1

Seminari 2

MATEMATIČKE METODE ZA INFORMATIČARE

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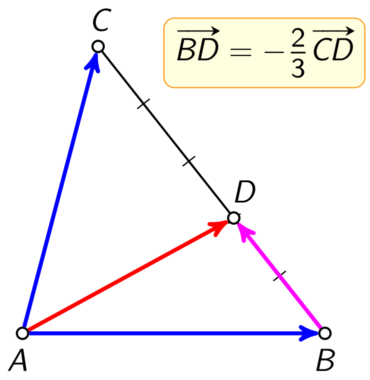
Zadatak 2

Na stranici $\overline{BC}$ trokuta ABC zadana je točka M takva da je $|BM| = \frac{2}{3}|BC|$ i točka N na stranici $\overline{AC}$ takva da je $|CA| = 4|CN|$. Neka je S presjek dužina $\overline{AM}$ i $\overline{BN}$. Nađite omjere u kojima točka S dijeli dužine $\overline{AM}$ i $\overline{BN}$.

Zadatak 1

Točka D leži na stranici $\overline{BC}$ trokuta ABC i dijeli tu stranicu u omjeru $2:3$. Prikažite vektor $\overrightarrow{AD}$ kao linearnu kombinaciju vektora $\overrightarrow{AB}$ i $\overrightarrow{AC}$.

Rješenje



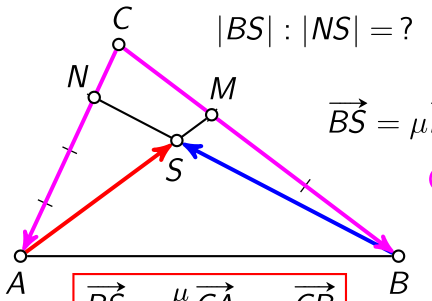
$|BD| : |CD| = 2 : 3$

$$\overrightarrow{BD} = -\frac{2}{3}\overrightarrow{CD}$$

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AB} + \frac{2}{5}\overrightarrow{BC} =$$
$$= \overrightarrow{AB} + \frac{2}{5}\left(\overrightarrow{BA} + \overrightarrow{AC}\right) =$$
$$= \overrightarrow{AB} + \frac{2}{5}\overrightarrow{BA} + \frac{2}{5}\overrightarrow{AC} =$$
$$= \overrightarrow{AB} - \frac{2}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AC} =$$
$$= \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AC}$$

$$\boxed{\overrightarrow{AD} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AC}}$$

Rješenje



$|AS| : |MS| = ?$

$|BS| : |NS| = ?$

$|BM| = \frac{2}{3}|BC| \rightsquigarrow \overrightarrow{CM} = \frac{1}{3}\overrightarrow{CB}$

$|CA| = 4|CN| \rightsquigarrow \overrightarrow{CN} = \frac{1}{4}\overrightarrow{CA}$

$\overrightarrow{BS} = \mu\overrightarrow{BN},\ \mu \in \mathbb{R}$ $\qquad$ $\overrightarrow{AS} = \lambda\overrightarrow{AM},\ \lambda \in \mathbb{R}$

Odabrana baza: $\mathcal{B} = \left(\overrightarrow{CA}, \overrightarrow{CB}\right)$

$$\boxed{\overrightarrow{BS} = \frac{\mu}{4}\overrightarrow{CA} - \mu\overrightarrow{CB}}$$

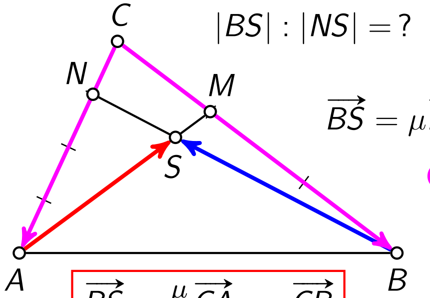
$$\boxed{\overrightarrow{BS} = (1-\lambda)\overrightarrow{CA} + \left(\frac{\lambda}{3} - 1\right)\overrightarrow{CB}}$$

$$\overrightarrow{BS} = \mu\overrightarrow{BN} = \mu\left(\overrightarrow{BC} + \overrightarrow{CN}\right) = \mu\left(-\overrightarrow{CB} + \frac{1}{4}\overrightarrow{CA}\right) = \frac{\mu}{4}\overrightarrow{CA} - \mu\overrightarrow{CB}$$

$$\overrightarrow{BS} = \overrightarrow{BA} + \overrightarrow{AS} = \left(\overrightarrow{BC} + \overrightarrow{CA}\right) + \lambda\overrightarrow{AM} =$$
$$= -\overrightarrow{CB} + \overrightarrow{CA} + \lambda\left(\overrightarrow{AC} + \overrightarrow{CM}\right) = -\overrightarrow{CB} + \overrightarrow{CA} - \lambda\overrightarrow{CA} + \lambda\overrightarrow{CM} =$$
$$= (1-\lambda)\overrightarrow{CA} - \overrightarrow{CB} + \lambda \cdot \frac{1}{3}\overrightarrow{CB} = (1-\lambda)\overrightarrow{CA} + \left(\frac{\lambda}{3} - 1\right)\overrightarrow{CB}$$

$|AS| : |MS| = ?$

$|BS| : |NS| = ?$



$|BM| = \frac{2}{3}|BC| \rightsquigarrow \overrightarrow{CM} = \frac{1}{3}\overrightarrow{CB}$

$|CA| = 4|CN| \rightsquigarrow \overrightarrow{CN} = \frac{1}{4}\overrightarrow{CA}$

$\overrightarrow{BS} = \mu\overrightarrow{BN},\ \mu \in \mathbb{R} \qquad \overrightarrow{AS} = \lambda\overrightarrow{AM},\ \lambda \in \mathbb{R}$

Odabrana baza: $\mathcal{B} = \left(\overrightarrow{CA}, \overrightarrow{CB}\right)$

$$\boxed{\overrightarrow{BS} = (1-\lambda)\overrightarrow{CA} + \left(\frac{\lambda}{3} - 1\right)\overrightarrow{CB}}$$

$$\boxed{\overrightarrow{BS} = \frac{\mu}{4}\overrightarrow{CA} - \mu\overrightarrow{CB}}$$

$$\left.\begin{array}{l} \frac{\mu}{4} = 1 - \lambda \\ -\mu = \frac{\lambda}{3} - 1 \end{array}\right\} \begin{array}{l} /\cdot 4 \\ /\cdot 3 \end{array} \rightsquigarrow \left.\begin{array}{l} 4\lambda + \mu = 4 \\ \lambda + 3\mu = 3 \end{array}\right\} \rightsquigarrow \begin{array}{l} \lambda = \frac{9}{11} \\ \mu = \frac{8}{11} \end{array}$$

$$\overrightarrow{AS} = \frac{9}{11}\overrightarrow{AM} \rightsquigarrow |AS| : |MS| = 9 : 2 \rightsquigarrow \overrightarrow{AS} = -\frac{9}{2}\overrightarrow{MS}$$

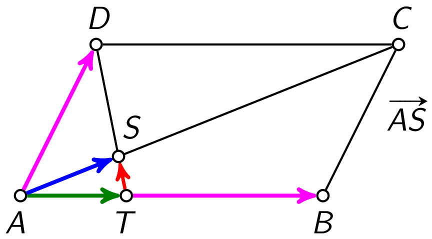
$$\overrightarrow{BS} = \frac{8}{11}\overrightarrow{BN} \rightsquigarrow |BS| : |NS| = 8 : 3 \rightsquigarrow \overrightarrow{BS} = -\frac{8}{3}\overrightarrow{NS}$$

Zadatak 3

Zadan je paralelogram $ABCD$ i točka T na stranici $\overline{AB}$ takva da je $|AT| = \frac{1}{n}|AB|$ za neki realni broj $n > 1$. Neka je S presjek dužina $\overline{AC}$ i $\overline{TD}$. Odredite omjer u kojemu točka S dijeli dužinu $\overline{AC}$.

Rješenje

$|AS| : |CS| = ?$



$|AT| = \frac{1}{n}|AB| \rightsquigarrow \overrightarrow{AT} = \frac{1}{n}\overrightarrow{AB}$

Odabrana baza: $\mathcal{B} = \left(\overrightarrow{AB}, \overrightarrow{AD}\right)$

$\overrightarrow{AS} = \lambda\overrightarrow{AC},\ \lambda \in \mathbb{R} \qquad \overrightarrow{TS} = \mu\overrightarrow{TD},\ \mu \in \mathbb{R}$

$$\boxed{\overrightarrow{AS} = \lambda\overrightarrow{AB} + \lambda\overrightarrow{AD}}$$

$$\boxed{\overrightarrow{AS} = \frac{1-\mu}{n}\overrightarrow{AB} + \mu\overrightarrow{AD}}$$

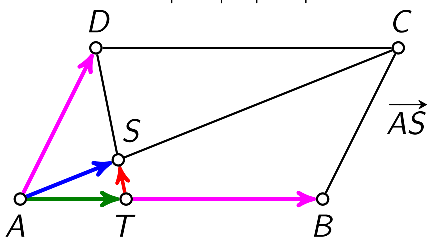
$$\overrightarrow{AS} = \lambda\overrightarrow{AC} = \lambda\left(\overrightarrow{AB} + \overrightarrow{AD}\right) = \lambda\overrightarrow{AB} + \lambda\overrightarrow{AD}$$

$$\overrightarrow{AS} = \overrightarrow{AT} + \overrightarrow{TS} = \frac{1}{n}\overrightarrow{AB} + \mu\overrightarrow{TD} = \frac{1}{n}\overrightarrow{AB} + \mu\left(\overrightarrow{TA} + \overrightarrow{AD}\right) =$$

$$= \frac{1}{n}\overrightarrow{AB} + \mu\overrightarrow{TA} + \mu\overrightarrow{AD} = \frac{1}{n}\overrightarrow{AB} + \mu \cdot \frac{-1}{n}\overrightarrow{AB} + \mu\overrightarrow{AD} =$$

$$= \frac{1-\mu}{n}\overrightarrow{AB} + \mu\overrightarrow{AD}$$

$|AS| : |CS| = ?$



$|AT| = \frac{1}{n}|AB| \rightsquigarrow \overrightarrow{AT} = \frac{1}{n}\overrightarrow{AB}$

Odabrana baza: $\mathcal{B} = \left(\overrightarrow{AB}, \overrightarrow{AD}\right)$

$\overrightarrow{AS} = \lambda\overrightarrow{AC},\ \lambda \in \mathbb{R} \qquad \overrightarrow{TS} = \mu\overrightarrow{TD},\ \mu \in \mathbb{R}$

$$\boxed{\overrightarrow{AS} = \lambda\overrightarrow{AB} + \lambda\overrightarrow{AD}}$$

$$\boxed{\overrightarrow{AS} = \frac{1-\mu}{n}\overrightarrow{AB} + \mu\overrightarrow{AD}}$$

$$\lambda = \frac{1-\mu}{n}$$

$$\lambda = \mu$$

$$\lambda = \frac{1-\lambda}{n} \Big/ \cdot n$$

$$n\lambda = 1 - \lambda$$

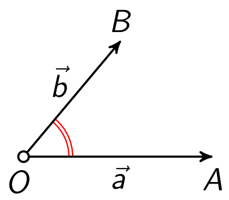
$$(n+1)\lambda = 1$$

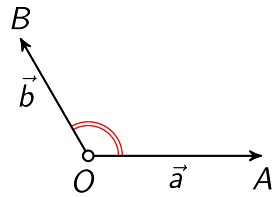
$$\boxed{\lambda = \frac{1}{n+1}}$$

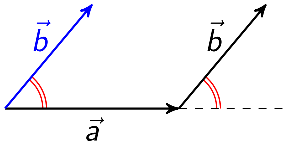
$$\overrightarrow{AS} = \frac{1}{n+1}\overrightarrow{AC} \rightsquigarrow |AS| : |CS| = 1 : n \rightsquigarrow \overrightarrow{AS} = -\frac{1}{n}\overrightarrow{CS}$$

Skalarni produkt vektora

- $\vec{a}\,\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\vec{a}, \vec{b}), \quad \vec{a}, \vec{b} \neq \vec{0}$
- $\cos(\vec{a}, \vec{b}) = \dfrac{\vec{a}\,\vec{b}}{|\vec{a}|\,|\vec{b}|}$
- $\vec{a}^{\,2} = |\vec{a}|^2$
- $(\vec{a} \pm \vec{b})^2 = \vec{a}^{\,2} \pm 2\vec{a}\,\vec{b} + \vec{b}^{\,2}$
- $\vec{a} \perp \vec{b} \;\Leftrightarrow\; \vec{a}\,\vec{b} = 0$







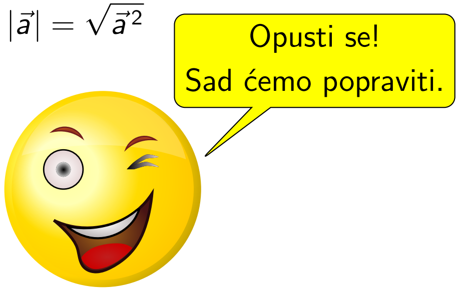
OPREZ

$|\vec{a}|^2 = \vec{a}^{\,2} \;/\sqrt{}$

$\sqrt{|\vec{a}|^2} = \sqrt{\vec{a}^{\,2}}$

~~$|\vec{a}| = \vec{a}$~~

$|\vec{a}| = \sqrt{\vec{a}^{\,2}}$



Zadatak 4

Zadani su okomiti vektori $\vec{a} = \vec{m} + 2\vec{n}$ i $\vec{b} = 5\vec{m} - 4\vec{n}$ pri čemu su $\vec{m}$ i $\vec{n}$ jedinični vektori.

a) *Izračunajte kut između vektora $\vec{m}$ i $\vec{n}$.*

b) *Odredite duljinu vektora $\vec{a} + \vec{b}$.*

Rješenje

$\vec{a} = \vec{m} + 2\vec{n}, \quad \vec{b} = 5\vec{m} - 4\vec{n}, \quad |\vec{m}| = 1, \quad |\vec{n}| = 1, \quad \vec{a} \perp \vec{b}$

a) $\sphericalangle(\vec{m}, \vec{n}) = ?$

$$\vec{a} \cdot \vec{b} = 0$$
$$(\vec{m} + 2\vec{n}) \cdot (5\vec{m} - 4\vec{n}) = 0$$
$$5\vec{m}^{\,2} - 4\vec{m}\vec{n} + 10\vec{m}\vec{n} - 8\vec{n}^{\,2} = 0$$
$$5|\vec{m}|^2 + 6\vec{m}\vec{n} - 8|\vec{n}|^2 = 0$$
$$5 \cdot 1^2 + 6\vec{m}\vec{n} - 8 \cdot 1^2 = 0$$
$$6\vec{m}\vec{n} - 3 = 0$$
$$\boxed{\vec{m}\vec{n} = \frac{1}{2}}$$

$$\cos(\vec{m}, \vec{n}) = \frac{\vec{m} \cdot \vec{n}}{|\vec{m}| \cdot |\vec{n}|}$$
$$\cos(\vec{m}, \vec{n}) = \frac{\frac{1}{2}}{1 \cdot 1}$$
$$\cos(\vec{m}, \vec{n}) = \frac{1}{2}$$
$$\boxed{\sphericalangle(\vec{m}, \vec{n}) = 60^\circ}$$

$\vec{a} = \vec{m} + 2\vec{n}, \quad \vec{b} = 5\vec{m} - 4\vec{n}, \quad |\vec{m}| = 1, \quad |\vec{n}| = 1, \quad \vec{a} \perp \vec{b}$

b) $\left|\vec{a} + \vec{b}\right| = ?$

$$\vec{a} + \vec{b} = (\vec{m} + 2\vec{n}) + (5\vec{m} - 4\vec{n}) = 6\vec{m} - 2\vec{n}$$

$$\left|\vec{a} + \vec{b}\right|^2 = \left(\vec{a} + \vec{b}\right)^2 = (6\vec{m} - 2\vec{n})^2 =$$

$$= 36\vec{m}^2 - 24\vec{m}\vec{n} + 4\vec{n}^2 =$$

$\boxed{\vec{m}\vec{n} = \frac{1}{2}}$

$$= 36|\vec{m}|^2 - 24\vec{m}\vec{n} + 4|\vec{n}|^2 =$$

$$= 36 \cdot 1^2 - 24 \cdot \frac{1}{2} + 4 \cdot 1^2 = 28$$

$$\left|\vec{a} + \vec{b}\right|^2 = 28 \rightsquigarrow \left|\vec{a} + \vec{b}\right| = \sqrt{28} \rightsquigarrow \boxed{\left|\vec{a} + \vec{b}\right| = 2\sqrt{7}}$$

Zadatak 5

Zadan je paralelogram ABCD s duljinama stranica $|AB| = 5$, $|AD| = 3$ i kutom $\sphericalangle DAB = 60^\circ$. Na dijagonali $\overline{AC}$ zadana je točka E takva da je $|AE| = \frac{4}{5}|AC|$.

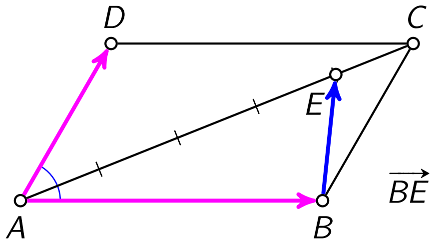
a) *Prikažite vektor $\overrightarrow{BE}$ u bazi $\mathcal{B} = \left(\overrightarrow{AB}, \overrightarrow{AD}\right)$. Koje su koordinate vektora $\overrightarrow{BE}$ u bazi $\mathcal{B}$?*

b) *Izračunajte skalarni produkt vektora $\overrightarrow{BE}$ i $\overrightarrow{BA}$.*

c) *Izračunajte duljinu vektora $\overrightarrow{BE}$.*

d) *Izračunajte kut $\sphericalangle ABE$.*

Rješenje



$|AB| = 5, \quad |AD| = 3, \quad \sphericalangle DAB = 60^\circ$

$|AE| = \frac{4}{5}|AC| \rightsquigarrow \overrightarrow{AE} = \frac{4}{5}\overrightarrow{AC}$

Zadana baza: $\mathcal{B} = \left(\overrightarrow{AB}, \overrightarrow{AD}\right)$

$\overrightarrow{BE} = -\frac{1}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD}$

a)

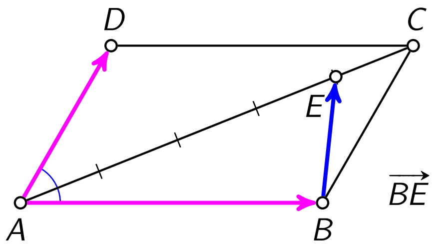
$$\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE} = -\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AC} = -\overrightarrow{AB} + \frac{4}{5}\left(\overrightarrow{AB} + \overrightarrow{AD}\right) =$$

$$= -\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD} = -\frac{1}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD}$$

$$\overrightarrow{BE} = -\frac{1}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD} \rightsquigarrow \overrightarrow{BE} = \left(-\frac{1}{5}, \frac{4}{5}\right)$$

koordinate vektora $\overrightarrow{BE}$ u bazi $\mathcal{B}$

$\overrightarrow{BE} \cdot \overrightarrow{BA} = -1$



$|AB| = 5, \quad |AD| = 3, \quad \sphericalangle DAB = 60^\circ$

$|AE| = \frac{4}{5}|AC| \rightsquigarrow \overrightarrow{AE} = \frac{4}{5}\overrightarrow{AC}$

Zadana baza: $\mathcal{B} = \left(\overrightarrow{AB}, \overrightarrow{AD}\right)$

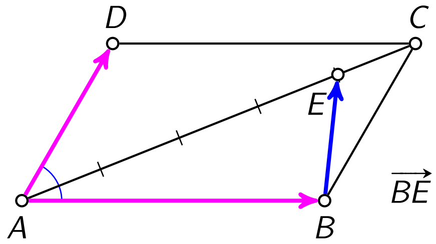
$\overrightarrow{BE} = -\frac{1}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD} \qquad \overrightarrow{AB} \cdot \overrightarrow{AD} = \frac{15}{2}$

b)

$$\overrightarrow{BE} \cdot \overrightarrow{BA} = \left(-\frac{1}{5}\overrightarrow{AB} + \frac{4}{5}\overrightarrow{AD}\right) \cdot \left(-\overrightarrow{AB}\right) = \frac{1}{5}\overrightarrow{AB}^2 - \frac{4}{5}\overrightarrow{AB} \cdot \overrightarrow{AD} =$$

$$= \frac{1}{5}\left|\overrightarrow{AB}\right|^2 - \frac{4}{5}\overrightarrow{AB} \cdot \overrightarrow{AD} = \frac{1}{5} \cdot 5^2 - \frac{4}{5} \cdot \frac{15}{2} = -1$$

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AD}\right| \cdot \cos\left(\overrightarrow{AB}, \overrightarrow{AD}\right) = 5 \cdot 3 \cdot \cos 60^\circ = \frac{15}{2}$$



$$\overrightarrow{BE}\cdot\overrightarrow{BA}=-1$$

$|AB|=5,\quad |AD|=3,\quad \sphericalangle DAB=60^\circ$

$|AE|=\frac{4}{5}|AC| \rightsquigarrow \overrightarrow{AE}=\frac{4}{5}\overrightarrow{AC}$

Zadana baza: $\mathcal{B}=\left(\overrightarrow{AB},\overrightarrow{AD}\right)$

$\overrightarrow{BE}=-\frac{1}{5}\overrightarrow{AB}+\frac{4}{5}\overrightarrow{AD} \qquad \overrightarrow{AB}\cdot\overrightarrow{AD}=\frac{15}{2}$

$\left|\overrightarrow{BE}\right|=\frac{\sqrt{109}}{5}$

c)

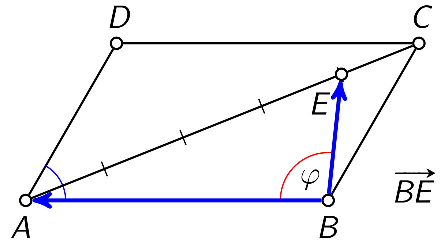
$$\left|\overrightarrow{BE}\right|^2=\overrightarrow{BE}^{\,2}=\left(-\frac{1}{5}\overrightarrow{AB}+\frac{4}{5}\overrightarrow{AD}\right)^2=$$

$$=\frac{1}{25}\overrightarrow{AB}^{\,2}-\frac{8}{25}\overrightarrow{AB}\cdot\overrightarrow{AD}+\frac{16}{25}\overrightarrow{AD}^{\,2}=$$

$$=\frac{1}{25}\left|\overrightarrow{AB}\right|^2-\frac{8}{25}\overrightarrow{AB}\cdot\overrightarrow{AD}+\frac{16}{25}\left|\overrightarrow{AD}\right|^2=$$

$$=\frac{1}{25}\cdot 5^2-\frac{8}{25}\cdot\frac{15}{2}+\frac{16}{25}\cdot 3^2=\frac{109}{25}$$

$$\left|\overrightarrow{BE}\right|^2=\frac{109}{25} \rightsquigarrow \left|\overrightarrow{BE}\right|=\frac{\sqrt{109}}{5}$$



$$\overrightarrow{BE}\cdot\overrightarrow{BA}=-1$$

$|AB|=5,\quad |AD|=3,\quad \sphericalangle DAB=60^\circ$

$|AE|=\frac{4}{5}|AC| \rightsquigarrow \overrightarrow{AE}=\frac{4}{5}\overrightarrow{AC}$

Zadana baza: $\mathcal{B}=\left(\overrightarrow{AB},\overrightarrow{AD}\right)$

$\overrightarrow{BE}=-\frac{1}{5}\overrightarrow{AB}+\frac{4}{5}\overrightarrow{AD} \qquad \overrightarrow{AB}\cdot\overrightarrow{AD}=\frac{15}{2}$

$\left|\overrightarrow{BE}\right|=\frac{\sqrt{109}}{5}$

d) $\sphericalangle ABE=?$

$$\varphi=\sphericalangle ABE=\sphericalangle\left(\overrightarrow{BA},\overrightarrow{BE}\right)$$

$$\cos\varphi=\frac{\overrightarrow{BA}\cdot\overrightarrow{BE}}{\left|\overrightarrow{BA}\right|\cdot\left|\overrightarrow{BE}\right|}=\frac{-1}{5\cdot\frac{\sqrt{109}}{5}}=-\frac{1}{\sqrt{109}}$$

$$\varphi=\arccos\left(-\frac{1}{\sqrt{109}}\right) \rightsquigarrow \boxed{\varphi=95^\circ\,29'\,47''}$$