Seminari 14

Matematičke metode za informatičare

Damir Horvat

FOI, Varaždin

Zadatak 1

Odredite lokalne ekstreme funkcije $f(x,y) = (x+2y)e^{-xy}$.

Rješenje

$$(e^x)' = e^x \qquad \left(e^{\text{nešto}}\right)' = e^{\text{nešto}} \cdot (\text{nešto})'$$

$$f_x = 1 \cdot e^{-xy} + (x+2y) \cdot e^{-xy} \cdot (-y)$$

$$\boxed{f_x = \left(1 - xy - 2y^2\right)e^{-xy}}$$

$$f_y = 2 \cdot e^{-xy} + (x+2y) \cdot e^{-xy} \cdot (-x)$$

$$\boxed{f_y = \left(2 - x^2 - 2xy\right)e^{-xy}}$$

$$\left(1 - xy - 2y^2\right)e^{-xy} = 0$$

$$\left(2 - x^2 - 2xy\right)e^{-xy} = 0$$

$$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$$\left(1 - xy - 2y^2\right)e^{-xy} = 0 \;/ : e^{-xy}$$

$$\left(2 - x^2 - 2xy\right)e^{-xy} = 0 \;/ : e^{-xy}$$

$$1 - xy - 2y^2 = 0$$

$$2 - x^2 - 2xy = 0$$

$$xy = 1 - 2y^2 \;/ : y, \; y \neq 0$$

$$\boxed{x = \frac{1}{y} - 2y}$$

$$2 - \left(\frac{1}{y} - 2y\right)^2 - 2\left(\frac{1}{y} - 2y\right)y = 0$$

$$2 - \frac{1}{y^2} + 4 - 4y^2 - 2 + 4y^2 = 0$$

$$-\frac{1}{y^2} = -4$$

$$y^2 = \frac{1}{4}$$

$$\boxed{y_1 = \frac{1}{2}} \qquad \boxed{y_2 = -\frac{1}{2}}$$

$$x_1 = \frac{1}{\frac{1}{2}} - 2 \cdot \frac{1}{2}$$

$$x_1 = 2 - 1 \qquad \boxed{x_1 = 1}$$

$$x_2 = \frac{1}{-\frac{1}{2}} - 2 \cdot \frac{-1}{2}$$

$$x_2 = -2 + 1 \qquad \boxed{x_2 = -1}$$

Stacionarne točke

$$\left(1, \frac{1}{2}\right) \qquad \left(-1, -\frac{1}{2}\right)$$

$$f_x = \left(1 - xy - 2y^2\right)e^{-xy} \qquad f_y = \left(2 - x^2 - 2xy\right)e^{-xy}$$

$$f_{xx} = -ye^{-xy} + \left(1 - xy - 2y^2\right)e^{-xy} \cdot (-y)$$

$$\boxed{f_{xx} = \left(2y^2 + xy - 2\right)ye^{-xy}}$$

$$f_{xy} = (-x - 4y)e^{-xy} + \left(1 - xy - 2y^2\right)e^{-xy} \cdot (-x)$$

$$\boxed{f_{xy} = \left(2xy^2 + x^2y - 2x - 4y\right)e^{-xy}}$$

$$f_{yy} = -2xe^{-xy} + \left(2 - x^2 - 2xy\right)e^{-xy} \cdot (-x)$$

$$\boxed{f_{yy} = \left(x^2 + 2xy - 4\right)xe^{-xy}}$$

$$(e^x)' = e^x \qquad \left(e^{\text{nešto}}\right)' = e^{\text{nešto}} \cdot (\text{nešto})' \qquad f(x,y) = (x+2y)e^{-xy}$$

$$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$f_{xx} = \left(2y^2 + xy - 2\right) ye^{-xy}$

$f_{xy} = \left(2xy^2 + x^2y - 2x - 4y\right) e^{-xy}$

$f_{yy} = \left(x^2 + 2xy - 4\right) xe^{-xy}$

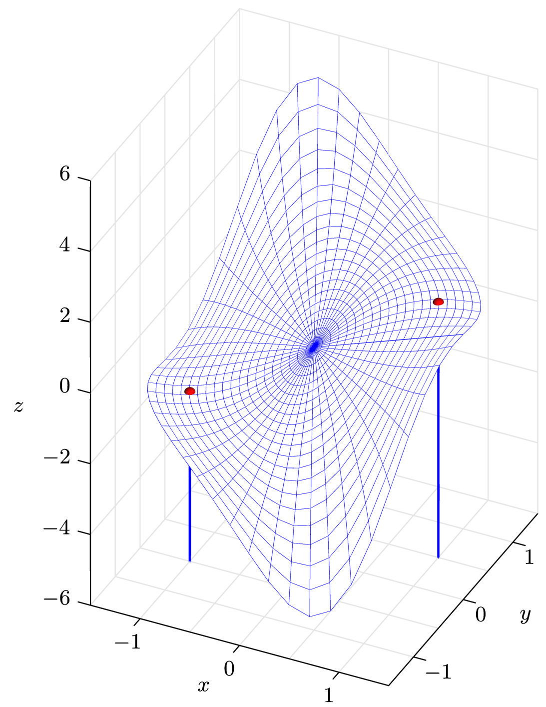
$$H(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{vmatrix}$$

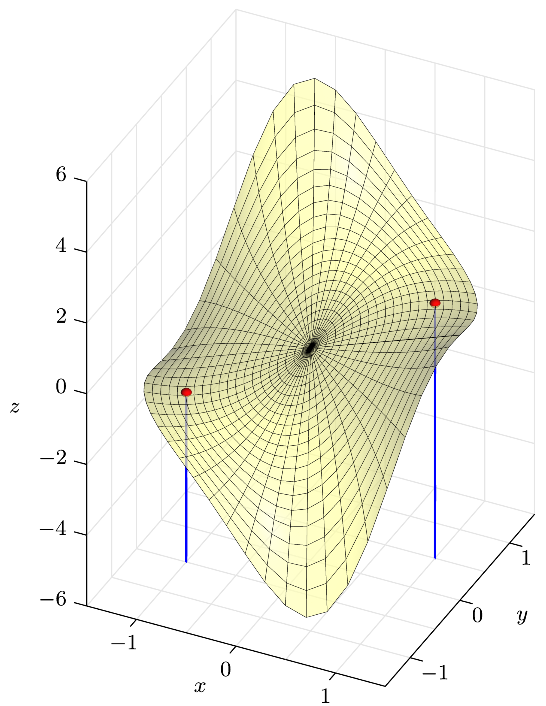
$$H\left(\overset{x}{1}, \overset{y}{\tfrac{1}{2}}\right) = \begin{vmatrix} -\frac{1}{2}e^{-\frac{1}{2}} & -3e^{-\frac{1}{2}} \\ -3e^{-\frac{1}{2}} & -2e^{-\frac{1}{2}} \end{vmatrix} = e^{-1} - 9e^{-1} = -8e^{-1} < 0$$

sedlasta točka

$$H\left(\overset{x}{-1}, \overset{y}{-\tfrac{1}{2}}\right) = \begin{vmatrix} \frac{1}{2}e^{-\frac{1}{2}} & 3e^{-\frac{1}{2}} \\ 3e^{-\frac{1}{2}} & 2e^{-\frac{1}{2}} \end{vmatrix} = e^{-1} - 9e^{-1} = -8e^{-1} < 0$$

sedlasta točka





Zadatak 2

Odredite lokalne ekstreme i sedlaste točke funkcije

$$f(x,y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2.$$

Rješenje

$f_x = 6x^2 + 9y^2 + 30x$ $\qquad$ $f_y = 18xy + 54y$

$$6x^2 + 9y^2 + 30x = 0 \;/:3$$
$$18xy + 54y = 0 \;/:18$$

$$2x^2 + 3y^2 + 10x = 0$$
$$xy + 3y = 0 \qquad y(x+3) = 0$$

$y = 0$ $\qquad$ $x = -3$

$2x^2 + 10x = 0 \;/:2$ $\qquad$ $18 + 3y^2 - 30 = 0$

$x(x+5) = 0$ $\qquad$ $3y^2 = 12 \;/:3 \qquad y^2 = 4$

$x_1 = 0, \quad x_2 = -5$ $\qquad$ $y_1 = 2, \quad y_2 = -2$

Stacionarne točke

$\overset{x_1\ y}{(0,0)}$ $\quad$ $\overset{x_2\ y}{(-5,0)}$

$\overset{x\ y_1}{(-3,2)}$ $\quad$ $\overset{x\ y_2}{(-3,-2)}$

$f_x = 6x^2 + 9y^2 + 30x$ $f_y = 18xy + 54y$

$f_{xx} = 12x + 30$

$f_{xy} = 18y$

$f_{yy} = 18x + 54$

$$H(x, y) = \begin{vmatrix} 12x + 30 & 18y \\ 18y & 18x + 54 \end{vmatrix}$$

$$H(-3, 2) = \begin{vmatrix} -6 & 36 \\ 36 & 0 \end{vmatrix} = -1296 < 0$$ → sedlasta točka

$$H(-3, -2) = \begin{vmatrix} -6 & -36 \\ -36 & 0 \end{vmatrix} = -1296 < 0$$ → sedlasta točka

Stacionarne točke

$(0, 0)$ $(-5, 0)$

$(-3, 2)$ $(-3, -2)$

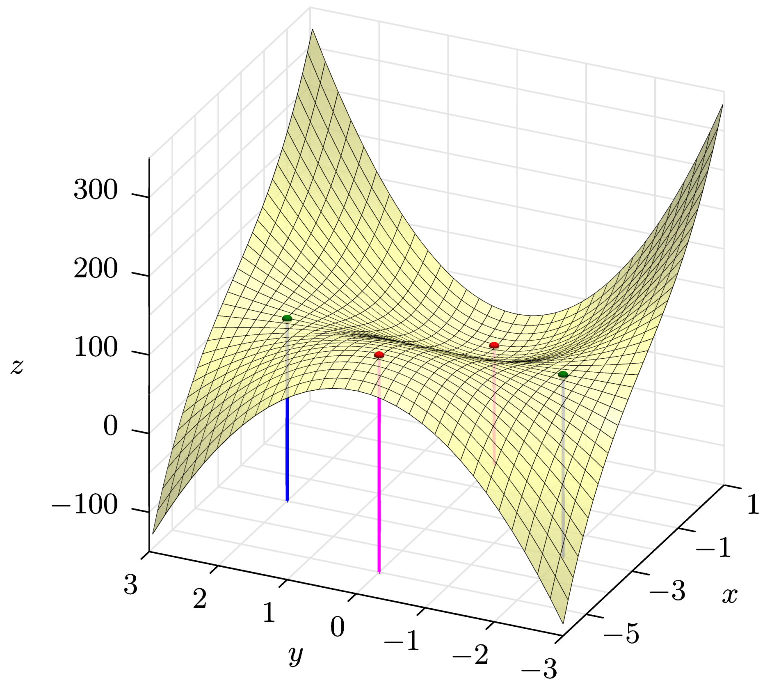
$$H(0, 0) = \begin{vmatrix} 30 & 0 \\ 0 & 54 \end{vmatrix} = 1620 > 0$$ ($30 > 0$) → lokalni minimum

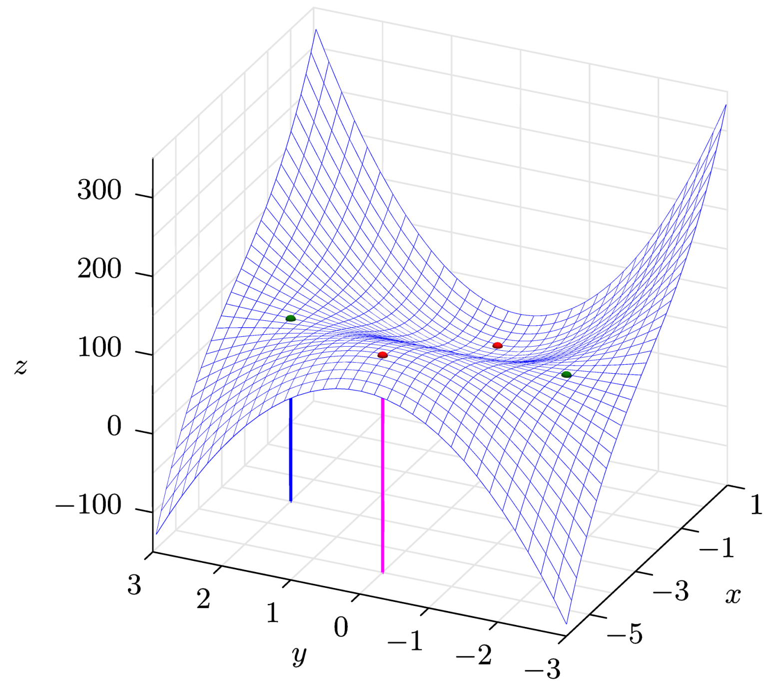
$f(0, 0) = 0$

$$H(-5, 0) = \begin{vmatrix} -30 & 0 \\ 0 & -36 \end{vmatrix} = 1080 > 0$$ ($-30 < 0$) → lokalni maksimum

$f(-5, 0) = 125$

$f(x, y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$





Zadatak 3

Odredite lokalne ekstreme funkcije $f(x, y) = 2x + 3y$ uz uvjet $xy = 2$.

(funkcija: $2x + 3y$)

Rješenje

$xy = 2$ → $xy - 2 = 0$ (uvjet: $xy - 2$)

- Lagrangeova funkcija

$$L(x, y, \lambda) = \text{funkcija} + \lambda \cdot \text{uvjet}$$

$$L(x, y, \lambda) = 2x + 3y + \lambda(xy - 2)$$

- Parcijalne derivacije Lagrangeove funkcije

$L_x = 2 + \lambda y$ $2 + \lambda y = 0$

$L_y = 3 + \lambda x$ $3 + \lambda x = 0$

$L_\lambda = xy - 2$ $xy - 2 = 0$

$$2+\lambda y = 0 \longrightarrow \lambda y = -2 \longrightarrow \lambda = -\frac{2}{y}$$
$$3+\lambda x = 0 \longrightarrow \lambda x = -3 \longrightarrow \lambda = -\frac{3}{x}$$
$$\Longrightarrow \quad -\frac{2}{y} = -\frac{3}{x}$$
$$3y = 2x$$
$$\boxed{y = \frac{2}{3}x}$$
$$xy - 2 = 0$$

$$x \cdot \frac{2}{3}x - 2 = 0$$
$$\frac{2}{3}x^2 = 2 \;\Big/ \cdot \frac{3}{2}$$
$$x^2 = 3$$
$$x_1 = \sqrt{3}, \quad x_2 = -\sqrt{3}$$
$$y_1 = \frac{2}{3}\sqrt{3}, \quad y_2 = -\frac{2}{3}\sqrt{3}$$
$$\lambda_1 = -\frac{3}{x_1} = -\frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\sqrt{3}$$
$$\lambda_2 = -\frac{3}{x_2} = -\frac{3}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

Stacionarne točke

$$\left(\overset{x_1}{\sqrt{3}},\ \overset{y_1}{\tfrac{2}{3}\sqrt{3}},\ \overset{\lambda_1}{-\sqrt{3}}\right)$$
$$\left(\overset{x_2}{-\sqrt{3}},\ \overset{y_2}{-\tfrac{2}{3}\sqrt{3}},\ \overset{\lambda_2}{\sqrt{3}}\right)$$

$$L(x,y,\lambda) = 2x + 3y + \lambda(xy-2) \qquad L_x = 2 + \lambda y \qquad L_y = 3 + \lambda x$$
$$f(x,y) = 2x + 3y \qquad g(x,y) = xy - 2$$
$$g_x = y, \quad g_y = x$$
$$L_{xx} = 0, \quad L_{xy} = \lambda, \quad L_{yy} = 0$$
$$\Delta(x,y,\lambda) = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{xy} & L_{yy} \end{vmatrix} = \begin{vmatrix} 0 & y & x \\ y & 0 & \lambda \\ x & \lambda & 0 \end{vmatrix}$$

$$\Delta\left(\overset{x}{\sqrt{3}},\ \overset{y}{\tfrac{2}{3}\sqrt{3}},\ \overset{\lambda}{-\sqrt{3}}\right) = \begin{vmatrix} 0 & \frac{2}{3}\sqrt{3} & \sqrt{3} \\ \frac{2}{3}\sqrt{3} & 0 & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 \end{vmatrix} = -4\sqrt{3} < 0$$

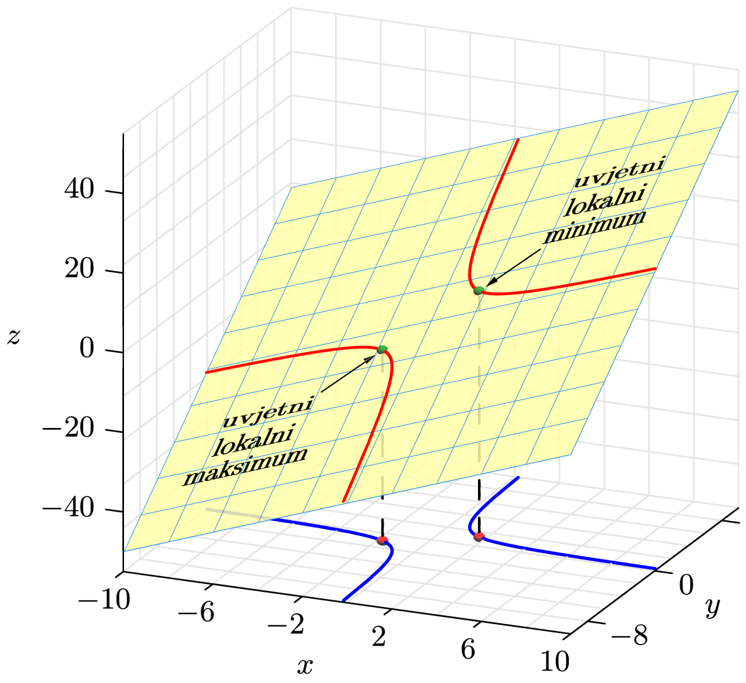
uvjetni lokalni minimum

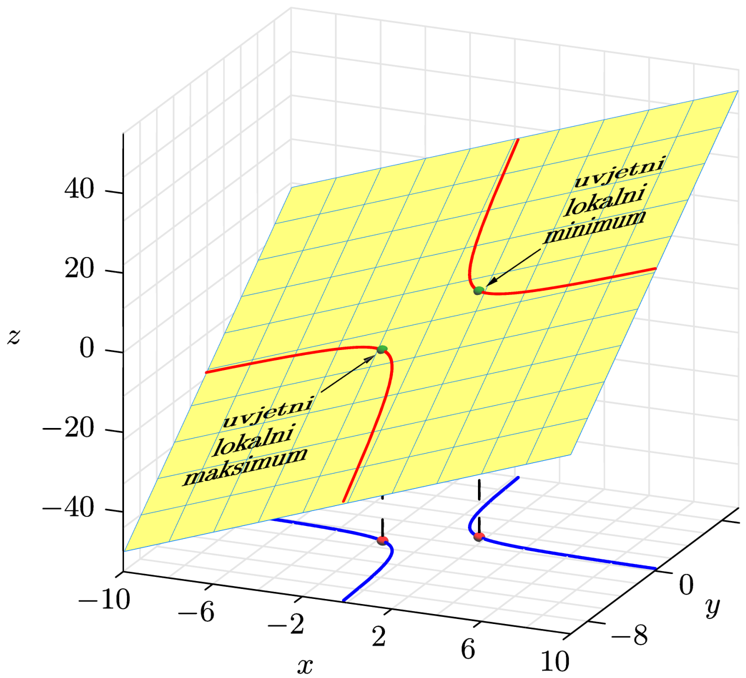
$$f\left(\sqrt{3}, \tfrac{2}{3}\sqrt{3}\right) = 4\sqrt{3}$$

$$\Delta\left(\overset{x}{-\sqrt{3}},\ \overset{y}{-\tfrac{2}{3}\sqrt{3}},\ \overset{\lambda}{\sqrt{3}}\right) = \begin{vmatrix} 0 & -\frac{2}{3}\sqrt{3} & -\sqrt{3} \\ -\frac{2}{3}\sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{3} & \sqrt{3} & 0 \end{vmatrix} = 4\sqrt{3} > 0$$

uvjetni lokalni maksimum

$$f\left(-\sqrt{3}, -\tfrac{2}{3}\sqrt{3}\right) = -4\sqrt{3}$$





2. način

$f(x,y) = 2x + 3y$

$xy = 2$

$y = \frac{2}{x}$

$f\left(x, \frac{2}{x}\right) = 2x + 3 \cdot \frac{2}{x} = 2x + 6x^{-1}$

$h(x) = 2x + 6x^{-1}$

$h'(x) = 2 - 6x^{-2}$

$2 - 6x^{-2} = 0 \; / \cdot x^2$

$2x^2 - 6 = 0$

$x^2 = 3$

$x_1 = \sqrt{3}, \quad x_2 = -\sqrt{3}$

$f\left(\sqrt{3}, \frac{2}{3}\sqrt{3}\right) = 4\sqrt{3}$ — uvjetni lokalni minimum

$f\left(-\sqrt{3}, -\frac{2}{3}\sqrt{3}\right) = -4\sqrt{3}$ — uvjetni lokalni maksimum

$h''(x) = 12x^{-3}$

$h''(\sqrt{3}) = 12\sqrt{3}^{-3} = \frac{4}{\sqrt{3}} > 0$

$h(\sqrt{3}) = 4\sqrt{3}$ lokalni minimum

$h''(-\sqrt{3}) = 12 \cdot (-\sqrt{3})^{-3} = \frac{-4}{\sqrt{3}} < 0$

$h(-\sqrt{3}) = -4\sqrt{3}$ lokalni maksimum

$y_1 = \frac{2}{x_1} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

$y_2 = \frac{2}{x_2} = \frac{2}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2}{3}\sqrt{3}$

Zadatak 4

Na elipsi $x^2 + 4y^2 = 4$ pronađite najbliže i najdalje točke od pravca $2x + 3y - 6 = 0$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Rješenje

$x^2 + 4y^2 = 4 \rightsquigarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$

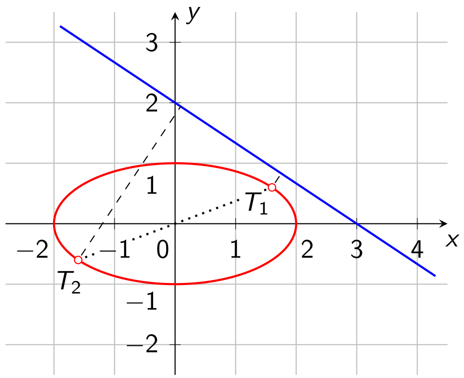
$2x + 3y - 6 = 0 \rightsquigarrow \frac{x}{3} + \frac{y}{2} = 1$

$d = \frac{|2x + 3y - 6|}{\sqrt{2^2 + 3^2}}$

$d = \frac{|2x + 3y - 6|}{\sqrt{13}}$

$f(x,y) = 2x + 3y - 6$

Tražimo ekstreme funkcije f uz uvjet $x^2 + 4y^2 = 4$.



Udaljenost točke od pravca

$T_0(x_0, y_0) \quad p \ldots Ax + By + C = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$f(x,y) = 2x + 3y - 6$ (funkcija) $\qquad x^2 + 4y^2 = 4 \longrightarrow x^2 + 4y^2 - 4 = 0$ (uvjet)

- Lagrangeova funkcija

$$L(x,y,\lambda) = \text{funkcija} + \lambda \cdot \text{uvjet}$$

$$L(x,y,\lambda) = 2x + 3y - 6 + \lambda(x^2 + 4y^2 - 4)$$

- Parcijalne derivacije Lagrangeove funkcije

$$L_x = 2 + 2\lambda x \qquad 2 + 2\lambda x = 0$$

$$L_y = 3 + 8\lambda y \qquad 3 + 8\lambda y = 0$$

$$L_\lambda = x^2 + 4y^2 - 4 \qquad x^2 + 4y^2 - 4 = 0$$

Tražimo ekstreme funkcije f uz uvjet $x^2 + 4y^2 = 4$.

$2 + 2\lambda x = 0 \longrightarrow \lambda x = -1 \rightarrow \lambda = -\frac{1}{x}$

$3 + 8\lambda y = 0 \longrightarrow 8\lambda y = -3 \rightarrow \lambda = -\frac{3}{8y}$

$x^2 + 4y^2 - 4 = 0$

$\Rightarrow -\frac{1}{x} = -\frac{3}{8y}$

$8y = 3x$

$y = \frac{3}{8}x$

$x^2 + 4 \cdot \left(\frac{3}{8}x\right)^2 - 4 = 0$

$x^2 + 4 \cdot \frac{9}{64}x^2 - 4 = 0$

$x^2 + \frac{9}{16}x^2 = 4$

$\frac{25}{16}x^2 = 4 \; / \cdot \frac{16}{25}$

$x^2 = \frac{64}{25}$

$x_1 = \frac{8}{5}, \quad x_2 = -\frac{8}{5}$

$y_1 = \frac{3}{8}x_1 = \frac{3}{8} \cdot \frac{8}{5} = \frac{3}{5}$

$y_2 = \frac{3}{8}x_2 = \frac{3}{8} \cdot \frac{-8}{5} = -\frac{3}{5}$

Stacionarne točke

$T_1\left(\frac{8}{5}, \frac{3}{5}\right) \qquad T_2\left(-\frac{8}{5}, -\frac{3}{5}\right)$

$f(x, y) = 2x + 3y - 6$ ← neprekidna funkcija

$x^2 + 4y^2 = 4$ → elipsa je kompaktni skup u ravnini

$$d = \frac{|2x + 3y - 6|}{\sqrt{13}}$$

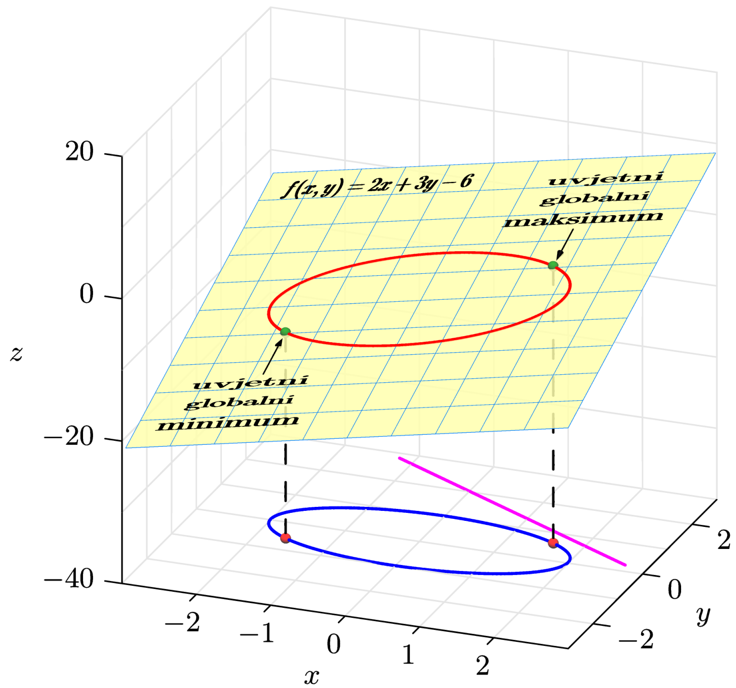
$$f\left(\frac{8}{5}, \frac{3}{5}\right) = 2 \cdot \frac{8}{5} + 3 \cdot \frac{3}{5} - 6 = -1 \rightsquigarrow d_1 = \frac{1}{\sqrt{13}}$$

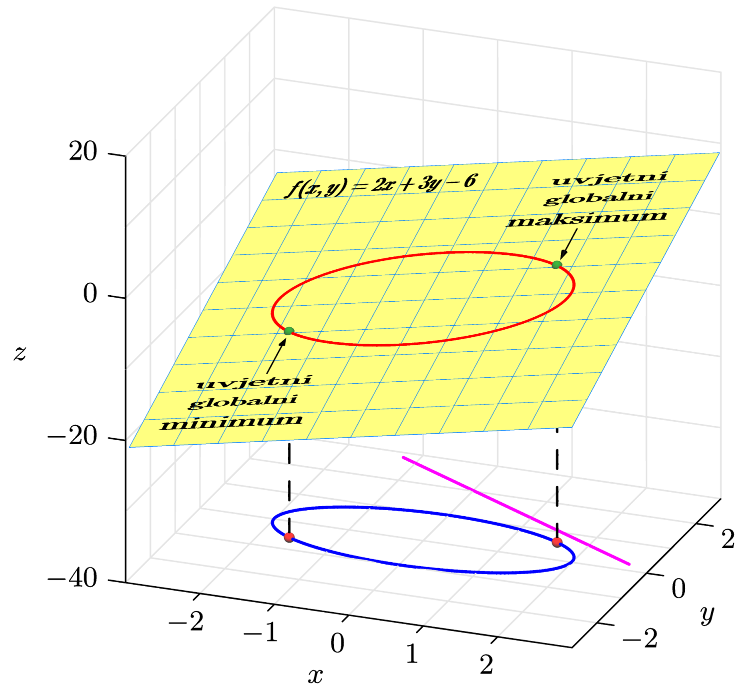
$$f\left(-\frac{8}{5}, -\frac{3}{5}\right) = 2 \cdot \frac{-8}{5} + 3 \cdot \frac{-3}{5} - 6 = -11 \rightsquigarrow d_2 = \frac{11}{\sqrt{13}}$$

Stacionarne točke

najbliža točka → $T_1\left(\frac{8}{5}, \frac{3}{5}\right)$ (x_1, y_1)

najudaljenija točka → $T_2\left(-\frac{8}{5}, -\frac{3}{5}\right)$ (x_2, y_2)

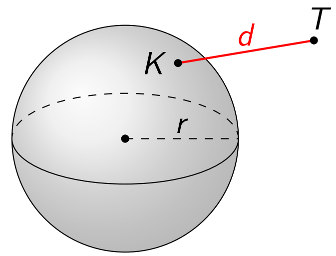




Zadatak 5

Pronađite na sferi $x^2 + y^2 + z^2 = 4$ točke koje su najbliže i najdalje od točke $T(3, 1, -1)$.

Rješenje



$T(3, 1, -1)$, $r = 2$, $K(x, y, z)$

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

$$d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$$

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

$x^2 + y^2 + z^2 = 4$ ⟵ točka K mora biti na sferi

Tražimo ekstreme funkcije f uz uvjet $x^2 + y^2 + z^2 = 4$.

$f(x,y,z) = \boxed{(x-3)^2+(y-1)^2+(z+1)^2} \longleftarrow$ funkcija

$x^2+y^2+z^2=4 \longrightarrow \boxed{x^2+y^2+z^2-4} = 0$

uvjet

- Lagrangeova funkcija

  $L(x,y,z,\lambda) = \text{funkcija} + \lambda\cdot\text{uvjet}$

  $L(x,y,z,\lambda) = (x-3)^2+(y-1)^2+(z+1)^2+\lambda(x^2+y^2+z^2-4)$

- Parcijalne derivacije Lagrangeove funkcije

$$L_x = 2(x-3)+2\lambda x \qquad 2(x-3)+2\lambda x = 0$$
$$L_y = 2(y-1)+2\lambda y \qquad 2(y-1)+2\lambda y = 0$$
$$L_z = 2(z+1)+2\lambda z \qquad 2(z+1)+2\lambda z = 0$$
$$L_\lambda = x^2+y^2+z^2-4 \qquad x^2+y^2+z^2-4 = 0$$

$$2(x-3)+2\lambda x = 0 \;/:2$$
$$x-3+\lambda x = 0$$
$$(\lambda+1)x = 3$$
$$\boxed{x = \frac{3}{\lambda+1}}$$

$$2(y-1)+2\lambda y = 0 \;/:2$$
$$y-1+\lambda y = 0$$
$$(\lambda+1)y = 1$$
$$\boxed{y = \frac{1}{\lambda+1}}$$

$$2(z+1)+2\lambda z = 0 \;/:2$$
$$z+1+\lambda z = 0$$
$$(\lambda+1)z = -1$$
$$\boxed{z = \frac{-1}{\lambda+1}}$$

$$x^2+y^2+z^2 = 4$$
$$\frac{9}{(\lambda+1)^2}+\frac{1}{(\lambda+1)^2}+\frac{1}{(\lambda+1)^2} = 4$$
$$\frac{11}{(\lambda+1)^2} = 4$$
$$(\lambda+1)^2 = \frac{11}{4}$$
$$\lambda+1 = \pm\frac{\sqrt{11}}{2}$$

$$\boxed{\lambda_1 = \frac{-2+\sqrt{11}}{2} \qquad \lambda_2 = \frac{-2-\sqrt{11}}{2}}$$

Stacionarne točke

$T_1\left(\overset{x_1}{\frac{6}{\sqrt{11}}}, \overset{y_1}{\frac{2}{\sqrt{11}}}, \overset{z_1}{-\frac{2}{\sqrt{11}}}\right) \qquad T_2\left(\overset{x_2}{-\frac{6}{\sqrt{11}}}, \overset{y_2}{-\frac{2}{\sqrt{11}}}, \overset{z_2}{\frac{2}{\sqrt{11}}}\right)$

$f(x,y,z) = (x-3)^2+(y-1)^2+(z+1)^2 \longleftarrow$ neprekidna funkcija

$d = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$

$x^2+y^2+z^2 = 4$

sfera je kompaktni skup u $\mathbb{R}^3$

$$f\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) = 15-4\sqrt{11} \rightsquigarrow d_1 = \sqrt{15-4\sqrt{11}} \approx 1.32$$
$$f\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right) = 15+4\sqrt{11} \rightsquigarrow d_2 = \sqrt{15+4\sqrt{11}} \approx 5.32$$

najbliža točka

najudaljenija točka

Stacionarne točke

$T_1\left(\overset{x_1}{\frac{6}{\sqrt{11}}}, \overset{y_1}{\frac{2}{\sqrt{11}}}, \overset{z_1}{-\frac{2}{\sqrt{11}}}\right) \qquad T_2\left(\overset{x_2}{-\frac{6}{\sqrt{11}}}, \overset{y_2}{-\frac{2}{\sqrt{11}}}, \overset{z_2}{\frac{2}{\sqrt{11}}}\right)$