Seminari 5

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

Zadan je pravac $p \ldots \dfrac{x-2}{1} = \dfrac{y+4}{-2} = \dfrac{z+1}{1}$ *i točka* $A(3,4,2)$.

a) *Odredite jednadžbu normale n iz točke A na pravac p.*

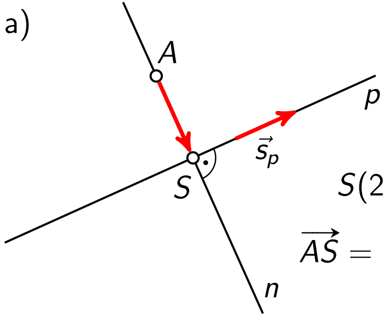
b) *Odredite simetričnu točku točke A s obzirom na pravac p.*

c) *Odredite sve točke na pravcu p koje su od točke A udaljene* $10\sqrt{2}$.

Rješenje

$A(3,4,2) \qquad \vec{s}_p = (1,-2,1) \qquad p \ldots \frac{x-2}{1} = \frac{y+4}{-2} = \frac{z+1}{1}$

a)



$\overrightarrow{AS} = (-3,-4,-5)$

$S(0,0,-3)$

$$p \ldots \begin{cases} x = 2+t \\ y = -4-2t \\ z = -1+t \end{cases}$$

$S(2+t, -4-2t, -1+t)$

$\overrightarrow{AS} = ((2+t)-3, (-4-2t)-4, (-1+t)-2)$

$\overrightarrow{AS} = (t-1, -8-2t, t-3)$

$$\overrightarrow{AS} \perp \vec{s}_p \;\Rightarrow\; \boxed{\overrightarrow{AS} \cdot \vec{s}_p = 0} \rightsquigarrow$$

$(t-1, -8-2t, t-3) \cdot (1,-2,1) = 0$

$(t-1)\cdot 1 + (-8-2t)\cdot(-2) + (t-3)\cdot 1 = 0$

$t - 1 + 16 + 4t + t - 3 = 0$

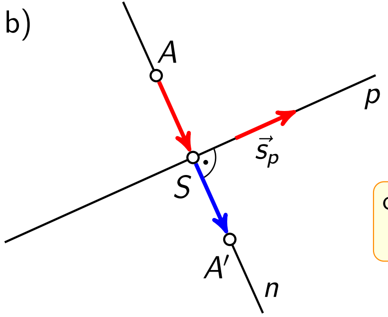
$6t + 12 = 0 \rightsquigarrow \boxed{t=-2}$

$$\boxed{n \ldots \frac{x-3}{-3} = \frac{y-4}{-4} = \frac{z-2}{-5}}$$

$n \ldots A,\ \overrightarrow{AS}$

$A(3,4,2) \qquad \vec{s}_p = (1,-2,1) \qquad p \ldots \frac{x-2}{1} = \frac{y+4}{-2} = \frac{z+1}{1}$

b)



$\overrightarrow{AS} = (-3,-4,-5)$

$S(0,0,-3)$

ortogonalna projekcija točke A na pravac p

$$p \ldots \begin{cases} x = 2+t \\ y = -4-2t \\ z = -1+t \end{cases}$$

$\overrightarrow{SA'} = \overrightarrow{AS}$

$\vec{r}_{A'} - \vec{r}_S = \overrightarrow{AS}$

$\vec{r}_{A'} = \vec{r}_S + \overrightarrow{AS}$

$\vec{r}_{A'} = (0,0,-3) + (-3,-4,-5)$

$\vec{r}_{A'} = (-3,-4,-8)$

$$\boxed{A'(-3,-4,-8)}$$

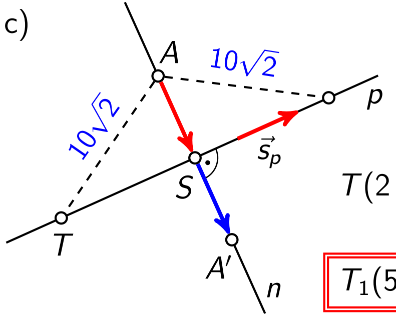
udaljenost točke od pravca

$d(A,p) = |AS| = |\overrightarrow{AS}|$

$d(A,p) = \sqrt{9+16+25}$

$d(A,p) = 5\sqrt{2}$

c)



$A(3,4,2)$ $\quad \vec{s}_p = (1,-2,1)$ $\quad p \ldots \frac{x-2}{1} = \frac{y+4}{-2} = \frac{z+1}{1}$

$\overrightarrow{AS} = (-3,-4,-5)$

$S(0,0,-3)$

$$p \ldots \begin{cases} x = 2+t \\ y = -4-2t \\ z = -1+t \end{cases}$$

$T(2+t, -4-2t, -1+t)$

$t = 3$ $\quad t = -7$

$\boxed{T_1(5,-10,2)}$ $\quad \boxed{T_2(-5,10,-8)}$

$d(A,T) = 10\sqrt{2}$

$$\sqrt{((2+t)-3)^2 + ((-4-2t)-4)^2 + ((-1+t)-2)^2} = 10\sqrt{2}$$

$$\sqrt{(t-1)^2 + (-2t-8)^2 + (t-3)^2} = 10\sqrt{2} \;/^2$$

$$(t-1)^2 + (-2t-8)^2 + (t-3)^2 = 200$$

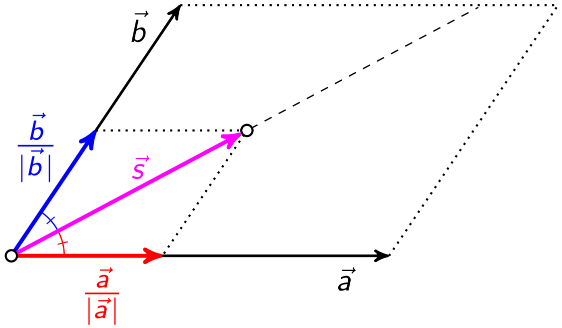
$$6t^2 + 24t + 74 = 200$$

$$6t^2 + 24t - 126 = 0 \;/:6$$

$$t^2 + 4t - 21 = 0$$

$$\boxed{t_1 = 3, \;\; t_2 = -7}$$

Simetrala kuta između dva vektora



$$\vec{s} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$$

Zadatak 2

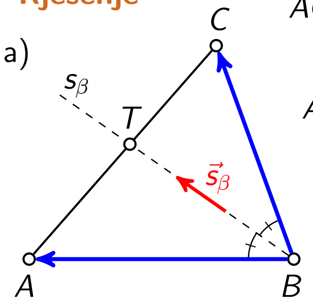
Zadane su točke $A(0,4,5)$, $B(0,0,2)$ i $C(6,0,2)$.

a) *Odredite točku T u kojoj simetrala s_β unutarnjeg kuta trokuta ABC pri vrhu B siječe stranicu $\overline{AC}$.*

b) *Odredite u kojem omjeru točka T dijeli dužinu $\overline{AC}$.*

Rješenje

a)



$AC \ldots A, \overrightarrow{AC}$ $\quad A(0,4,5),\; B(0,0,2),\; C(6,0,2)$

$$AC \ldots \begin{cases} x = 0 + 6 \cdot v \\ y = 4 + (-4) \cdot v \\ z = 5 + (-3) \cdot v \end{cases}$$

$\overrightarrow{AC} = (6,-4,-3)$

$\overrightarrow{BA} = (0,4,3)$

$\overrightarrow{BC} = (6,0,0)$

$$\left|\overrightarrow{BA}\right| = \sqrt{0^2 + 4^2 + 3^2} = 5$$

$$\left|\overrightarrow{BC}\right| = \sqrt{6^2 + 0^2 + 0^2} = 6$$

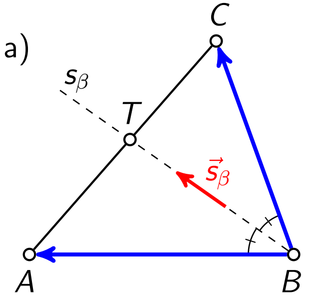
$$AC \ldots \begin{cases} x = 6v \\ y = 4 - 4v \\ z = 5 - 3v \end{cases}$$

$$\vec{s} = \frac{\overrightarrow{BA}}{\left|\overrightarrow{BA}\right|} + \frac{\overrightarrow{BC}}{\left|\overrightarrow{BC}\right|} = \frac{1}{5} \cdot (0,4,3) + \frac{1}{6} \cdot (6,0,0)$$

$$\vec{s} = \left(1, \frac{4}{5}, \frac{3}{5}\right) \qquad \vec{s}_\beta = 5\vec{s} = (5,4,3)$$

$$s_\beta \ldots \begin{cases} x = 5u \\ y = 4u \\ z = 2 + 3u \end{cases} \qquad s_\beta \ldots B, \vec{s}_\beta \qquad s_\beta \ldots \begin{cases} x = 0 + 5 \cdot u \\ y = 0 + 4 \cdot u \\ z = 2 + 3 \cdot u \end{cases}$$

a)



$$\boxed{T = s_\beta \cap AC}$$

$$\begin{aligned} 5u &= 6v \\ 4u &= 4 - 4v \\ 2 + 3u &= 5 - 3v \\ \hline 5u - 6v &= 0 \\ 4u + 4v &= 4 \\ 3u + 3v &= 3 \end{aligned}$$

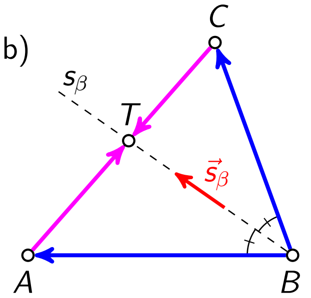
$$\begin{array}{cc|cl} u & v & & \\ \hline 5 & -6 & 0 & \\ 4 & 4 & 4 & /:4 \\ 3 & 3 & 3 & /:3 \\ \hline 5 & -6 & 0 & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ \hline 5 & -6 & 0 & \xleftarrow{+} \\ 1 & \textcircled{1} & 1 & /\cdot 6 \\ \hline \textcircled{11} & 0 & 6 & /\cdot \frac{-1}{11} \\ 1 & 1 & 1 & \xleftarrow{+} \\ \hline 11 & 0 & 6 & \\ 0 & 1 & \frac{5}{11} & \end{array}$$

$$AC \ldots \begin{cases} x = 6v \\ y = 4 - 4v \\ z = 5 - 3v \end{cases} \leftsquigarrow \boxed{v = \frac{5}{11}}$$

$$\boxed{\boxed{T\left(\frac{30}{11}, \frac{24}{11}, \frac{40}{11}\right)}}$$

$$s_\beta \ldots \begin{cases} x = 5u \\ y = 4u \\ z = 2 + 3u \end{cases} \leftsquigarrow \boxed{u = \frac{6}{11}}$$

b)



$A(0,4,5),\ B(0,0,2),\ C(6,0,2)$

$$T\left(\frac{30}{11}, \frac{24}{11}, \frac{40}{11}\right)$$

$$|\overrightarrow{BA}| = 5$$

$$|\overrightarrow{BC}| = 6$$

Simetrala unutarnjeg kuta trokuta
dijeli tom kutu nasuprotnu stranicu
u omjeru preostale dvije stranice.

$$\overrightarrow{AT} = \lambda\overrightarrow{CT} \rightsquigarrow \overrightarrow{AT} = -\frac{5}{6}\overrightarrow{CT} \rightsquigarrow \boxed{\boxed{|AT| : |CT| = 5 : 6}}$$

$$\overrightarrow{AT} = \left(\frac{30}{11}, -\frac{20}{11}, -\frac{15}{11}\right)$$

$$\overrightarrow{CT} = \left(-\frac{36}{11}, \frac{24}{11}, \frac{18}{11}\right)$$

$$\frac{\frac{30}{11}}{-\frac{36}{11}} = \frac{-\frac{20}{11}}{\frac{24}{11}} = \frac{-\frac{15}{11}}{\frac{18}{11}}$$

$$\boxed{\lambda = -\frac{5}{6}} \leftsquigarrow -\frac{5}{6} = -\frac{5}{6} = -\frac{5}{6}$$

Zadatak 3

Zadani su pravci

$$p_1 \ldots \frac{x}{-2} = \frac{y-1}{2} = \frac{z-2}{1} \quad i \quad p_2 \ldots \frac{x-1}{2} = \frac{y-1}{0} = \frac{z-3}{-2}.$$

a) *Pokažite da su p_1 i p_2 mimosmjerni pravci.*

b) *Odredite zajedničku normalu pravaca p_1 i p_2.*

c) *Izračunajte udaljenost pravaca p_1 i p_2.*

Rješenje

$$p_1 \ldots \frac{x}{-2} = \frac{y-1}{2} = \frac{z-2}{1} \qquad p_2 \ldots \frac{x-1}{2} = \frac{y-1}{0} = \frac{z-3}{-2}$$

a) Prvi način

$$\overset{x_1\ y_1\ z_1}{T_1(0,1,2)} \qquad \vec{s}_1 = \overset{\alpha_1\ \beta_1\ \gamma_1}{(-2,2,1)}$$

$$\overset{x_2\ y_2\ z_2}{T_2(1,1,3)} \qquad \vec{s}_2 = \overset{\alpha_2\ \beta_2\ \gamma_2}{(2,0,-2)}$$

$$\alpha_1 : \alpha_2 \neq \beta_1 : \beta_2 \implies p_1 \nparallel p_2$$

Uvjet komplanarnosti

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1-0 & 1-1 & 3-2 \\ -2 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = -8 \neq 0$$

p_1 i p_2 su mimosmjerni pravci

$$p_1 \ldots \frac{x}{-2} = \frac{y-1}{2} = \frac{z-2}{1} \qquad p_2 \ldots \frac{x-1}{2} = \frac{y-1}{0} = \frac{z-3}{-2}$$

a) **Drugi način**

$$p_1 \ldots \begin{cases} x = -2u \\ y = 1 + 2u \\ z = 2 + u \end{cases}$$

$$p_2 \ldots \begin{cases} x = 1 + 2v \\ y = 1 \\ z = 3 - 2v \end{cases}$$

$\boxed{p_1 \cap p_2}$

$$\begin{aligned} -2u &= 1 + 2v \\ 1 + 2u &= 1 \\ 2 + u &= 3 - 2v \\ \hline 2u + 2v &= -1 \\ 2u &= 0 \\ u + 2v &= 1 \end{aligned}$$

$$\begin{array}{cc|cl} u & v & & \\ \hline 2 & 2 & -1 & \\ 2 & 0 & 0 & /:2 \\ 1 & 2 & 1 & \\ \hline 2 & 2 & -1 & \xleftarrow{+} \\ \textcircled{1} & 0 & 0 & /\cdot(-2)\ /\cdot(-1) \\ 1 & 2 & 1 & \xleftarrow{+} \\ \hline 0 & \textcircled{2} & -1 & /\cdot(-1) \\ 1 & 0 & 0 & \\ 0 & 2 & 1 & \xleftarrow{+} \\ \hline 0 & 2 & -1 & \\ 1 & 0 & 0 & \\ \boxed{0} & \boxed{0} & \boxed{2} & \end{array}$$

$\boxed{0 = 2} \leftsquigarrow$

sustav nema rješenja

$$\left.\begin{aligned} p_1 \cap p_2 &= \emptyset \\ p_1 &\nparallel p_2 \end{aligned}\right\} \Downarrow$$

p_1 i p_2 su mimosmjerni pravci

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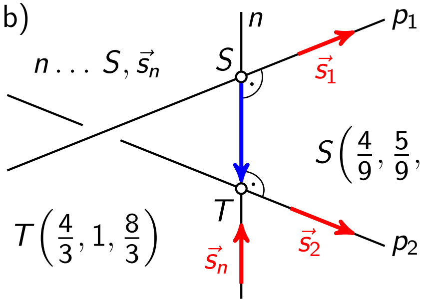
b)



$n \ldots S, \vec{s}_n$

$T\left(\frac{4}{3}, 1, \frac{8}{3}\right)$

$$\left.\begin{aligned} n \perp p_1 &\Rightarrow \vec{s}_n \perp \vec{s}_1 \\ n \perp p_2 &\Rightarrow \vec{s}_n \perp \vec{s}_2 \end{aligned}\right\} \Rightarrow \vec{s}_n = \vec{s}_1 \times \vec{s}_2$$

$$\boxed{\begin{aligned} \vec{s}_1 &= (-2, 2, 1) \\ \vec{s}_2 &= (2, 0, -2) \end{aligned}}$$

$$S\left(\frac{4}{9}, \frac{5}{9}, \frac{16}{9}\right)$$

$$\vec{s}_n = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = (-4, -2, -4)$$

$$\overrightarrow{ST} = ((1 + 2v) - (-2u),\ 1 - (1 + 2u),\ (3 - 2v) - (2 + u))$$

$$\boxed{\overrightarrow{ST} = (1 + 2u + 2v, -2u, 1 - u - 2v)}$$

$$\boxed{\begin{aligned} &S(-2u, 1 + 2u, 2 + u) \\ &T(1 + 2v, 1, 3 - 2v) \end{aligned}}$$

$$\overrightarrow{ST} = \lambda \vec{s}_n = \lambda \cdot (-4, -2, -4)$$

$$\lambda = -\frac{2}{9}$$

$$\boxed{\overrightarrow{ST} = (-4\lambda, -2\lambda, -4\lambda)}$$

$$\left.\begin{aligned} 1 + 2u + 2v &= -4\lambda \\ -2u &= -2\lambda \\ 1 - u - 2v &= -4\lambda \end{aligned}\right\}$$

$$\boxed{v = \frac{1}{6}}$$

$$\boxed{u = -\frac{2}{9}}$$

$$\boxed{n \ldots \frac{x - \frac{4}{9}}{-4} = \frac{y - \frac{5}{9}}{-2} = \frac{z - \frac{16}{9}}{-4}}$$

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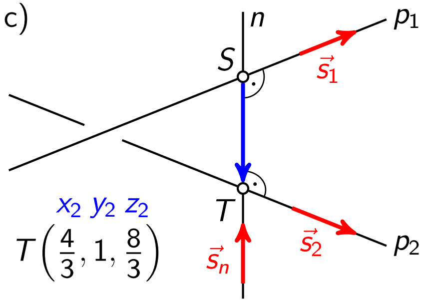
c)



$$T\left(\overset{x_2}{\frac{4}{3}}, \overset{y_2}{1}, \overset{z_2}{\frac{8}{3}}\right)$$

$$S\left(\overset{x_1}{\frac{4}{9}}, \overset{y_1}{\frac{5}{9}}, \overset{z_1}{\frac{16}{9}}\right)$$

Udaljenost mimosmjernih pravaca

$$d(p_1, p_2) = \frac{\left|(\vec{r}_2 - \vec{r}_1, \vec{s}_1, \vec{s}_2)\right|}{|\vec{s}_1 \times \vec{s}_2|}$$

$$d(p_1, p_2) = |ST|$$

$$d(p_1, p_2) = \sqrt{\left(\frac{4}{3} - \frac{4}{9}\right)^2 + \left(1 - \frac{5}{9}\right)^2 + \left(\frac{8}{3} - \frac{16}{9}\right)^2}$$

$$\boxed{d(p_1, p_2) = \frac{4}{3}}$$

$$|ST| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Zadatak 4

Zraka svjetlosti prolazi točkom $T(-2, -1, 1)$ *i kreće se u smjeru vektora* $\vec{v} = (-1, 0, -1)$ *te se reflektira na ravnini*

$$\pi_1 \ldots x + y - 2z = 0.$$

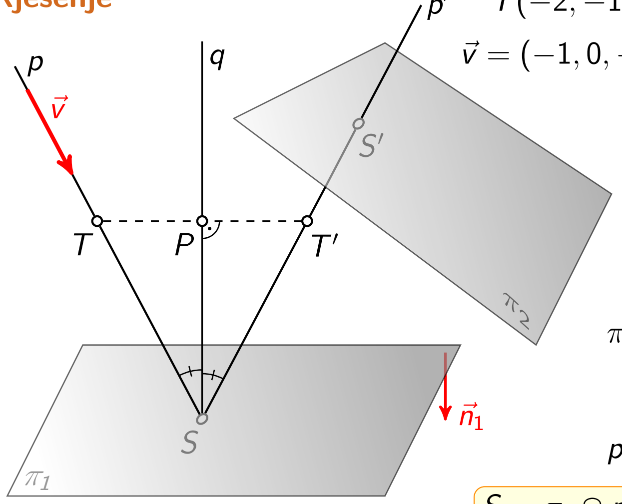
U kojoj točki reflektirana zraka siječe ravninu

$$\pi_2 \ldots x + y + z + 18 = 0?$$

Rješenje

$T(-2,-1,1)$ $\vec{n}_1 = (1,1,-2)$

$\vec{v} = (-1,0,-1)$



Zraka (polupravac) siječe ravninu jedino ako je $t > 0$. Osim smjera, moramo poštivati i orijentaciju vektora $\vec{v}$.

$\pi_1 \ldots x + y - 2z = 0$

$$p \ldots \begin{cases} x = -2 - t \\ y = -1 \\ z = 1 - t \end{cases}$$

$S = \pi_1 \cap p$

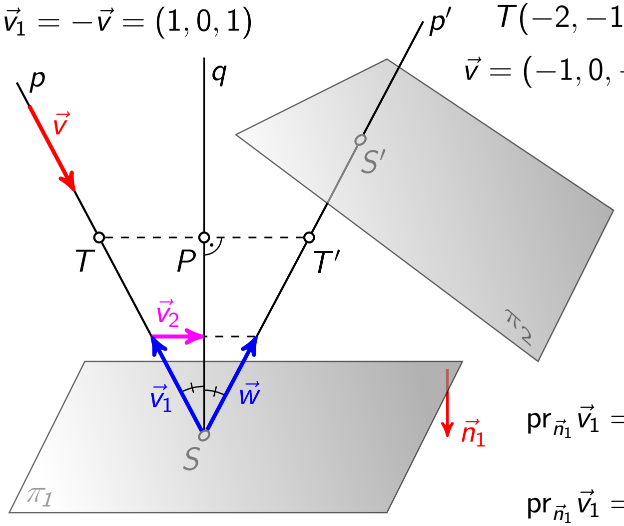
$$x + y - 2z = 0$$
$$(-2 - t) + (-1) - 2(1 - t) = 0$$
$$t - 5 = 0$$
$$\boxed{t = 5}$$

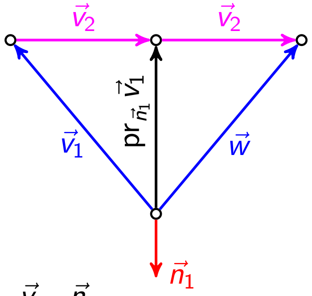
$$\boxed{S(-7,-1,-4)}$$

$\vec{v}_1 = -\vec{v} = (1,0,1)$

$T(-2,-1,1)$ $\vec{n}_1 = (1,1,-2)$

$\vec{v} = (-1,0,-1)$ $S(-7,-1,-4)$





$$\text{pr}_{\vec{n}_1} \vec{v}_1 = \frac{\vec{v}_1 \cdot \vec{n}_1}{|\vec{n}_1|^2} \vec{n}_1$$

$$\text{pr}_{\vec{n}_1} \vec{v}_1 = \frac{-1}{\sqrt{6}^2}(1,1,-2)$$

$$\boxed{\text{pr}_{\vec{n}_1} \vec{v}_1 = \left(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\right)}$$

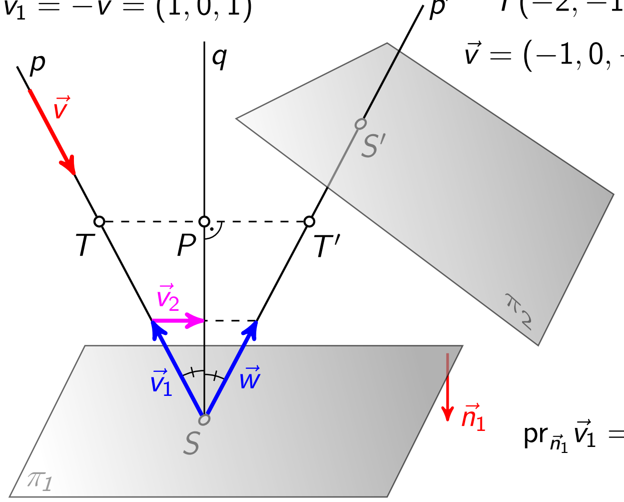
$$|\vec{n}_1| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$$

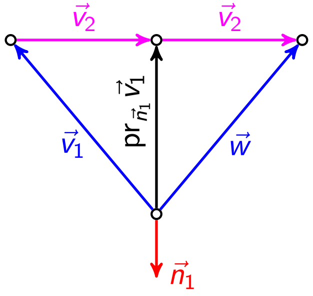
$$\vec{v}_1 \cdot \vec{n}_1 = (1,0,1) \cdot (1,1,-2) = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot (-2) = -1$$

$\vec{v}_1 = -\vec{v} = (1,0,1)$

$T(-2,-1,1)$ $\vec{n}_1 = (1,1,-2)$

$\vec{v} = (-1,0,-1)$ $S(-7,-1,-4)$





$$\text{pr}_{\vec{n}_1} \vec{v}_1 = \left(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\right)$$

$$\vec{v}_2 = \text{pr}_{\vec{n}_1} \vec{v}_1 - \vec{v}_1 = \left(-\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}\right) - (1,0,1) = \left(-\frac{7}{6}, -\frac{1}{6}, -\frac{2}{3}\right)$$

$$\vec{w} = \vec{v}_1 + 2\vec{v}_2 = (1,0,1) + \left(-\frac{7}{3}, -\frac{1}{3}, -\frac{4}{3}\right) = \left(-\frac{4}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$$

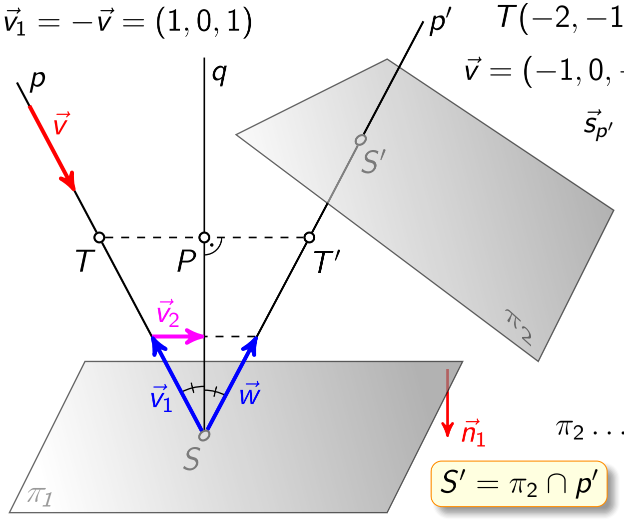
$\vec{v}_1 = -\vec{v} = (1,0,1)$

$T(-2,-1,1)$ $\vec{n}_1 = (1,1,-2)$

$\vec{v} = (-1,0,-1)$ $S(-7,-1,-4)$

$\vec{s}_{p'} = 3\vec{w} = (-4,-1,-1)$

$\vec{w} = \left(-\frac{4}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$



Kod skaliranja vektora smjera zrake moramo sačuvati orijentaciju vektora.

$\pi_2 \ldots x + y + z + 18 = 0$

$S' = \pi_2 \cap p'$

$$p' \ldots \begin{cases} x = -7 - 4t \\ y = -1 - t \\ z = -4 - t \end{cases}$$

$$x + y + z + 18 = 0$$
$$(-7 - 4t) + (-1 - t) + (-4 - t) + 18 = 0$$
$$-6t + 6 = 0$$
$$\boxed{t = 1}$$

$$\boxed{S'(-11,-2,-5)}$$