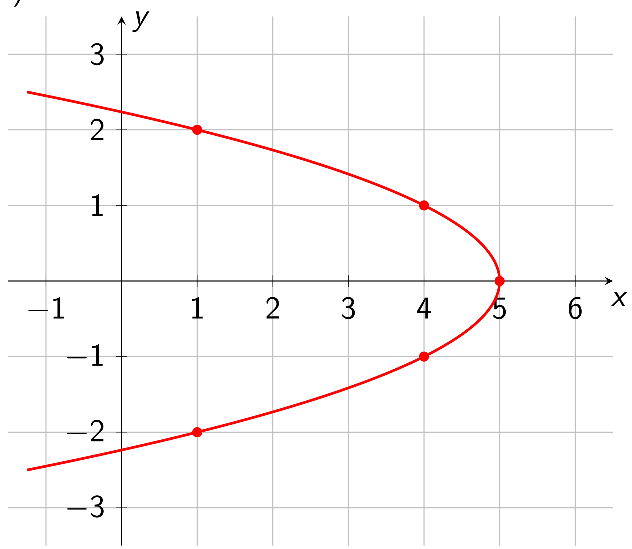
$$y^2 = -x$$

$$x = -y^2$$

y	-2	-1	0	1	2
x	-4	-1	0	-1	-4

b)

$f(x,y) = \ln\left(x + y^2\right)$



$$\ln\left(x + y^2\right) = C$$

$$x + y^2 = e^C$$

$$y^2 = -x + e^C, \quad C \in \mathbb{R}$$

nivo-linije su parabole

$$C = \ln 5$$

$$y^2 = -x + e^{\ln 5}$$

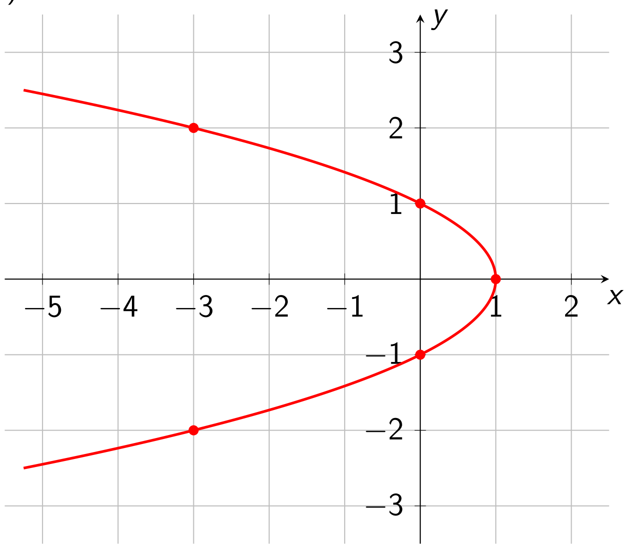
$$y^2 = -x + 5$$

$$x = 5 - y^2$$

y	-2	-1	0	1	2
x	1	4	5	4	1

c)

$f(x,y) = \ln\left(x+y^2\right)$



$y^2 = -x + e^C, \;\; C \in \mathbb{R}$

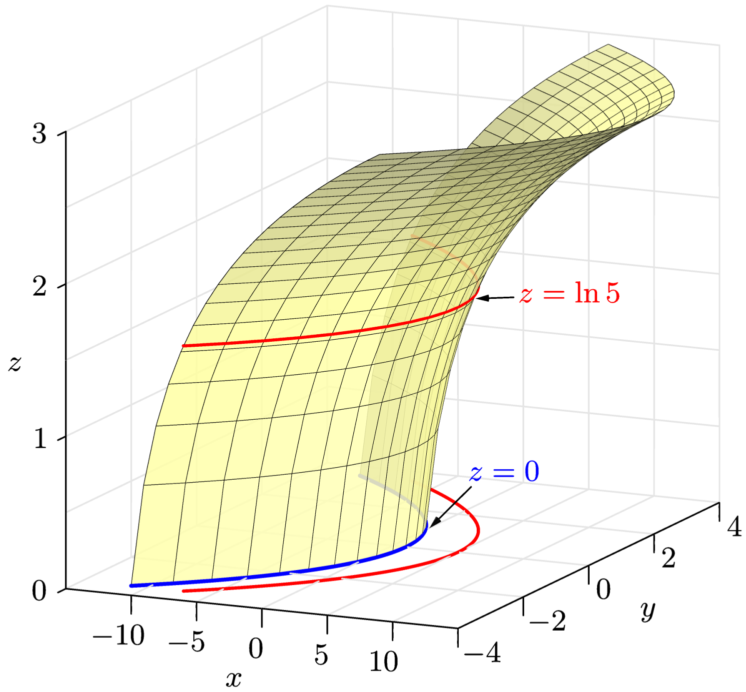
nultočke $C = 0$

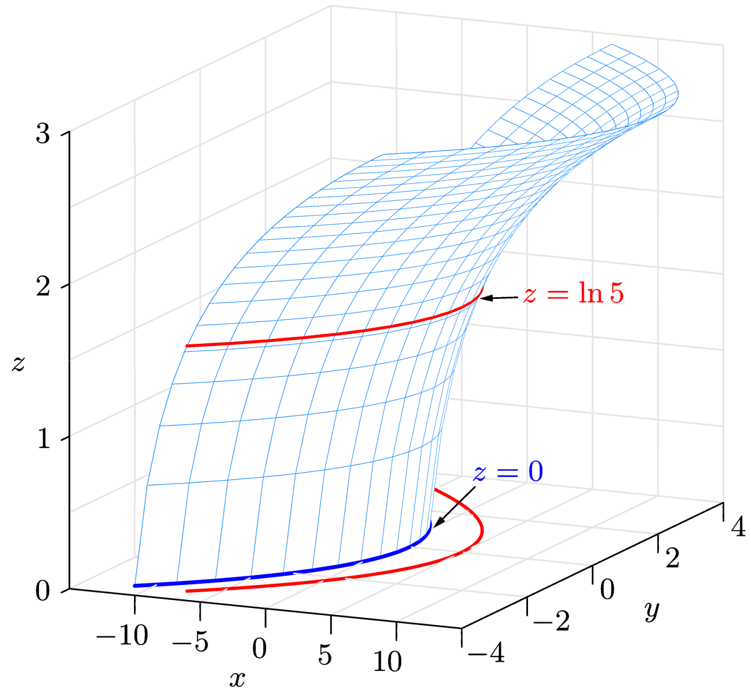
$y^2 = -x + e^0$

$y^2 = -x + 1$

$x = 1 - y^2$

y	-2	-1	0	1	2
x	-3	0	1	0	-3





d)

$f(x,y) = \ln\left(x+y^2\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{x+y^2}\cdot\left(x+y^2\right)'_x = \frac{1}{x+y^2}\cdot 1 = \frac{1}{x+y^2} = \left(x+y^2\right)^{-1}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x+y^2}\cdot\left(x+y^2\right)'_y = \frac{1}{x+y^2}\cdot 2y = \frac{2y}{x+y^2}$$

$(\ln x)' = \dfrac{1}{x}$

$(x^n)' = nx^{n-1}$

e)

$\dfrac{\partial^4 f}{\partial x^3 \partial y} \;\to\; f_{xxxy}$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = (f_x)_x = -\left(x+y^2\right)^{-2}\cdot\underbrace{\left(x+y^2\right)'_x}_{=1} = -\left(x+y^2\right)^{-2}$$

$$\frac{\partial^3 f}{\partial x^3} = f_{xxx} = (f_{xx})_x = -(-2)\left(x+y^2\right)^{-3}\cdot\underbrace{\left(x+y^2\right)'_x}_{=1} = 2\left(x+y^2\right)^{-3}$$

$$\frac{\partial^4 f}{\partial x^3 \partial y} = f_{xxxy} = (f_{xxx})_y = -6\left(x+y^2\right)^{-4}\cdot\underbrace{\left(x+y^2\right)'_y}_{=2y} = \frac{-12y}{\left(x+y^2\right)^4}$$

Zadatak 2

Zadana je ploha $z = x^3 + y^3$.

a) *Odredite na zadanoj plohi sve točke kojima je* x*-koordinata jednaka* 1 *i u kojima su tangencijalne ravnine plohe okomite na ravninu* $x + y + 51z = 0$.

b) *U svim tako pronađenim točkama napišite jednadžbe tangencijalnih ravnina i jednadžbe normala zadane plohe.*

Rješenje

a) $z = x^3 + y^3, \quad T(1, y, z)$

$$\Sigma \ldots x + y + 51z = 0$$

$$\Pi_t \perp \Sigma \quad \Leftrightarrow \quad \boxed{\vec{n}_t \cdot \vec{n}_\Sigma = 0}$$

$$\vec{n}_t = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

$$\vec{n}_t = \left(3x^2, 3y^2, -1\right)$$

$$\vec{n}_\Sigma = (1, 1, 51)$$

$$\left(3x^2, 3y^2, -1\right) \cdot (1, 1, 51) = 0$$

$$3x^2 + 3y^2 - 51 = 0$$

$$x = 1 \qquad 3y^2 = 48 \rightsquigarrow y^2 = 16$$

$$y_1 = 4, \quad y_2 = -4, \quad z_1 = 1^3 + 4^3 = 65, \quad z_2 = 1^3 + (-4)^3 = -63$$

$$\boxed{T_1(1, 4, 65)} \qquad \boxed{T_2(1, -4, -63)}$$

b) $T_1(\overset{x_0}{1}, \overset{y_0}{4}, \overset{z_0}{65}) \qquad \vec{n}_{t_1} = (\overset{A}{3}, \overset{B}{48}, \overset{C}{-1})$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

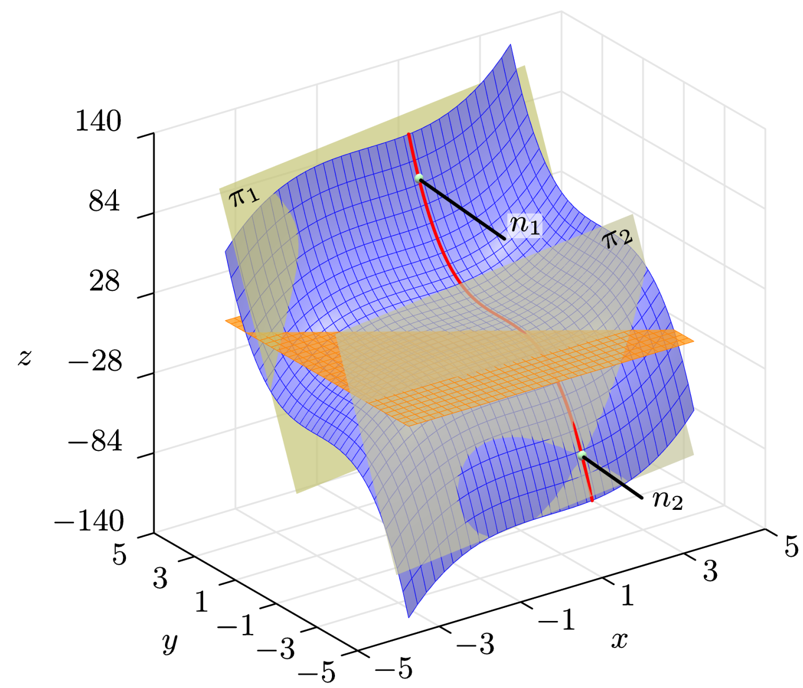
$$3(x - 1) + 48(y - 4) - 1 \cdot (z - 65) = 0$$

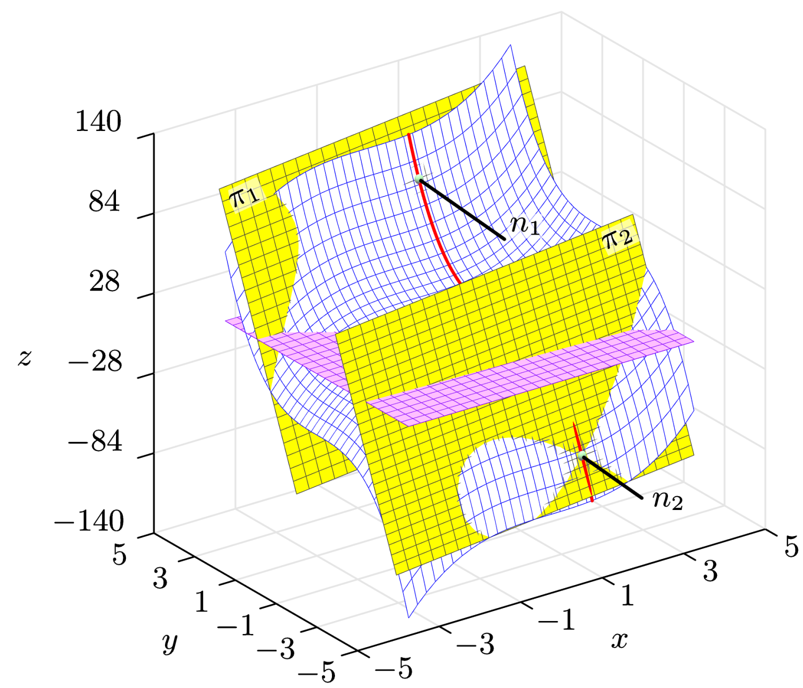
$$\boxed{\begin{array}{l} \Pi_1 \ldots 3x + 48y - z - 130 = 0 \\ n_1 \ldots \dfrac{x-1}{3} = \dfrac{y-4}{48} = \dfrac{z-65}{-1} \end{array}}$$

$T_2(\overset{x_0}{1}, \overset{y_0}{-4}, \overset{z_0}{-63}) \qquad \vec{n}_{t_2} = (\overset{A}{3}, \overset{B}{48}, \overset{C}{-1})$

$$3(x - 1) + 48(y + 4) - 1 \cdot (z + 63) = 0$$

$$\boxed{\begin{array}{l} \Pi_2 \ldots 3x + 48y - z + 126 = 0 \\ n_2 \ldots \dfrac{x-1}{3} = \dfrac{y+4}{48} = \dfrac{z+63}{-1} \end{array}}$$





Zadatak 3

Zadana je ploha $x^2z + y^2z = 9$ *i pravac*

$$p \ldots \frac{x-1}{1} = \frac{y+2}{1} = \frac{z+1}{1}.$$

Odredite jednadžbe tangencijalnih ravnina i normala na zadanu plohu u točkama u kojima zadani pravac siječe tu plohu.

Rješenje

presjek pravca i plohe

$$\frac{x-1}{1} = \frac{y+2}{1} = \frac{z+1}{1} = t$$

$$x^2z + y^2z = 9$$

$$\begin{cases} x = t+1 \\ y = t-2 \\ z = t-1 \end{cases}$$

$$(t+1)^2(t-1) + (t-2)^2(t-1) = 9$$

$$(t-1)\big((t+1)^2 + (t-2)^2\big) = 9$$

$$(t-1)\big(t^2 + 2t + 1 + t^2 - 4t + 4\big) = 9$$

$$(t-1)\big(2t^2 - 2t + 5\big) = 9$$

$$2t^3 - 2t^2 + 5t - 2t^2 + 2t - 5 - 9 = 0$$

$$2t^3 - 4t^2 + 7t - 14 = 0$$

$1, -1, 2, -2, 7, -7, 14, -14$

	2	−4	7	−14
2	2	0	7	0

$$(t-2)\big(2t^2 + 0 \cdot t + 7\big) = 0$$

$$(t-2)\big(2t^2 + 7\big) = 0$$

$t = 2$

$2t^2 + 7 = 0$

nema realnih rješenja

$S(3, 0, 1)$

$$x^2z + y^2z = 9$$

$$x^2z + y^2z - 9 = 0$$

$$F(x,y,z) = x^2z + y^2z - 9$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$x_0\ y_0\ z_0$

$S(3, 0, 1)$

$$F_x = 2xz, \quad F_y = 2yz, \quad F_z = x^2 + y^2$$

$$\vec{n}_t = (F_x, F_y, F_z), \quad \vec{n}_t = \big(2xz,\ 2yz,\ x^2 + y^2\big)$$

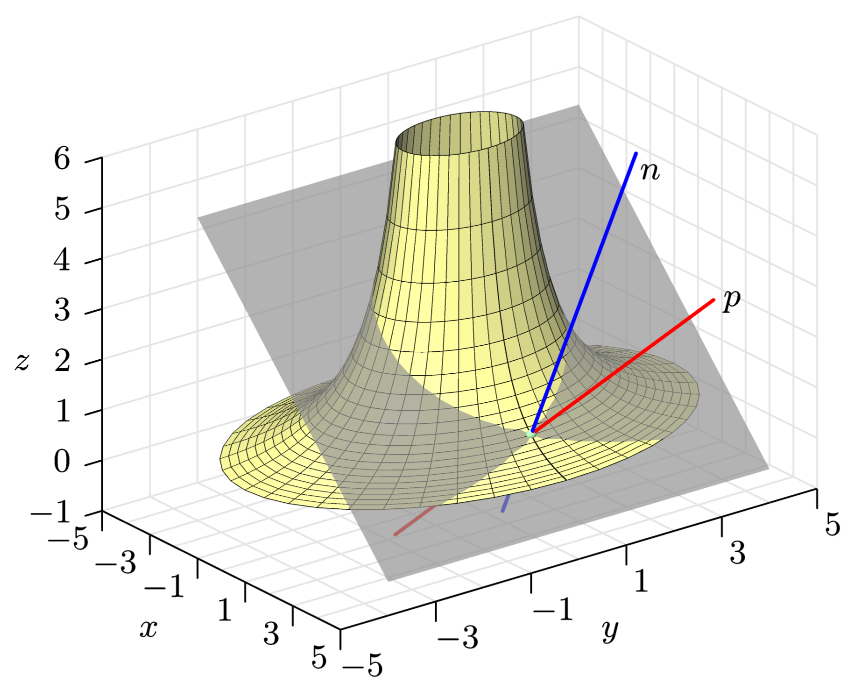
$A\ B\ C$

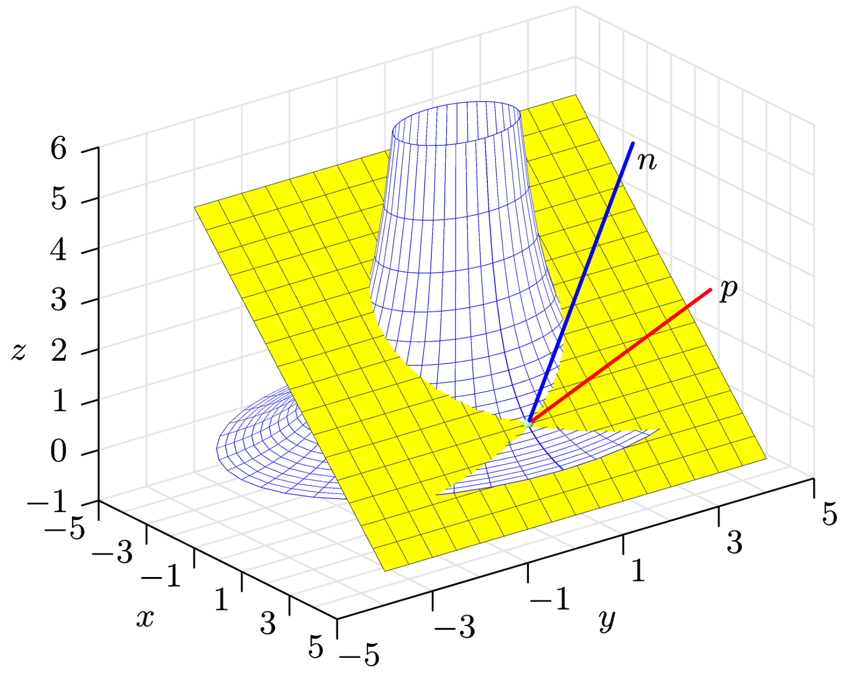
$$\vec{n}_t = (6, 0, 9) = 3 \cdot (2, 0, 3)$$

$$2 \cdot (x-3) + 0 \cdot (y-0) + 3 \cdot (z-1) = 0$$

$$\Pi_t \ldots 2x + 3z - 9 = 0$$

$$n \ldots \frac{x-3}{2} = \frac{y}{0} = \frac{z-1}{3}$$





Napomena

$z = f(x, y)$ ⟵ eksplicitni oblik jednadžbe plohe

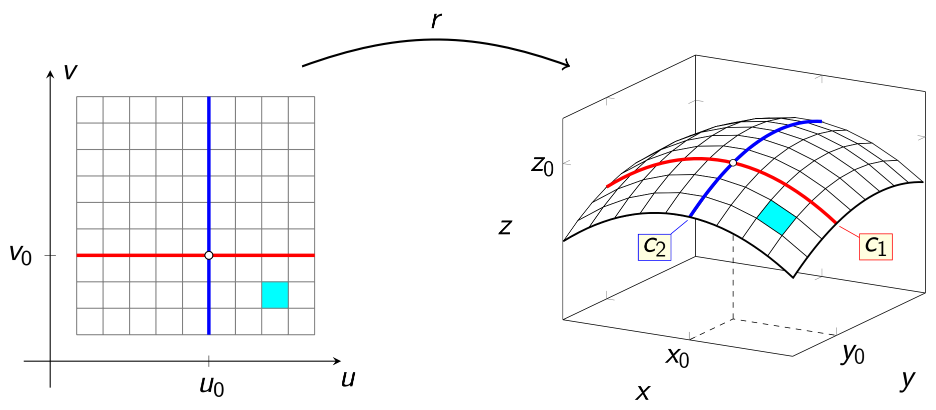
$f(x, y) - z = 0$ ⟵ implicitni oblik jednadžbe plohe

$F(x, y, z) = f(x, y) - z$

$F_x = f_x, \quad F_y = f_y, \quad F_z = -1$

$\vec{n}_t = \big(f_x, f_y, -1\big)$ ⟵ vektor normale tangencijalne ravnine

Parametrizacija plohe



$r(u, v) = \big(x(u, v),\, y(u, v),\, z(u, v)\big)$ $\qquad r(u_0, v_0) = (x_0, y_0, z_0)$

$c_1(u) = r(u, v_0)$ ⟵ parametarska *u*-crta

$c_2(v) = r(u_0, v)$ ⟵ parametarska *v*-crta

Zadatak 4

Zadana je ploha

$$r(u, v) = \big(\sin u,\, \sin v,\, \sin(u + v)\big)$$

i točka A na toj plohi s parametrima $u = \frac{\pi}{3}$, $v = \frac{\pi}{6}$.

a) *Odredite Kartezijeve koordinate točke A.*

b) *Odredite dva vektora koji razapinju tangencijalnu ravninu zadane plohe u točki A.*

c) *Nađite jednadžbu tangencijalne ravnine plohe u točki A.*

Rješenje $A \rightsquigarrow u = \frac{\pi}{3},\ v = \frac{\pi}{6}$

$$r(u,v) = \left(\sin u,\ \sin v,\ \sin(u+v)\right)$$

a)

$$r\left(\overset{u}{\frac{\pi}{3}}, \overset{v}{\frac{\pi}{6}}\right) = \left(\sin\frac{\pi}{3},\ \sin\frac{\pi}{6},\ \sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right)$$

$$r\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \left(\sin\frac{\pi}{3},\ \sin\frac{\pi}{6},\ \sin\frac{\pi}{2}\right)$$

$$r\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2},\ \frac{1}{2},\ 1\right) \qquad A\left(\overset{x_0}{\frac{\sqrt{3}}{2}},\ \overset{y_0}{\frac{1}{2}},\ \overset{z_0}{1}\right)$$

b)

$$r_u = \left(\cos u,\ 0,\ \cos(u+v)\right)$$

$$r_v = \left(0,\ \cos v,\ \cos(u+v)\right)$$

$$r_u\left(\overset{u}{\frac{\pi}{3}}, \overset{v}{\frac{\pi}{6}}\right) = \left(\frac{1}{2},\ 0,\ 0\right)$$

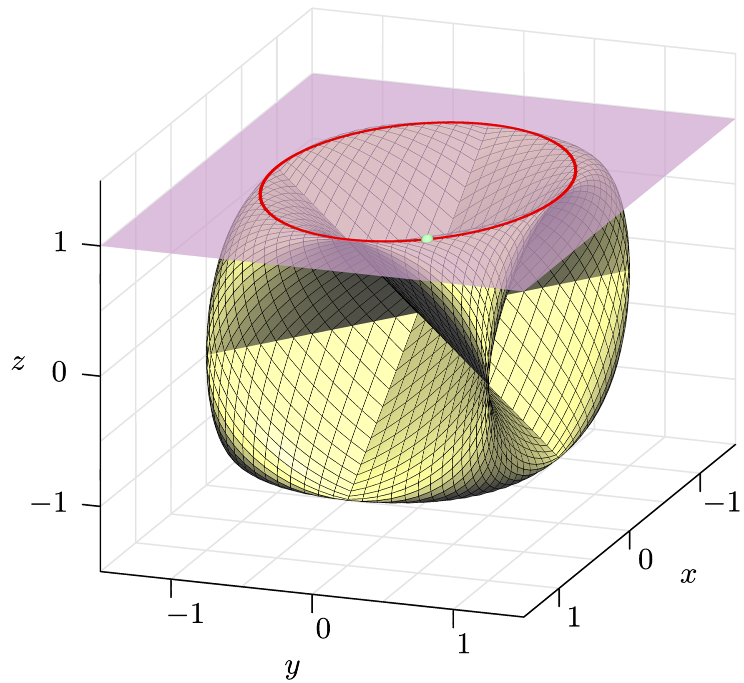
$$r_v\left(\overset{u}{\frac{\pi}{3}}, \overset{v}{\frac{\pi}{6}}\right) = \left(0,\ \frac{\sqrt{3}}{2},\ 0\right)$$

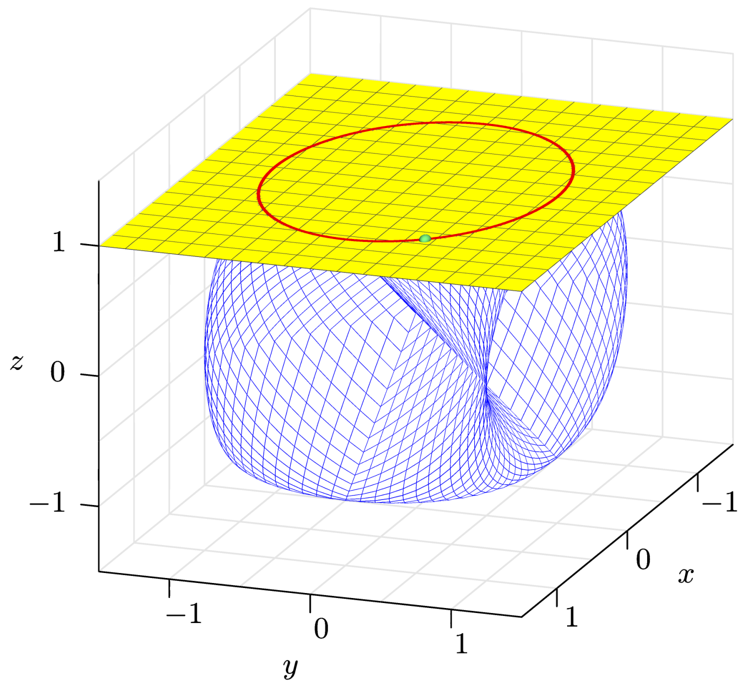
c)

$$r_u\left(\frac{\pi}{3}, \frac{\pi}{6}\right) \times r_v\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \end{vmatrix} = \left(0,\ 0,\ \frac{\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{4} \cdot (\overset{A}{0}, \overset{B}{0}, \overset{C}{1})$$

$$\Pi_t \ldots A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \qquad \boxed{\Pi_t \ldots z = 1}$$

$$0 \cdot \left(x - \frac{\sqrt{3}}{2}\right) + 0 \cdot \left(y - \frac{1}{2}\right) + 1 \cdot (z-1) = 0$$

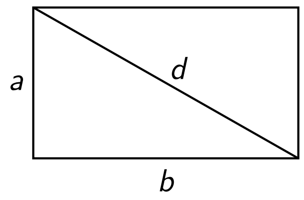




Zadatak 5

Susjedne stranice pravokutnika imaju duljine 10 cm i 24 cm. Kako će se promijeniti duljina dijagonale tog pravokutnika ako prvu stranicu produljimo za 4 mm, a drugu stranicu skratimo za 1 mm? Usporedite približnu promjenu dobivenu pomoću diferencijala sa stvarnom promjenom.

Rješenje



$$d = \sqrt{a^2 + b^2}$$

$$f(x,y) = \sqrt{x^2 + y^2}$$

$$x = 10\,\text{cm}, \quad y = 24\,\text{cm}$$

$$(\overset{x}{10}, \overset{y}{24})$$

$$\Delta x = 0.4\,\text{cm}, \quad \Delta y = -0.1\,\text{cm}$$

$$f(x,y)=\sqrt{x^2+y^2},\quad x=10,\ y=24,\quad \Delta x=0.4,\ \Delta y=-0.1$$

točna promjena dijagonale

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\Delta f = f(10.4, 23.9) - f(10, 24)$$

$$\Delta f = \sqrt{10.4^2+23.9^2} - \sqrt{10^2+24^2}$$

$$\Delta f = \sqrt{679.37} - \sqrt{676}$$

$$\boxed{\Delta f = 0.064727\cdots}$$

$$\mathrm{d}f = f_x\,\mathrm{d}x + f_y\,\mathrm{d}y$$

približna promjena dijagonale

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Delta f \approx \mathrm{d}f$$

$$\Delta f \approx \frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y$$

$$\boxed{\Delta f \approx 0.061538\cdots}$$

$$\Delta f \approx \frac{10}{\sqrt{10^2+24^2}}\cdot 0.4 + \frac{24}{\sqrt{10^2+24^2}}\cdot(-0.1)$$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$