Seminari 8

MATEMATIČKE METODE ZA INFORMATIČARE

Damir Horvat

FOI, Varaždin

Zadatak 1

Odredite dimenziju i jednu bazu vektorskog prostora R svih realnih rješenja homogenog sustava linearnih jednadžbi

$$\begin{aligned} x+2y+z-3t&=0\\ 2x+4y+4z-t&=0\\ 3x+6y+7z+t&=0 \end{aligned}$$

i nadopunite dobivenu bazu do baze za $\mathbb{R}^4$.

Rješenje

x	$\boxed{y}$	$\boxed{z}$	t		
1	2	1	−3	0	+
2	4	4	−1	0	+
3	6	7	1	0	$/\cdot 1$ $/\cdot 3$
10	20	22	0	0	$/:2$
5	10	11	0	0	
3	6	7	1	0	
5	10	11	0	0	
5	10	11	0	0	
3	6	7	1	0	
5	10	11	0	0	$/\cdot \frac{-3}{5}$
3	6	7	1	0	+

x	$\boxed{y}$	$\boxed{z}$	t		
5	10	11	0	0	
0	0	$\frac{2}{5}$	1	0	$/\cdot 5$
5	10	11	0	0	
0	0	2	5	0	

$$\left.\begin{aligned} 5x+10y+11z&=0\\ 2z+5t&=0 \end{aligned}\right\}$$

$$\begin{cases} x=-2u-\frac{11}{5}v\\ y=u\\ z=v & u,v\in\mathbb{R}\\ t=-\frac{2}{5}v \end{cases}$$

$\dim \mathbb{R}^4 = 4$

$$R < \mathbb{R}^4$$

$$\begin{cases} x=-2u-\frac{11}{5}v\\ y=u\\ z=v\\ t=-\frac{2}{5}v \end{cases}$$

$$(x,y,z,t)=\left(-2u-\frac{11}{5}v,\ u,\ v,\ -\frac{2}{5}v\right)=$$

$$=u\cdot(-2,1,0,0)+v\cdot\left(-\frac{11}{5},0,1,-\frac{2}{5}\right)$$

$$\mathcal{B}_R=\left\{(-2,1,0,0),\ \left(-\frac{11}{5},0,1,-\frac{2}{5}\right)\right\} \qquad \dim R = 2$$

$$\left[\begin{array}{cc|cccc} -2 & -\frac{11}{5} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & -\frac{11}{5} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} /\cdot 2 \\ + \\ \\ \\ \end{matrix} \sim$$

$$\sim \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{11}{5} & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & 1 \end{array}\right]\begin{matrix} \\ /\cdot 5 \\ \\ /\cdot 5 \end{matrix} \sim \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -11 & 5 & 10 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 5 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -11 & 5 & 10 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -11 & 5 & 10 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 5 \end{array}\right]\begin{matrix} \\ /\cdot 11 \; /\cdot 2 \\ + \\ + \end{matrix} \sim$$

$$\sim \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 10 & 11 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|cccc} -2 & -\frac{11}{5} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & 1 \end{array}\right]$$

Jedna nadopuna do baze za $\mathbb{R}^4$

$$\left\{(-2,1,0,0),\ \left(-\frac{11}{5},0,1,-\frac{2}{5}\right),\ (1,0,0,0),\ (0,0,1,0)\right\}$$

Zadatak 2

U $\mathcal{P}_3(t)$ zadan je skup $\mathcal{B} = \{t+2,\ t^2,\ t^2+t\}$.

a) *Dokažite da je $\mathcal{B}$ baza za $\mathcal{P}_3(t)$.*

b) *Bez korištenja matrice prijelaza pronađite koordinate polinoma $p(t) = t^2 + 3t - 5$ u bazi $\mathcal{B}$.*

c) *Pomoću matrice prijelaza pronađite koordinate polinoma $p(t) = t^2 + 3t - 5$ u bazi $\mathcal{B}$.*

Rješenje

a) $\mathcal{B} = \{t+2,\ t^2,\ t^2+t\}$ $\qquad$ $\mathcal{B}_{\text{kan}} = \{1, t, t^2\}$

$$t+2 \longrightarrow (2,1,0)$$
$$t^2 \longrightarrow (0,0,1)$$
$$t^2+t \longrightarrow (0,1,1)$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{matrix} /:2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{matrix} /\cdot(-1) \\ + \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ $\mathcal{B}$ je baza za $\mathcal{P}_3(t)$

b) $p(t) = t^2 + 3t - 5$ $\qquad \mathcal{B} = \{t+2,\ t^2,\ t^2+t\}$

$$t^2 + 3t - 5 = \alpha_1 \cdot (t+2) + \alpha_2 \cdot t^2 + \alpha_3 \cdot (t^2+t)$$
$$t^2 + 3t - 5 = (\alpha_2 + \alpha_3)t^2 + (\alpha_1 + \alpha_3)t + 2\alpha_1$$

$$\left.\begin{aligned} \alpha_2 + \alpha_3 &= 1 \\ \alpha_1 + \alpha_3 &= 3 \\ 2\alpha_1 &= -5 \end{aligned}\right\}$$

$$\begin{array}{ccc|cl}
\alpha_1 & \alpha_2 & \alpha_3 & & \\
\hline
0 & 1 & 1 & 1 & \\
1 & 0 & 1 & 3 & /\cdot(-1) \\
2 & 0 & 0 & -5 & \\
\hline
-1 & 1 & 0 & -2 & \\
1 & 0 & 1 & 3 & \\
2 & 0 & 0 & -5 & /:2 \\
\hline
\end{array}
\qquad
\begin{array}{ccc|cl}
\alpha_1 & \alpha_2 & \alpha_3 & & \\
\hline
-1 & 1 & 0 & -2 & \\
1 & 0 & 1 & 3 & \\
1 & 0 & 0 & -\frac{5}{2} & /\cdot(-1) \ /\cdot 1 \\
\hline
0 & 1 & 0 & -\frac{9}{2} & \\
0 & 0 & 1 & \frac{11}{2} & \\
1 & 0 & 0 & -\frac{5}{2} & \\
\end{array}$$

$\alpha_2 = -\frac{9}{2} \qquad \alpha_3 = \frac{11}{2} \qquad \alpha_1 = -\frac{5}{2}$

c) $p(t) = t^2 + 3t - 5$

$$\boxed{X_{\mathcal{B}_1} = T_{\mathcal{B}_1 \to \mathcal{B}_2} X_{\mathcal{B}_2}}$$

$\mathcal{B} = \{t+2,\ t^2,\ t^2+t\} \qquad \mathcal{B}_{\text{kan}} = \{1, t, t^2\}$

$$t+2 \longrightarrow (2,1,0)$$
$$t^2 \longrightarrow (0,0,1)$$
$$t^2+t \longrightarrow (0,1,1)$$

$$X_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathcal{B}_{\text{kan}} \xrightarrow{T} \mathcal{B} \qquad T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$X_{\mathcal{B}_{\text{kan}}} = T_{\mathcal{B}_{\text{kan}} \to \mathcal{B}} X_{\mathcal{B}} \rightsquigarrow X_{\mathcal{B}} = T^{-1} X_{\mathcal{B}_{\text{kan}}}$$

$$\mathcal{B} \xrightarrow{T^{-1}} \mathcal{B}_{\text{kan}}$$

$$X_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} = \begin{array}{l} \begin{bmatrix} -\frac{5}{2} \\ -\frac{9}{2} \\ \frac{11}{2} \end{bmatrix} \end{array} \begin{array}{l} \leftsquigarrow \alpha_1 \\ \leftsquigarrow \alpha_2 \\ \leftsquigarrow \alpha_3 \end{array}$$

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc|ccc l}
\hline
2 & 0 & 0 & 1 & 0 & 0 & \\
1 & 0 & 1 & 0 & 1 & 0 & \\
0 & 1 & 1 & 0 & 0 & 1 & \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & /\cdot(-2) \\
2 & 0 & 0 & 1 & 0 & 0 & \\
0 & 1 & 1 & 0 & 0 & 1 & \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & \\
0 & 0 & -2 & 1 & -2 & 0 & \\
0 & 1 & 1 & 0 & 0 & 1 & \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & \\
0 & 1 & 1 & 0 & 0 & 1 & \\
0 & 0 & -2 & 1 & -2 & 0 & /:(-2) \\
\hline
\end{array}
\qquad
\begin{array}{ccc|ccc l}
\hline
1 & 0 & 1 & 0 & 1 & 0 & \\
0 & 1 & 1 & 0 & 0 & 1 & \\
0 & 0 & 1 & -\frac{1}{2} & 1 & 0 & /\cdot(-1)/\cdot(-1) \\
\hline
1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \\
0 & 1 & 0 & \frac{1}{2} & -1 & 1 & \\
0 & 0 & 1 & -\frac{1}{2} & 1 & 0 & \\
\end{array}$$

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Zadatak 3

Neka je $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$.

a) *Dokažite da je* V *potprostor od* $\mathbb{R}^3$.

b) *Provjerite da je skup* $\mathcal{B}_1 = \{(1,0,1),\ (0,1,1)\}$ *baza za* V.

c) *Dokažite da je* $\mathcal{B}_2 = \{(1,1,2),\ (-2,1,-1)\}$ *također baza za* V.

d) *Odredite matricu prijelaza iz baze* $\mathcal{B}_1$ *u bazu* $\mathcal{B}_2$.

e) *Odredite koordinate vektora* $(-3,2,-1) \in V$ *u bazi* $\mathcal{B}_2$.

Rješenje $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$

a) $\alpha, \beta \in \mathbb{R},\ a, b \in V \overset{?}{\Longrightarrow} \alpha a + \beta b \in V$

$$a \in V \implies a = (x_1, y_1, x_1 + y_1), \quad x_1, y_1 \in \mathbb{R}$$
$$b \in V \implies b = (x_2, y_2, x_2 + y_2), \quad x_2, y_2 \in \mathbb{R}$$

$$\begin{aligned}\alpha a + \beta b &= \alpha \cdot (x_1, y_1, x_1 + y_1) + \beta \cdot (x_2, y_2, x_2 + y_2) = \\ &= \big(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ \alpha(x_1 + y_1) + \beta(x_2 + y_2)\big) = \\ &= \big(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2)\big)\end{aligned}$$

$$\implies \alpha a + \beta b \in V \implies V < \mathbb{R}^3$$

$\mathcal{B}_1 = \{(1,0,1),\ (0,1,1)\}$ $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$

$\mathcal{B}_2 = \{(1,1,2),\ (-2,1,-1)\}$

b) $(x, y, x+y) = x \cdot (1,0,1) + y \cdot (0,1,1)$

Skup $\mathcal{B}_1 = \{(1,0,1),\ (0,1,1)\}$ je skup izvodnica za V, a očito je i linearno nezavisni pa je $\mathcal{B}_1$ jedna baza za V.

$(x, y) \leftrightsquigarrow$ koordinate vektora $(x, y, x+y)$ u bazi $\mathcal{B}_1$

c)
$$\left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right] /\cdot(-1)\ /\cdot(-2) \sim \left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 3 & -1 & 1 \end{array}\right] /\cdot(-1) \sim$$

$$\sim \left[\begin{array}{cc:cc} 1 & -2 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathcal{B}_1$ i $\mathcal{B}_2$ razapinju isti potprostor od $\mathbb{R}^3$, tj. $\mathcal{B}_2$ je također baza za V.

$\mathcal{B}_1 = \{(1,0,1),\ (0,1,1)\}$ $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$

$\mathcal{B}_2 = \{(1,1,2),\ (-2,1,-1)\}$

$\dim V = 2$

d) $\mathcal{B}_1 \xrightarrow{T} \mathcal{B}_2$

$$(1,1,2) = 1 \cdot (1,0,1) + 1 \cdot (0,1,1)$$
$$(-2,1,-1) = -2 \cdot (1,0,1) + 1 \cdot (0,1,1)$$

$$T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

e) $(-3, 2, -1)$ **1. način:** pomoću matrice prijelaza

$$X_{\mathcal{B}_1} = T_{\mathcal{B}_1 \to \mathcal{B}_2} X_{\mathcal{B}_2} \rightsquigarrow X_{\mathcal{B}_2} = T^{-1} X_{\mathcal{B}_1}$$

$$X_{\mathcal{B}_1} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$X_{\mathcal{B}_2} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$X_{\mathcal{B}_2} = \begin{bmatrix} \frac{1}{3} \\ \frac{5}{3} \end{bmatrix}$$

$(x, y, x+y) = x \cdot (1,0,1) + y \cdot (0,1,1)$

$\mathcal{B}_1 = \{(1,0,1),\ (0,1,1)\}$ $V = \{(x, y, x+y) : x, y \in \mathbb{R}\}$

$\mathcal{B}_2 = \{(1,1,2),\ (-2,1,-1)\}$

$\dim V = 2$

2. način: bez korištenja matrice prijelaza

$$(-3, 2, -1) = \alpha_1 \cdot (1,1,2) + \alpha_2 \cdot (-2,1,-1)$$

$$\left.\begin{aligned} \alpha_1 - 2\alpha_2 &= -3 \\ \alpha_1 + \alpha_2 &= 2 \\ 2\alpha_1 - \alpha_2 &= -1 \end{aligned}\right\}$$

$$\begin{array}{cc|c} \alpha_1 & \alpha_2 & \\ \hline 1 & -2 & -3 \\ 1 & 1 & 2 \\ 2 & -1 & -1 \\ \hline 1 & -2 & -3 \\ 0 & 3 & 5 \\ 0 & 3 & 5 \\ \hline \end{array} \quad /\cdot(-1)\ /\cdot(-2)$$

$$\begin{array}{cc|c} \alpha_1 & \alpha_2 & \\ \hline 1 & -2 & -3 \\ 0 & 3 & 5 \\ \hline 1 & 0 & \frac{1}{3} \\ 0 & 3 & 5 \end{array} \quad /\cdot \frac{2}{3}$$

$$\rightsquigarrow \boxed{\alpha_1 = \frac{1}{3}} \qquad \rightsquigarrow \boxed{\alpha_2 = \frac{5}{3}}$$

Geometrijska interpretacija

$\mathcal{B}_1 = \{(1,0,1),\,(0,1,1)\}$

$\mathcal{B}_2 = \{(1,1,2),\,(-2,1,-1)\}$

$$V = \{(x, y, \boxed{x+y}) : x, y \in \mathbb{R}\}$$

$$z = x + y$$

ravnina kroz ishodište $\longrightarrow$ $x + y - z = 0$

- Potprostor V je ravnina kroz ishodište s jednadžbom $x + y - z = 0$.
- Potprostor V je skup svih rješenja homogenog sustava $x + y - z = 0$ koji se sastoji od jedne linearne jednadžbe s tri nepoznanice.
- Vektorska jednadžba ravnine s istaknutim vektorima iz baze $\mathcal{B}_1$

$$\vec{r} = u \cdot (1,0,1) + v \cdot (0,1,1), \quad u, v \in \mathbb{R}$$

- Vektorska jednadžba ravnine s istaknutim vektorima iz baze $\mathcal{B}_2$

$$\vec{r} = u \cdot (1,1,2) + v \cdot (-2,1,-1), \quad u, v \in \mathbb{R}$$

Geometrijska interpretacija

- Je li U potprostor od $\mathbb{R}^3$?

$$U = \{(x, y, \boxed{x+y+1}) : x, y \in \mathbb{R}\}$$

$$z = x + y + 1$$

ravnina koja ne prolazi kroz ishodište $\longrightarrow$ $x + y - z + 1 = 0$

Skup U nije potprostor od $\mathbb{R}^3$ jer ne sadrži nulvektor.

- U n-dimenzionalnom afinom prostoru k-ravnina je zadana s točkom i k linearno nezavisnih vektora. Svaka k-ravnina je rješenje sustava od $n - k$ nezavisnih linearnih jednadžbi s n nepoznanica.
- k-ravnina je potprostor jedino ako prolazi kroz ishodište, tj. ako sadrži nulvektor. Ravnine koje nisu potprostori zovemo **linearnim mnogostrukostima**.