Seminari 6

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

U $\mathcal{P}_3(t)$ zadani su polinomi

$$p_1(t) = t^2 + t, \quad p_2(t) = t^2 - 2t + 3.$$

Prikažite, ako je moguće, polinome

$$p_3(t) = -t^2 + 8t - 9 \quad i \quad p_4(t) = t + 2$$

kao linearne kombinacije polinoma p_1 i p_2.

Rješenje

$$p_3(t) = \alpha_1 \cdot p_1(t) + \alpha_2 \cdot p_2(t)$$

$$-t^2 + 8t - 9 = \alpha_1 \cdot (t^2 + t) + \alpha_2 \cdot (t^2 - 2t + 3)$$

$$-t^2 + 8t - 9 = (\alpha_1 + \alpha_2)t^2 + (\alpha_1 - 2\alpha_2)t + 3\alpha_2$$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= -1 \\ \alpha_1 - 2\alpha_2 &= 8 \\ 3\alpha_2 &= -9 \end{aligned}\right\}$$

$$\begin{array}{cc|cl}
\alpha_1 & \alpha_2 & & \\
\hline
1 & 1 & -1 & \\
1 & -2 & 8 & \\
0 & 3 & -9 & /:3 \\
\hline
1 & 1 & -1 & \leftarrow + \\
\textcircled{1} & -2 & 8 & /\cdot(-1) \\
0 & 1 & -3 & \\
\hline
0 & 3 & -9 & /:3 \\
1 & -2 & 8 & \\
0 & 1 & -3 & \\
\hline
\end{array}
\qquad
\begin{array}{cc|cl}
\alpha_1 & \alpha_2 & & \\
\hline
0 & 1 & -3 & \\
1 & -2 & 8 & \\
0 & 1 & -3 & \\
\hline
0 & \textcircled{1} & -3 & /\cdot 2 \\
1 & -2 & 8 & \leftarrow + \\
\hline
0 & 1 & -3 & \rightsquigarrow \boxed{\alpha_2 = -3} \\
1 & 0 & 2 & \rightsquigarrow \boxed{\alpha_1 = 2}
\end{array}$$

$$p_3(t) = 2 \cdot p_1(t) - 3 \cdot p_2(t)$$

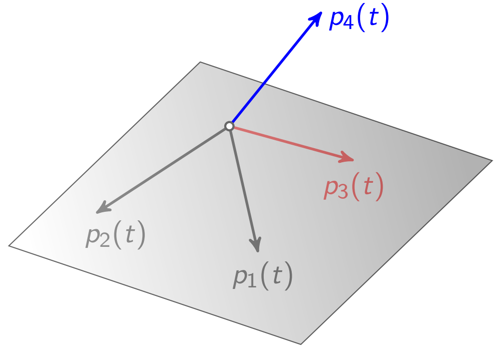
$$p_4(t) = \alpha_1 \cdot p_1(t) + \alpha_2 \cdot p_2(t)$$

$$t + 2 = \alpha_1 \cdot (t^2 + t) + \alpha_2 \cdot (t^2 - 2t + 3)$$

$$t + 2 = (\alpha_1 + \alpha_2)t^2 + (\alpha_1 - 2\alpha_2)t + 3\alpha_2$$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 - 2\alpha_2 &= 1 \\ 3\alpha_2 &= 2 \end{aligned}\right\}$$

$$\begin{array}{cc|cl}
\alpha_1 & \alpha_2 & & \\
\hline
\textcircled{1} & 1 & 0 & /\cdot(-1) \\
1 & -2 & 1 & \leftarrow + \\
0 & 3 & 2 & \\
\hline
1 & 1 & 0 & \leftarrow + \\
0 & \textcircled{-3} & 1 & /\cdot 1 \Big/ \cdot\frac{1}{3} \\
0 & 3 & 2 & \leftarrow + \\
\hline
1 & 0 & \frac{1}{3} & \\
0 & -3 & 1 & \\
\boxed{0} & \boxed{0} & \boxed{3} & \rightsquigarrow \boxed{0 = 3}
\end{array}$$



Polinom p_4 se ne može napisati kao linearna kombinacija polinoma p_1 i p_2.

Zadatak 2

U $\mathbb{R}^3$ ispitajte linearnu nezavisnost vektora

$$(2,1,2),\ (1,0,1),\ (-1,1,1),\ (4,-1,0)$$

i prikažite pojedini vektor kao linearnu kombinaciju preostalih kad god je to moguće.

Rješenje

$$\alpha_1\cdot(2,1,2)+\alpha_2\cdot(1,0,1)+\alpha_3\cdot(-1,1,1)+\alpha_4\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$\big(2\alpha_1+\alpha_2-\alpha_3+4\alpha_4,\ \alpha_1+\alpha_3-\alpha_4,\ 2\alpha_1+\alpha_2+\alpha_3\big)=(0,0,0)$$

$$\left.\begin{aligned}2\alpha_1+\alpha_2-\alpha_3+4\alpha_4&=0\\ \alpha_1+\alpha_3-\alpha_4&=0\\ 2\alpha_1+\alpha_2+\alpha_3&=0\end{aligned}\right\}$$

$$\begin{array}{cccc|c|l}
\boxed{\alpha_1} & \alpha_2 & \alpha_3 & \alpha_4 & & \\
\hline
2 & 1 & -1 & 4 & 0 & \leftarrow + \\
1 & 0 & 1 & \textcircled{-1} & 0 & /\cdot 4 \\
2 & 1 & 1 & 0 & 0 & \\
\hline
6 & 1 & 3 & 0 & 0 & \leftarrow + \\
1 & 0 & 1 & -1 & 0 & \\
2 & \textcircled{1} & 1 & 0 & 0 & /\cdot(-1) \\
\hline
4 & 0 & 2 & 0 & 0 & /:2 \\
1 & 0 & 1 & -1 & 0 & \\
2 & 1 & 1 & 0 & 0 & \\
\hline
2 & 0 & \textcircled{1} & 0 & 0 & /\cdot(-1)\ /\cdot(-1) \\
1 & 0 & 1 & -1 & 0 & \leftarrow + \\
2 & 1 & 1 & 0 & 0 & \leftarrow + \\
\hline
\end{array}$$

$$\begin{array}{cccc|c}
\boxed{\alpha_1} & \alpha_2 & \alpha_3 & \alpha_4 & \\
\hline
2 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{array}$$

$$\left.\begin{aligned}2\alpha_1+\alpha_3&=0\\ -\alpha_1-\alpha_4&=0\\ \alpha_2&=0\end{aligned}\right\}\qquad \left\{\begin{aligned}\alpha_1&=t\\ \alpha_2&=0\\ \alpha_3&=-2t\\ \alpha_4&=-t\end{aligned}\right.$$

$$t\in\mathbb{R}$$

Kako dobiveni homogeni sustav linearnih jednadžbi ima i netrivijalnih rješenja, zadani vektori su linearno zavisni u $\mathbb{R}^3$.

$$\alpha_1\cdot(2,1,2)+\alpha_2\cdot(1,0,1)+\alpha_3\cdot(-1,1,1)+\alpha_4\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$t\cdot(2,1,2)+0\cdot(1,0,1)+(-2t)\cdot(-1,1,1)+(-t)\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$t\in\mathbb{R}$

$\boxed{t=1}$

$$(2,1,2)+0\cdot(1,0,1)-2\cdot(-1,1,1)-1\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$(2,1,2)=0\cdot(1,0,1)+2\cdot(-1,1,1)+1\cdot(4,-1,0)$$

$\boxed{t=-\frac{1}{2}}$

$$-\frac{1}{2}\cdot(2,1,2)+0\cdot(1,0,1)+(-1,1,1)+\frac{1}{2}\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$(-1,1,1)=\frac{1}{2}\cdot(2,1,2)+0\cdot(1,0,1)-\frac{1}{2}\cdot(4,-1,0)$$

$\boxed{t=-1}$

$$-1\cdot(2,1,2)+0\cdot(1,0,1)+2\cdot(-1,1,1)+(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$(4,-1,0)=1\cdot(2,1,2)+0\cdot(1,0,1)-2\cdot(-1,1,1)$$

$$\alpha_1\cdot(2,1,2)+\alpha_2\cdot(1,0,1)+\alpha_3\cdot(-1,1,1)+\alpha_4\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$$t\cdot(2,1,2)+0\cdot(1,0,1)+(-2t)\cdot(-1,1,1)+(-t)\cdot(4,-1,0)=\Theta_{\mathbb{R}^3}$$

$t\in\mathbb{R}$

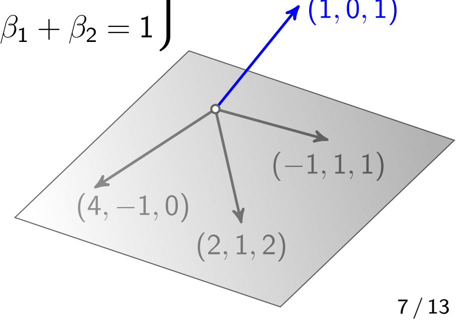
$$(1,0,1)=\beta_1\cdot(2,1,2)+\beta_2\cdot(-1,1,1)+\beta_3\cdot(4,-1,0)$$

Vektor $(1,0,1)$ se ne može prikazati kao linearna kombinacija preostalih vektora.

$$\begin{array}{ccc|c|l}
\beta_1 & \beta_2 & \beta_3 & & \\
\hline
2 & -1 & 4 & 1 & \leftarrow + \\
1 & 1 & \textcircled{-1} & 0 & /\cdot 4 \\
2 & 1 & 0 & 1 & \\
\hline
6 & 3 & 0 & 1 & \leftarrow + \\
1 & 1 & -1 & 0 & \leftarrow + \\
2 & \textcircled{1} & 0 & 1 & /\cdot(-1)\ /\cdot(-3) \\
\hline
\boxed{0 \quad 0 \quad 0} & & & \boxed{-2} & \rightsquigarrow \boxed{0=-2} \\
-1 & 0 & -1 & -1 & \\
2 & 1 & 0 & 1 &
\end{array}$$

sustav nema rješenja

$$\left.\begin{aligned}2\beta_1-\beta_2+4\beta_3&=1\\ \beta_1+\beta_2-\beta_3&=0\\ 2\beta_1+\beta_2&=1\end{aligned}\right\}$$



Zadatak 3

U $\mathbb{R}^2$ ispitajte linearnu nezavisnost vektora

$$(1,1),\ (2,3),\ (1,0),\ (-2,1)$$

i prikažite pojedini vektor kao linearnu kombinaciju preostalih kad god je to moguće.

Rješenje

$$\alpha_1\cdot(1,1)+\alpha_2\cdot(2,3)+\alpha_3\cdot(1,0)+\alpha_4\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$\big(\alpha_1+2\alpha_2+\alpha_3-2\alpha_4,\ \alpha_1+3\alpha_2+\alpha_4\big)=(0,0)$$

$$\begin{array}{cccc|cl}
\alpha_1 & \boxed{\alpha_2} & \alpha_3 & \boxed{\alpha_4} & & \\
\hline
1 & 2 & 1 & -2 & 0 & \leftarrow + \\
\textcircled{1} & 3 & 0 & 1 & 0 & /\cdot(-1) \\
\hline
0 & -1 & 1 & -3 & 0 & \\
1 & 3 & 0 & 1 & 0 &
\end{array}$$

$$\left.\begin{aligned}\alpha_1+2\alpha_2+\alpha_3-2\alpha_4&=0\\ \alpha_1+3\alpha_2+\alpha_4&=0\end{aligned}\right\}$$

$$\left.\begin{aligned}-\alpha_2+\alpha_3-3\alpha_4&=0\\ \alpha_1+3\alpha_2+\alpha_4&=0\end{aligned}\right\}$$

$$\left\{\begin{aligned}\alpha_1&=-3u-v\\ \alpha_2&=u\\ \alpha_3&=u+3v\\ \alpha_4&=v\end{aligned}\right.\qquad u,v\in\mathbb{R}$$

Kako dobiveni homogeni sustav linearnih jednadžbi ima i netrivijalnih rješenja, zadani vektori su linearno zavisni u $\mathbb{R}^2$.

$$\alpha_1\cdot(1,1)+\alpha_2\cdot(2,3)+\alpha_3\cdot(1,0)+\alpha_4\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$(-3u-v)\cdot(1,1)+u\cdot\underline{\underline{(2,3)}}+(u+3v)\cdot(1,0)+v\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$u,v\in\mathbb{R}$

$\boxed{u=1,\ v=0}$

$$-3\cdot(1,1)+(2,3)+1\cdot(1,0)+0\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$(2,3)=3\cdot(1,1)-1\cdot(1,0)+0\cdot(-2,1)$$

$\boxed{u=1,\ v=1}$

$$-4\cdot(1,1)+(2,3)+4\cdot(1,0)+1\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$(2,3)=4\cdot(1,1)-4\cdot(1,0)-1\cdot(-2,1)$$

$\boxed{u=1,\ v=-1}$

$$-2\cdot(1,1)+(2,3)-2\cdot(1,0)-1\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$(2,3)=2\cdot(1,1)+2\cdot(1,0)+1\cdot(-2,1)$$

$$\vdots$$

$$\alpha_1\cdot(1,1)+\alpha_2\cdot(2,3)+\alpha_3\cdot(1,0)+\alpha_4\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$$(-3u-v)\cdot(1,1)+u\cdot\underline{\underline{(2,3)}}+(u+3v)\cdot(1,0)+v\cdot(-2,1)=\Theta_{\mathbb{R}^2}$$

$u,v\in\mathbb{R}$

$$(2,3)=\beta_1\cdot(1,1)+\beta_2\cdot(1,0)+\beta_3\cdot(-2,1)$$

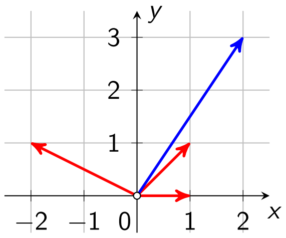
$$(2,3)=(3-t)\cdot(1,1)+(-1+3t)\cdot(1,0)+t\cdot(-2,1)\qquad t\in\mathbb{R}$$

Prikaz nije jedinstven jer su vektori $(1,1)$, $(1,0)$ i $(-2,1)$ linearno zavisni.

$$\left.\begin{aligned}\beta_1+\beta_2-2\beta_3&=2\\ \beta_1+\beta_3&=3\end{aligned}\right\}$$

$$\begin{array}{ccc|cl}
\beta_1 & \beta_2 & \boxed{\beta_3} & & \\
\hline
1 & 1 & -2 & 2 & \leftarrow + \\
\textcircled{1} & 0 & 1 & 3 & /\cdot(-1) \\
\hline
0 & 1 & -3 & -1 & \\
1 & 0 & 1 & 3 &
\end{array}$$

$$\left.\begin{aligned}\beta_2-3\beta_3&=-1\\ \beta_1+\beta_3&=3\end{aligned}\right\}$$



$$\left\{\begin{aligned}\beta_1&=3-t\\ \beta_2&=-1+3t\\ \beta_3&=t\end{aligned}\right.$$

Svaki vektor iz zadanog skupa može se prikazati kao linearna kombinacija preostalih vektora, ali prikazi nisu jedinstveni zbog linearne zavisnosti preostalih vektora.

Zadatak 4

Ispitajte jesu li sljedeće matrice linearno nezavisne u $M_2(\mathbb{R})$:

$$\begin{bmatrix}0&3\\1&1\end{bmatrix},\ \begin{bmatrix}0&1\\0&0\end{bmatrix},\ \begin{bmatrix}4&3\\1&3\end{bmatrix},\ \begin{bmatrix}-2&0\\3&0\end{bmatrix}.$$

Rješenje

$$\alpha\cdot\begin{bmatrix}0&3\\1&1\end{bmatrix}+\beta\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix}+\gamma\cdot\begin{bmatrix}4&3\\1&3\end{bmatrix}+\delta\cdot\begin{bmatrix}-2&0\\3&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}4\gamma-2\delta & 3\alpha+\beta+3\gamma\\ \alpha+\gamma+3\delta & \alpha+3\gamma\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}\qquad \left.\begin{aligned}4\gamma-2\delta&=0\\3\alpha+\beta+3\gamma&=0\\\alpha+\gamma+3\delta&=0\\\alpha+3\gamma&=0\end{aligned}\right\}$$

$$\begin{array}{cccc|cl}
\alpha & \beta & \gamma & \delta & & \\
\hline
0 & 0 & 4 & -2 & 0 & /:(-2)\\
3 & 1 & 3 & 0 & 0 & \\
1 & 0 & 1 & 3 & 0 & \\
1 & 0 & 3 & 0 & 0 & \\
\hline
0 & 0 & -2 & \textcircled{1} & 0 & /\cdot(-3)\\
3 & 1 & 3 & 0 & 0 & \\
1 & 0 & 1 & 3 & 0 & +\\
1 & 0 & 3 & 0 & 0 & \\
\hline
0 & 0 & -2 & 1 & 0 & \\
3 & 1 & 3 & 0 & 0 & +\\
1 & 0 & 7 & 0 & 0 & +\\
\textcircled{1} & 0 & 3 & 0 & 0 & /\cdot(-1)\ /\cdot(-3)\\
\hline
\end{array}
\qquad
\begin{array}{cccc|cl}
\alpha & \beta & \gamma & \delta & & \\
\hline
0 & 0 & -2 & 1 & 0 & \\
0 & 1 & -6 & 0 & 0 & \\
0 & 0 & 4 & 0 & 0 & /:4\\
1 & 0 & 3 & 0 & 0 & \\
\hline
0 & 0 & -2 & 1 & 0 & +\\
0 & 1 & -6 & 0 & 0 & +\\
0 & 0 & \textcircled{1} & 0 & 0 & /\cdot(-3)\ /\cdot 6\ /\cdot 2\\
1 & 0 & 3 & 0 & 0 & +\\
\hline
0 & 0 & 0 & 1 & 0 & \rightsquigarrow \boxed{\delta=0}\\
0 & 1 & 0 & 0 & 0 & \rightsquigarrow \boxed{\beta=0}\\
0 & 0 & 1 & 0 & 0 & \rightsquigarrow \boxed{\gamma=0}\\
1 & 0 & 0 & 0 & 0 & \rightsquigarrow \boxed{\alpha=0}
\end{array}$$

Zadane matrice su linearno nezavisne u $M_2(\mathbb{R})$ pa se niti jedna od njih ne može napisati kao linearna kombinacija preostalih.