Seminari 7

MATEMATIČKE METODE ZA INFORMATIČARE

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Zadatak 1

a) *U* $\mathbb{R}^3$ *nadopunite do baze skup vektora* $\{(5,0,2),\ (0,-5,0)\}$.

b) *U* $\mathcal{P}_4(x)$ *nadopunite do baze skup vektora*

$$\{6+2x-3x^2-x^3,\ x-7x^3\}.$$

Rješenje

Nadopuna do baze nije jedinstvena.

a) $\{(5,0,2),\ (0,-5,0)\}$

$$\alpha\cdot(5,0,2)+\beta\cdot(0,-5,0)=\Theta_{\mathbb{R}^3}$$

$$(5\alpha,\ -5\beta,2\alpha)=(0,0,0)$$

$$\left.\begin{aligned} 5\alpha&=0\\ -5\beta&=0\\ 2\alpha&=0 \end{aligned}\right\} \rightsquigarrow \boxed{\begin{aligned}\alpha&=0\\ \beta&=0\end{aligned}}$$

Zadani skup vektora je linearno nezavisan u $\mathbb{R}^3$ pa se može nadopuniti do neke baze vektorskog prostora $\mathbb{R}^3$.

$$\mathcal{B}_{\text{kan}}=\{(1,0,0),(0,1,0),(0,0,1)\}$$

$\dim\mathbb{R}^3=3$

$$\{(5,0,2),\ (0,-5,0),\ {\color{red}(1,0,0)}\}$$

jedna nadopuna do baze

$$\alpha\cdot(5,0,2)+\beta\cdot(0,-5,0)+\gamma\cdot(1,0,0)=\Theta_{\mathbb{R}^3}$$

$$(5\alpha+\gamma,\ -5\beta,2\alpha)=(0,0,0)$$

$$\left.\begin{aligned} 5\alpha+\gamma&=0\\ -5\beta&=0\\ 2\alpha&=0 \end{aligned}\right\} \begin{aligned} &\\ &\rightsquigarrow \beta=0\\ &\rightsquigarrow \alpha=0 \end{aligned} \quad \gamma=0$$

b) $\{6+2x-3x^2-x^3,\ x-7x^3\}$

$$\mathcal{B}_{\text{kan}}=\{1,x,x^2,x^3\}$$

$\dim\mathcal{P}_4(x)=4$

$$\alpha\cdot(6+2x-3x^2-x^3)+\beta\cdot(x-7x^3)=\Theta_{\mathcal{P}_4(x)}$$

$$(-\alpha-7\beta)x^3+(-3\alpha)x^2+(2\alpha+\beta)x+6\alpha=0$$

$$\left.\begin{aligned} -\alpha-7\beta&=0\\ -3\alpha&=0\\ 2\alpha+\beta&=0\\ 6\alpha&=0 \end{aligned}\right\} \begin{aligned} &\\ &\rightsquigarrow \boxed{\alpha=0}\\ &\quad\ \boxed{\beta=0} \end{aligned}$$

Zadani skup vektora je linearno nezavisan u $\mathcal{P}_4(x)$ pa se može nadopuniti do neke baze vektorskog prostora $\mathcal{P}_4(x)$.

$$\{6+2x-3x^2-x^3,\ x-7x^3,\ {\color{red}1}\}$$

$$\alpha\cdot(6+2x-3x^2-x^3)+\beta\cdot(x-7x^3)+\gamma\cdot 1=\Theta_{\mathcal{P}_4(x)}$$

$$(-\alpha-7\beta)x^3+(-3\alpha)x^2+(2\alpha+\beta)x+(6\alpha+\gamma)=0$$

$$\left.\begin{aligned} -\alpha-7\beta&=0\\ -3\alpha&=0\\ 2\alpha+\beta&=0\\ 6\alpha+\gamma&=0 \end{aligned}\right\} \begin{aligned} &\beta=0\\ &\rightsquigarrow \alpha=0\\ &\\ &\gamma=0 \end{aligned}$$

linearno nezavisni skup vektora, ali nije baza za $\mathcal{P}_4(x)$

b) $\{6+2x-3x^2-x^3,\ x-7x^3\}$ $\qquad \mathcal{B}_{\text{kan}} = \{1, x, x^2, x^3\}$

$\dim \mathcal{P}_4(x) = 4$

$$\{6+2x-3x^2-x^3,\ x-7x^3,\ 1,\ x\}$$

jedna nadopuna do baze

$$\alpha\cdot(6+2x-3x^2-x^3)+\beta\cdot(x-7x^3)+\gamma\cdot 1+\delta\cdot x = \Theta_{\mathcal{P}_4(x)}$$

$$(-\alpha-7\beta)x^3+(-3\alpha)x^2+(2\alpha+\beta+\delta)x+(6\alpha+\gamma)=0$$

$$\left.\begin{aligned} -\alpha-7\beta&=0\\ -3\alpha&=0\\ 2\alpha+\beta+\delta&=0\\ 6\alpha+\gamma&=0 \end{aligned}\right\}$$

$\alpha = 0 \rightsquigarrow \beta = 0, \ \gamma = 0$

$\boxed{\alpha = 0,\ \beta = 0} \xrightarrow{\delta = 0} 2\alpha+\beta+\delta=0$

Nadopuna do baze nije jedinstvena.

Sustav linearnih jednadžbi

$$\begin{aligned} x_1+2x_2-x_3&=5\\ 5x_1-3x_2+4x_3&=-8 \end{aligned}$$

Matrični zapis $\quad AX = B$

$$\begin{bmatrix} 1 & 2 & -1\\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \begin{bmatrix} 5\\ -8 \end{bmatrix}$$

Zapis pomoću linearne kombinacije vektora

$$x_1\cdot\begin{bmatrix} 1\\ 5 \end{bmatrix} + x_2\cdot\begin{bmatrix} 2\\ -3 \end{bmatrix} + x_3\cdot\begin{bmatrix} -1\\ 4 \end{bmatrix} = \begin{bmatrix} 5\\ -8 \end{bmatrix}$$

Proširena matrica sustava

$$A_p = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5\\ 5 & -3 & 4 & -8 \end{array}\right]$$

- Kronecker-Capellijev teorem
  Sustav linearnih jednadžbi $AX = B$ je rješiv akko $r(A_p) = r(A)$.
- Posljednji stupac u matrici A_p može se zapisati kao linearna kombinacija preostalih stupaca akko $r(A_p) = r(A)$.

Zadatak 2

U $\mathcal{P}_4(t)$ zadani su polinomi

$$p(t) = t^3+t^2+t, \quad q(t) = t^3-t+1, \quad r(t) = 2t^3-t^2+t-2.$$

a) *Ispitajte jesu li polinomi p, q i r linearno nezavisni u $\mathcal{P}_4(t)$.*

b) *Može li se polinom $f(t) = t^3+3t^2+3$ prikazati kao linearna kombinacija polinoma p, q i r?*

c) *Može li se polinom $g(t) = t^3+3t^2+t+3$ prikazati kao linearna kombinacija polinoma p, q i r?*

Rješenje

$\dim \mathcal{P}_4(t) = 4$

- Kanonska baza za $\mathcal{P}_4(t)$: $\{1, t, t^2, t^3\}$

$$\begin{aligned} p(t) = t^3+t^2+t &\longrightarrow p(t) = (0,1,1,1)\\ q(t) = t^3-t+1 &\longrightarrow q(t) = (1,-1,0,1)\\ r(t) = 2t^3-t^2+t-2 &\longrightarrow r(t) = (-2,1,-1,2)\\ f(t) = t^3+3t^2+3 &\longrightarrow f(t) = (3,0,3,1)\\ g(t) = t^3+3t^2+t+3 &\longrightarrow g(t) = (3,1,3,1) \end{aligned}$$

$$A = \left[\begin{array}{ccc|cc} 0 & 1 & -2 & 3 & 3\\ 1 & -1 & 1 & 0 & 1\\ 1 & 0 & -1 & 3 & 3\\ 1 & 1 & 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:cc} 0 & 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 & 3 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 & 3 \\ 1 & 0 & -1 & 3 & 3 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right] / \cdot (-1) / \cdot (-1) \sim$$

$$\sim \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 & 3 \\ 0 & 1 & -2 & 3 & 2 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right] / \cdot (-1) / \cdot (-2) \sim \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 5 & -5 & -6 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 & 3 \\ 0 & 0 & 5 & -5 & -6 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

$\boxed{r(A) = 4}$

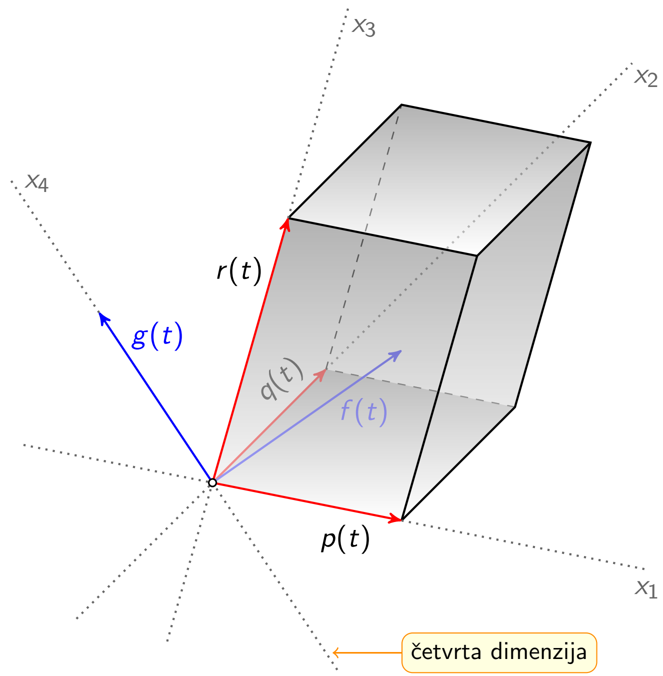
$\{p, q, r, g\}$ je jedna baza za vektorski prostor $\mathcal{P}_4(t)$.

$$A = \begin{array}{c} \begin{array}{ccccc} p & q & r & f & g \end{array} \\ \left[\begin{array}{ccc:cc} 0 & 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 & 3 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right] \end{array} \sim \cdots \sim \left[\begin{array}{ccc:cc} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 & 3 \\ 0 & 0 & 5 & -5 & -6 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

a) Polinomi p, q i r su linearno nezavisni u $\mathcal{P}_4(t)$ pa je $\mathcal{L}(p, q, r)$ potprostor dimenzije 3 u vektorskom prostoru $\mathcal{P}_4(t)$.

b) Polinom f se može prikazati kao linearna kombinacija polinoma p, q i r. Drugim riječima, $f \in \mathcal{L}(p, q, r)$.

c) Polinom g se ne može prikazati kao linearna kombinacija polinoma p, q i r. Drugim riječima, $g \notin \mathcal{L}(p, q, r)$.



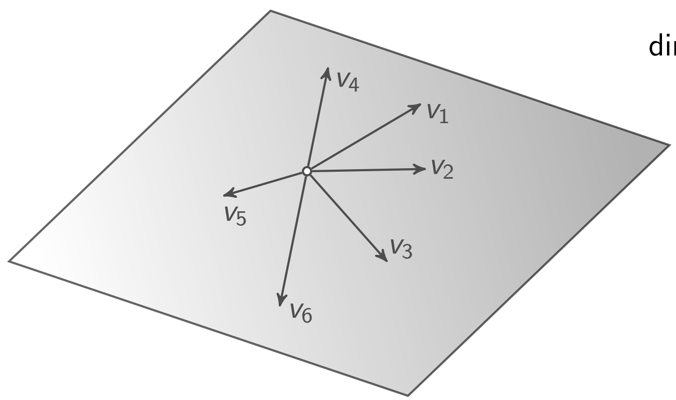
Skup izvodnica za potprostor

$$\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = \mathcal{L}(v_1, v_2, v_4) = \mathcal{L}(v_1, v_2) = \mathcal{L}(v_3, v_6) = \cdots$$

$$\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) \neq \mathcal{L}(v_4, v_6)$$

$$\dim \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = 2$$

$$\dim \mathcal{L}(v_4, v_6) = 1$$



Zadatak 3

Neka je W potprostor od $\mathbb{R}^5$ razapet s vektorima

$$u_1 = (1,2,-1,3,4),\ u_2 = (2,4,-2,6,8),\ u_3 = (1,3,2,2,6),$$
$$u_4 = (1,4,5,1,8),\ u_5 = (2,7,3,3,9).$$

Odredite jednu bazu i dimenziju vektorskog prostora W.

Rješenje

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} /\cdot(-2)\ /\cdot 1\ /\cdot(-3)\ /\cdot(-4) \sim$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} /\cdot(-3)\ /\cdot 1\ /\cdot(-2) \sim$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \Big/\cdot\frac{-5}{4} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{r(A) = 3}$$

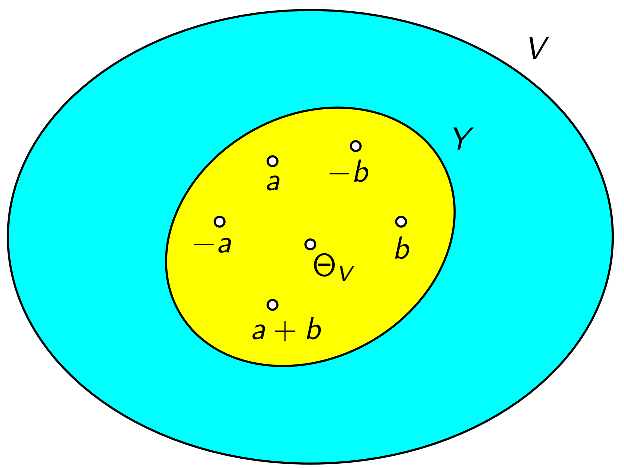
$$A = \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} \end{array} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}_W = \{u_1, u_3, u_5\} \qquad \dim W = 3$$

$$W = \mathcal{L}(u_1, u_2, u_3, u_4, u_5)$$

$$W = \mathcal{L}(u_1, u_3, u_5)$$

Potprostor vektorskog prostora



Karakterizacija vektorskog potprostora

Neka je V vektorski prostor nad poljem F. Neprazan podskup $Y \subseteq V$ je potprostor od V akko za svaki izbor $a, b \in Y$ i $\alpha, \beta \in F$ vrijedi $\alpha a + \beta b \in Y$.

Linearni omotač skupa

Neka je V vektorski prostor nad poljem F, a $S \subseteq V$ bilo koji podskup. Tada je $\mathcal{L}(S)$ najmanji potprostor od V koji sadrži skup S.

Zadatak 4

Zadani su sljedeći podskupovi od $M_2(\mathbb{R})$:

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \geqslant 0 \right\}, \; V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c + 2d = 0, \; a + b - 2c = 0 \right\}$$

a) *Dokažite da U nije potprostor od $M_2(\mathbb{R})$.*

b) *Dokažite da je V potprostor od $M_2(\mathbb{R})$ i odredite mu neku bazu i dimenziju.*

Rješenje

a)

$$U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \geqslant 0 \right\}$$

$$\alpha, \beta \in \mathbb{R}, \; A, B \in U \overset{?}{\Longrightarrow} \alpha A + \beta B \in U$$

$$A = \begin{bmatrix} 2 & -3 \\ \underset{\geqslant 0}{5} & 1 \end{bmatrix} \in U \qquad -2A = \begin{bmatrix} -4 & 6 \\ \underset{< 0}{-10} & -2 \end{bmatrix} \notin U$$

Skup U nije zatvoren na uzimanje linearnih kombinacija svojih elemenata pa stoga U nije potprostor od $M_2(\mathbb{R})$.

$$U \not< M_2(\mathbb{R})$$

b)

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c + 2d = 0, \; a + b - 2c = 0 \right\}$$

$$\alpha, \beta \in \mathbb{R}, \; A, B \in V \overset{?}{\Longrightarrow} \alpha A + \beta B \in V$$

$$A \in V \;\Rightarrow\; A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad c_1 + 2d_1 = 0, \quad a_1 + b_1 - 2c_1 = 0$$

$$B \in V \;\Rightarrow\; B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad c_2 + 2d_2 = 0, \; a_2 + b_2 - 2c_2 = 0$$

$$\alpha A + \beta B = \alpha \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta \cdot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{bmatrix}$$

$$(\alpha c_1 + \beta c_2) + 2(\alpha d_1 + \beta d_2) = \alpha(c_1 + 2d_1) + \beta(c_2 + 2d_2) =$$
$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$(\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2) - 2(\alpha c_1 + \beta c_2) =$$
$$= \alpha(a_1 + b_1 - 2c_1) + \beta(a_2 + b_2 - 2c_2) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

$$\Longrightarrow \; \alpha A + \beta B \in V \;\Longrightarrow\; V < M_2(\mathbb{R})$$

b)

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c + 2d = 0,\ a + b - 2c = 0 \right\}$$

$$\begin{array}{cccc|c} a & \boxed{b} & \boxed{c} & d & \\ \hline 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 & 0 \end{array}$$

$$\left.\begin{aligned} c + 2d &= 0 \\ a + b - 2c &= 0 \end{aligned}\right\} \qquad \begin{aligned} &\rightsquigarrow d = -\tfrac{1}{2}c \\ &\rightsquigarrow a = -b + 2c \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -b + 2c & b \\ c & -\frac{1}{2}c \end{bmatrix} = b \cdot \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$\mathcal{B}_V = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \right\} \qquad \dim V = 2$$