# Seminari 4

## MATEMATIČKE METODE ZA INFORMATIČARE

---

Damir Horvat

FOI, Varaždin

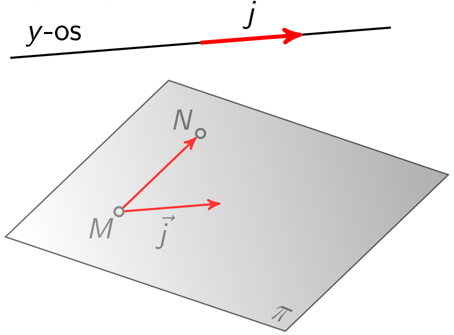
---

### Zadatak 1

*Odredite jednadžbu ravnine $\pi$ koja prolazi točkama $M(3,4,-1)$, $N(-2,-3,-2)$ i paralelna je s $y$-osi. Odredite točke u kojima ravnina $\pi$ siječe preostale koordinatne osi.*

---

### Rješenje



$M(3,4,-1)$, $N(-2,-3,-2)$

$\overrightarrow{MN} = (-5,-7,-1)$

$\vec{j} = (0,1,0)$

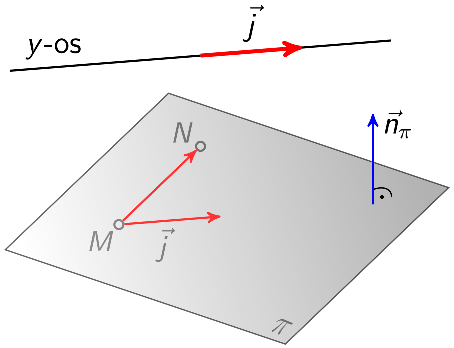
Parametarske jednadžbe

$\pi \ldots M,\ \overrightarrow{MN},\ \vec{j}$

$$\pi \ldots \begin{cases} x = 3 + (-5)\cdot u + 0\cdot v \\ y = 4 + (-7)\cdot u + 1\cdot v \\ z = -1 + (-1)\cdot u + 0\cdot v \end{cases}$$

$$\pi \ldots \begin{cases} x = 3 - 5u \\ y = 4 - 7u + v \\ z = -1 - u \end{cases} \qquad u, v \in \mathbb{R}$$

---



$M(3,4,-1)$, $N(-2,-3,-2)$

$\overrightarrow{MN} = (-5,-7,-1)$

$\vec{j} = (0,1,0)$

Opći oblik

$$M(\overset{x_0}{3}, \overset{y_0}{4}, \overset{z_0}{-1})$$

$\pi \ldots M,\ \vec{n}_\pi$

$$\vec{n}_\pi = \vec{j} \times \overrightarrow{MN} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -5 & -7 & -1 \end{vmatrix} = (\overset{A}{-1}, \overset{B}{0}, \overset{C}{5})$$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$-1\cdot(x-3) + 0\cdot(y-4) + 5\cdot(z-(-1)) = 0$$

Ova ravnina nema segmentni oblik

$$\boxed{\pi \ldots -x + 5z + 8 = 0}$$

**Normalni oblik**

$$x\cos\alpha + y\cos\beta + z\cos\gamma - \delta = 0$$

$$-x+5z+8=0 \Big/ \cdot \frac{-1}{\sqrt{26}}$$

$$\boxed{\frac{1}{\sqrt{26}}x - \frac{5}{\sqrt{26}}z - \frac{8}{\sqrt{26}} = 0}$$

$$\vec{n}_0 = -\frac{1}{\sqrt{26}}\vec{n}_\pi$$

$$\cos\alpha = \frac{1}{\sqrt{26}}$$

$$\cos\beta = 0$$

$$\cos\gamma = -\frac{5}{\sqrt{26}}$$

kosinusi smjera od $\vec{n}_0$ i od $-\vec{n}_\pi$

$$\boxed{\delta = \frac{8}{\sqrt{26}}}$$

udaljenost ravnine od ishodišta

**Opći oblik**

$$Ax+By+Cz+D=0$$

$$-x+5z+8=0$$

$$\vec{n}_\pi = (\overset{A}{-1}, \overset{B}{0}, \overset{C}{5}) \qquad D=8$$

$$\lambda = \frac{1}{-\operatorname{sign} D\cdot\sqrt{A^2+B^2+C^2}}$$

$$\lambda = \frac{1}{-\operatorname{sign} 8\cdot\sqrt{(-1)^2+0^2+5^2}}$$

$$\lambda = \frac{1}{-1\cdot\sqrt{26}} \qquad \boxed{\lambda = \frac{-1}{\sqrt{26}}}$$

---

**$\pi\ \cap$ $x$-os**

$\pi\ldots\ -x+5z+8=0$

$$x\text{-os}\ldots\begin{cases} x=t\\ y=0\\ z=0\end{cases}$$

$$-x+5z+8=0$$

$$-t+5\cdot 0+8=0$$

$$t=8$$

$$T_1(8,0,0)$$

**$\pi\ \cap$ $z$-os**

$\pi\ldots\ -x+5z+8=0$

$$z\text{-os}\ldots\begin{cases} x=0\\ y=0\\ z=t\end{cases}$$

$$-x+5z+8=0$$

$$0+5t+8=0$$

$$t=-\frac{8}{5}$$

$$T_2\left(0,0,-\frac{8}{5}\right)$$

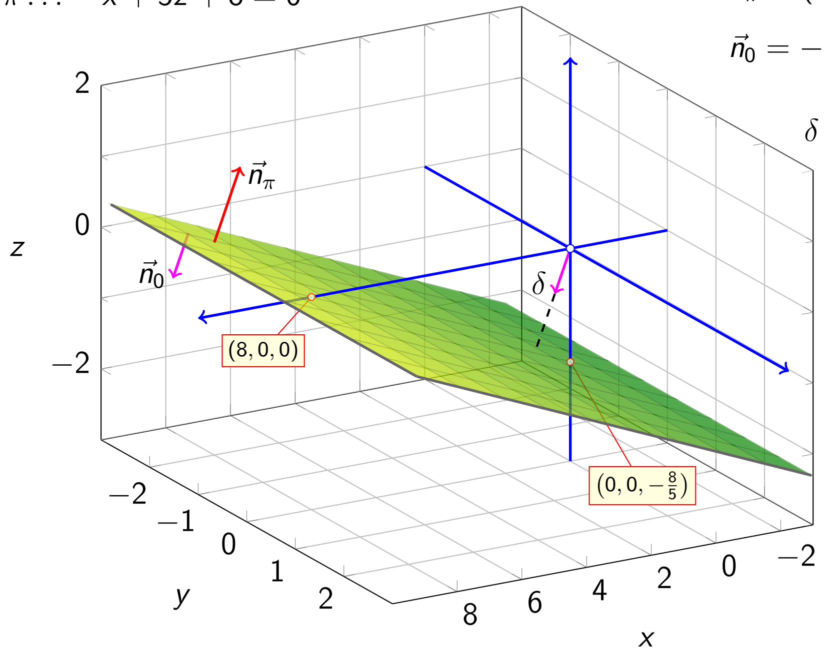
---

$\pi\ldots\ -x+5z+8=0$

$$\vec{n}_\pi = (-1,0,5)$$

$$\vec{n}_0 = -\frac{1}{\sqrt{26}}\vec{n}_\pi$$

$$\delta = \frac{8}{\sqrt{26}}$$



---

## Domaća zadaća

*Odredite vrijednosti parametara $u$ i $v$ za koje se dobivaju presjeci ravnine $\pi$ s koordinatnim osima u njezinim parametarskim jednadžbama*

$$\pi\ldots\begin{cases} x=3-5u\\ y=4-7u+v.\\ z=-1-u\end{cases}$$

$$T_1(8,0,0) \rightsquigarrow u=-1,\ \ v=-11$$

$$T_2\left(0,0,-\frac{8}{5}\right) \rightsquigarrow u=\frac{3}{5},\ \ v=\frac{1}{5}$$

Podrazumijevamo da se od ishodišta pomičemo u smjeru zadane normale poštujući njezinu orijentaciju jer u protivnom postoje dvije takve ravnine.

## Zadatak 2

*Nađite jednadžbu ravnine čija je normala* $\vec{n} = 8\vec{i} + 9\vec{j} + \vec{k}$*, a udaljenost od ishodišta iznosi* 1.

$$\pi' \dots \frac{8}{\sqrt{146}}x + \frac{9}{\sqrt{146}}y + \frac{1}{\sqrt{146}}z + 1 = 0$$

## Rješenje

1. način $\quad \vec{n} = (8, 9, 1), \ \delta = 1$

$$|\vec{n}| = \sqrt{8^2 + 9^2 + 1^2}$$

$$|\vec{n}| = \sqrt{146}$$

$$\pi \dots x \cos\alpha + y \cos\beta + z \cos\gamma - \delta = 0$$

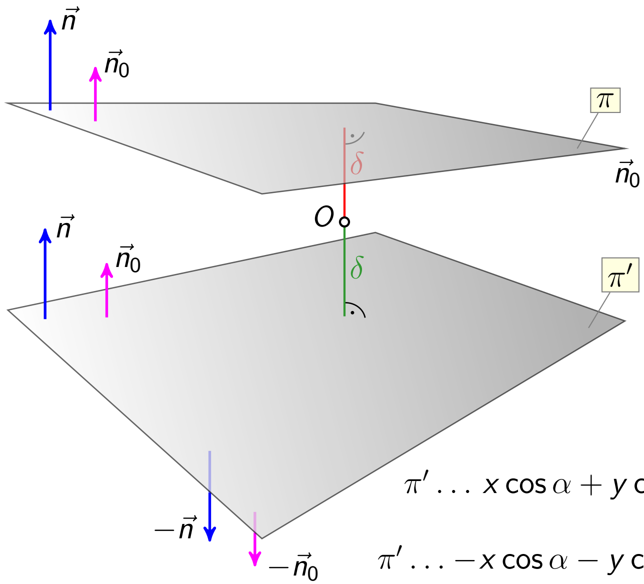
$$\boxed{\pi \dots \frac{8}{\sqrt{146}}x + \frac{9}{\sqrt{146}}y + \frac{1}{\sqrt{146}}z - 1 = 0}$$

$$\vec{n}_0 = \frac{\vec{n}}{|\vec{n}|} = \left(\frac{8}{\sqrt{146}}, \frac{9}{\sqrt{146}}, \frac{1}{\sqrt{146}}\right)$$

---

$$\pi \dots x \cos\alpha + y \cos\beta + z \cos\gamma - \delta = 0$$

normalni oblik



$$\vec{n}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$$

nije normalni oblik

$$\pi' \dots x \cos\alpha + y \cos\beta + z \cos\gamma + \delta = 0$$

$$\pi' \dots -x \cos\alpha - y \cos\beta - z \cos\gamma - \delta = 0$$

normalni oblik

---

2. način $\quad \vec{n} = (\overset{A}{8}, \overset{B}{9}, \overset{C}{1}), \ T_0(\overset{x_0}{0}, \overset{y_0}{0}, \overset{z_0}{0}), \ d(T_0, \pi) = 1$

$$\pi \dots Ax + By + Cz + D = 0$$

$$\boxed{\pi \dots 8x + 9y + z - \sqrt{146} = 0}$$

negativni predznak poštuje orijentaciju normale $\vec{n}$

$$d(T_0, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$1 = \frac{|8 \cdot 0 + 9 \cdot 0 + 1 \cdot 0 + D|}{\sqrt{8^2 + 9^2 + 1^2}}$$

$$1 = \frac{|D|}{\sqrt{146}} \rightsquigarrow |D| = \sqrt{146} \rightsquigarrow D = \pm\sqrt{146}$$

pozitivni predznak ne poštuje orijentaciju normale $\vec{n}$

$$\pi' \dots 8x + 9y + z + \sqrt{146} = 0$$

---

## Zadatak 3

*Ispitajte jesu li točke* $A(1, 2, 3)$ *i* $B(4, 0, -5)$ *s iste strane ravnine*

$$\Sigma \dots 2x - y + 5z - 1 = 0.$$

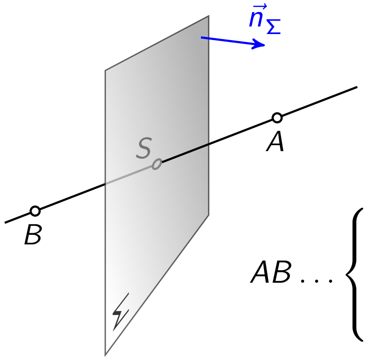
*Odredite točku* $S$ *u kojoj pravac* $AB$ *siječe ravninu* $\Sigma$*. Pripada li točka* $S$ *dužini* $\overline{AB}$*? Obrazložite svoj odgovor.*

## Rješenje

$$A(\overset{x}{1}, \overset{y}{2}, \overset{z}{3}) \rightsquigarrow 2x - y + 5z - 1 = 2 \cdot 1 - 2 + 5 \cdot 3 - 1 = 14 > 0$$

$$B(\overset{x}{4}, \overset{y}{0}, \overset{z}{-5}) \rightsquigarrow 2x - y + 5z - 1 = 2 \cdot 4 - 0 + 5 \cdot (-5) - 1 = -18 < 0$$

Točke $A$ i $B$ se nalaze s različitih strana ravnine $\Sigma$. Točka $A$ se nalazi na onoj strani na koju pokazuje normala $\vec{n}_\Sigma = (2, -1, 5)$.



$A(1,2,3),\ B(4,0,-5)$

$AB \ldots A, \overrightarrow{AB} \qquad \overrightarrow{AB}=(3,-2,-8)$

$\Sigma \ldots 2x-y+5z-1=0$

$$AB \ldots \begin{cases} x = 1 + 3 \cdot t \\ y = 2 + (-2) \cdot t \\ z = 3 + (-8) \cdot t \end{cases} \qquad AB \ldots \begin{cases} x = 1 + 3t \\ y = 2 - 2t \\ z = 3 - 8t \end{cases}$$

$$2x-y+5z-1=0$$
$$2 \cdot (1+3t) - (2-2t) + 5 \cdot (3-8t) - 1 = 0$$
$$2+6t-2+2t+15-40t-1=0$$
$$-32t+14=0$$
$$\boxed{t=\frac{7}{16}}$$

$$\boxed{S\left(\frac{37}{16}, \frac{9}{8}, -\frac{1}{2}\right)}$$

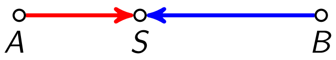
Točka $S$ pripada dužini $\overline{AB}$ jer se točke $A$ i $B$ nalaze s različitih strana ravnine $\Sigma$.

---

**2. način** *pomoću djelišnog omjera*

$A(1,2,3),\ B(4,0,-5)$

$S\left(\frac{37}{16}, \frac{9}{8}, -\frac{1}{2}\right)$



$\overrightarrow{AS} = \left(\frac{21}{16}, -\frac{7}{8}, -\frac{7}{2}\right) \qquad \overrightarrow{BS} = \left(-\frac{27}{16}, \frac{9}{8}, \frac{9}{2}\right)$

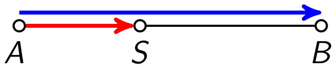
$\overrightarrow{AS} = \lambda \overrightarrow{BS}$

$$\frac{\frac{21}{16}}{-\frac{27}{16}} = \frac{-\frac{7}{8}}{\frac{9}{8}} = \frac{-\frac{7}{2}}{\frac{9}{2}} \rightsquigarrow -\frac{7}{9} = -\frac{7}{9} = -\frac{7}{9} \rightsquigarrow$$ točke $A$, $B$ i $S$ su kolinearne

Kako je $\lambda < 0$, točka $S$ pripada dužini $\overline{AB}$.

$$\boxed{\lambda = -\frac{7}{9}}$$

**3. način** *još jedna ideja*



$\overrightarrow{AS} = \mu \overrightarrow{AB}$

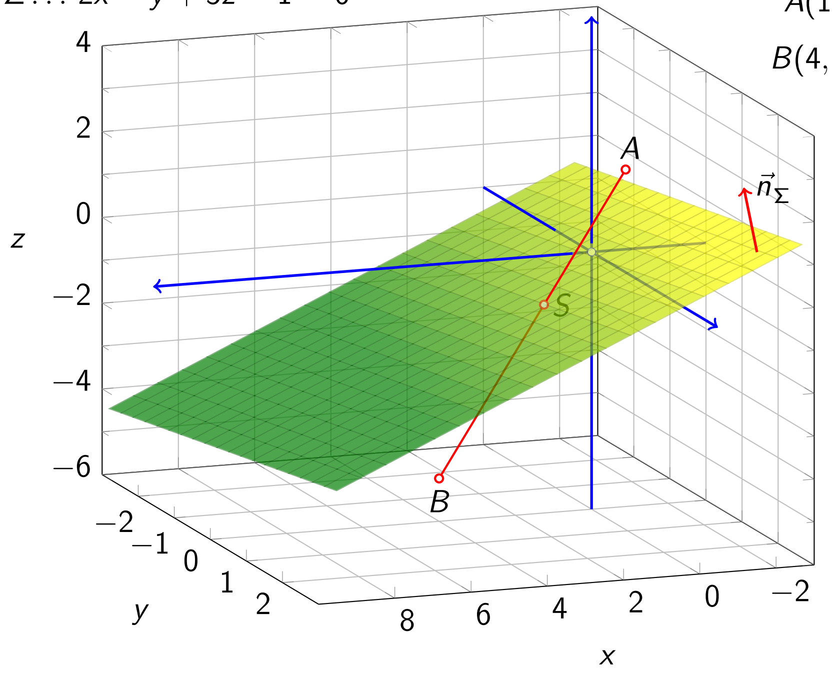
$S \in \overline{AB} \ \Leftrightarrow\ \mu \in [0,1]$

---

$\Sigma \ldots 2x-y+5z-1=0$

$A(1,2,3)$

$B(4,0,-5)$



---

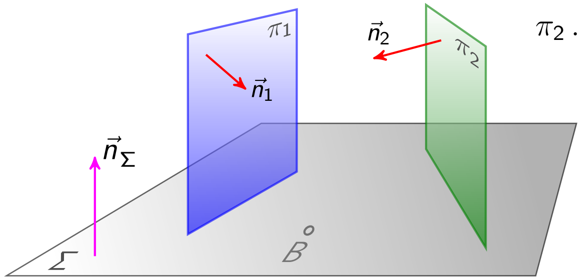
## Zadatak 4

*Napišite jednadžbu ravnine koja prolazi točkom* $B(-1,2,-4)$, *a okomita je na ravnine* $x+3y-2z+5=0$ *i* $-4x+5y-z+3=0$.

## Rješenje

$\pi_1 \ldots x+3y-2z+5=0$

$\pi_2 \ldots -4x+5y-z+3=0$



$B(\overset{x_0}{-1},\overset{y_0}{2},\overset{z_0}{-4})$

$\vec n_1=(1,3,-2)$

$\vec n_2=(-4,5,-1)$

$$\left.\begin{aligned}\Sigma\perp\pi_1 &\Rightarrow \vec n_\Sigma\perp\vec n_1\\ \Sigma\perp\pi_2 &\Rightarrow \vec n_\Sigma\perp\vec n_2\end{aligned}\right\}\Rightarrow \vec n_\Sigma=\vec n_1\times\vec n_2=\begin{vmatrix}\vec i&\vec j&\vec k\\ 1&3&-2\\ -4&5&-1\end{vmatrix}=(\overset{A}{7},\overset{B}{9},\overset{C}{17})$$

$$\Sigma\ldots A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$7\cdot(x-(-1))+9\cdot(y-2)+17\cdot(z-(-4))=0$$

$$\boxed{\Sigma\ldots 7x+9y+17z+57=0}$$

## Zadatak 5

*Zadani su pravac $p$ i ravnina $\Sigma$ svojim vektorskim jednadžbama*

$$p\ldots \vec r=(2,1,-1)+t\cdot(1,-1,0),$$

$$\Sigma\ldots \vec r=(2,1,3)+u\cdot(1,0,0)+v\cdot(-1,1,2).$$

a) *Napišite parametarske jednadžbe i opći oblik jednadžbe ravnine $\Sigma$.*

b) *Odredite pravac $q$ koji prolazi točkom $T(1,0,4)$ i siječe zadani pravac $p$ te je paralelan s ravninom $\Sigma$.*

## Rješenje

a) $$\Sigma\ldots \overset{(x,y,z)}{\vec r}=\underbrace{(2,1,3)}_{T_0(\overset{x_0}{2},\overset{y_0}{1},\overset{z_0}{3})}+u\cdot\underbrace{(1,0,0)}_{\vec a=(1,0,0)}+v\cdot\underbrace{(-1,1,2)}_{\vec b=(-1,1,2)}$$

$$\vec n_\Sigma=\vec a\times\vec b=\begin{vmatrix}\vec i&\vec j&\vec k\\ 1&0&0\\ -1&1&2\end{vmatrix}=(\overset{A}{0},\overset{B}{-2},\overset{C}{1})$$

Parametarske jednadžbe

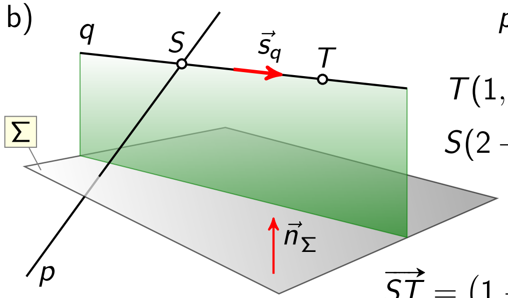
$$\Sigma\ldots\begin{cases}x=2+u-v\\ y=1+v\\ z=3+2v\end{cases}$$

$$\Sigma\ldots A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$0\cdot(x-2)+(-2)\cdot(y-1)+1\cdot(z-3)=0$$

$$\boxed{\Sigma\ldots -2y+z-1=0}\longleftarrow \text{opći oblik}$$

b)



$p\ldots \overset{(x,y,z)}{\vec r}=(2,1,-1)+t\cdot(1,-1,0)$

$T(1,0,4)$

$S(2+t,1-t,-1)$

$$p\ldots\begin{cases}x=2+t\\ y=1-t\\ z=-1\end{cases}$$

$\vec n_\Sigma=(0,-2,1)$

$\Sigma\ldots -2y+z-1=0$

$\overrightarrow{ST}=(1-(2+t),\ 0-(1-t),\ 4-(-1))$

$\overrightarrow{ST}=(-1-t,t-1,5)$

$$q\parallel\Sigma\ \Rightarrow\ \overrightarrow{ST}\perp\vec n_\Sigma\ \Rightarrow\ \boxed{\overrightarrow{ST}\cdot\vec n_\Sigma=0}$$

$$(-1-t,t-1,5)\cdot(0,-2,1)=0$$

$$(-1-t)\cdot 0+(t-1)\cdot(-2)+5\cdot 1=0$$

$$-2t+7=0\rightsquigarrow\boxed{t=\frac{7}{2}}$$

$S\left(\frac{11}{2},-\frac{5}{2},-1\right)$

$\overrightarrow{ST}=\left(-\frac{9}{2},\frac{5}{2},5\right)$

$\vec s_q=2\cdot\overrightarrow{ST}=(-9,5,10)$

$$\boxed{q\ldots\frac{x-1}{-9}=\frac{y}{5}=\frac{z-4}{10}}$$

$q\ldots T,\vec s_q$