Seminari 9

Matematičke metode za informatičare

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Zadatak 1

U vektorskom prostoru $\mathbb{R}^3$ zadane su dvije baze

$$\mathcal{B}_1 = \{(1,2,-1),\ (0,2,0),\ (1,1,1)\},$$

$$\mathcal{B}_2 = \{(0,2,1),\ (2,2,0),\ (1,-1,1)\}.$$

Vektor $\vec{v} \in \mathbb{R}^3$ u bazi $\mathcal{B}_1$ ima koordinate $(3,-1,2)$.

a) *Odredite koordinate vektora $\vec{v}$ u kanonskoj bazi vektorskog prostora $\mathbb{R}^3$.*

b) *Odredite matricu prijelaza iz baze $\mathcal{B}_2$ u bazu $\mathcal{B}_1$ i pomoću nje odredite koordinate vektora $\vec{v}$ u bazi $\mathcal{B}_2$.*

Rješenje

$$\mathcal{B}_1 = \{(1,2,-1),\ (0,2,0),\ (1,1,1)\}$$

$$\mathcal{B}_2 = \{(0,2,1),\ (2,2,0),\ (1,-1,1)\}$$

a) $\vec{v} = (3,-1,2)_{\mathcal{B}_1}$

$$\mathcal{B}_{\text{kan}} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$$

1. način

$$\vec{v} = (3,-1,2)_{\mathcal{B}_1} = 3\cdot(1,2,-1) + (-1)\cdot(0,2,0) + 2\cdot(1,1,1) =$$

$$= (5,6,-1)_{\mathcal{B}_{\text{kan}}} \rightsquigarrow 5\cdot(1,0,0) + 6\cdot(0,1,0) + (-1)\cdot(0,0,1)$$

2. način

$$X_{\mathcal{B}_{\text{kan}}} = MX_{\mathcal{B}_1}, \quad \mathcal{B}_{\text{kan}} \xrightarrow{M} \mathcal{B}_1$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad X_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$X_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

b) $(1,2,-1) = \alpha_1\cdot(0,2,1) + \alpha_2\cdot(2,2,0) + \alpha_3\cdot(1,-1,1)$

$$\begin{aligned} 2\alpha_2 + \alpha_3 &= 1 \\ 2\alpha_1 + 2\alpha_2 - \alpha_3 &= 2 \\ \alpha_1 + \alpha_3 &= -1 \end{aligned}$$

$$\mathcal{B}_1 = \{(1,2,-1),\ (0,2,0),\ (1,1,1)\}$$

$$\mathcal{B}_2 = \{(0,2,1),\ (2,2,0),\ (1,-1,1)\}$$

$(0,2,0) = \beta_1\cdot(0,2,1) + \beta_2\cdot(2,2,0) + \beta_3\cdot(1,-1,1)$

$$\begin{aligned} 2\beta_2 + \beta_3 &= 0 \\ 2\beta_1 + 2\beta_2 - \beta_3 &= 2 \\ \beta_1 + \beta_3 &= 0 \end{aligned}$$

$$\mathcal{B}_2 \xrightarrow{T} \mathcal{B}_1 \qquad T = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

$(1,1,1) = \gamma_1\cdot(0,2,1) + \gamma_2\cdot(2,2,0) + \gamma_3\cdot(1,-1,1)$

$$\begin{aligned} 2\gamma_2 + \gamma_3 &= 1 \\ 2\gamma_1 + 2\gamma_2 - \gamma_3 &= 1 \\ \gamma_1 + \gamma_3 &= 1 \end{aligned}$$

			α_i	β_i	γ_i	
0	2	1	1	0	1	
2	2	−1	2	2	1	+
(1)	0	1	−1	0	1	$/\cdot(-2)$
0	2	1	1	0	1	+
0	(2)	−3	4	2	−1	$/\cdot(-1)$
1	0	1	−1	0	1	
0	0	(4)	−3	−2	2	$/\cdot\frac{3}{4}$ $/\cdot\frac{-1}{4}$
0	2	−3	4	2	−1	+
1	0	1	−1	0	1	+
0	0	4	−3	−2	2	$/:4$
0	2	0	$\frac{7}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$/:2$
1	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	

			α_i	β_i	γ_i
1	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	0	$\frac{7}{8}$	$\frac{1}{4}$	$\frac{1}{4}$
0	0	1	$-\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$

$$T = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{8} & \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

$\vec{v} = (3, -1, 2)_{\mathcal{B}_1}$

$\mathcal{B}_2 \xrightarrow{T} \mathcal{B}_1$

$$X_{\mathcal{B}_2} = T_{\mathcal{B}_2 \to \mathcal{B}_1} X_{\mathcal{B}_1}$$

$$X_{\mathcal{B}_2} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{8} & \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$X_{\mathcal{B}_2} = \begin{bmatrix} -\frac{1}{4} \\ \frac{23}{8} \\ -\frac{3}{4} \end{bmatrix}$$

Napomena

$\mathcal{B}_2 \xrightarrow{T} \mathcal{B}_1$ $\quad \mathcal{B}_{\text{kan}} \xrightarrow{T_1} \mathcal{B}_1$ $\quad \mathcal{B}_{\text{kan}} \xrightarrow{T_2} \mathcal{B}_2$

$$T_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_2 \xrightarrow{T_2^{-1}} \mathcal{B}_{\text{kan}} \xrightarrow{T_1} \mathcal{B}_1, \qquad \mathcal{B}_2 \xrightarrow{T_2^{-1} T_1} \mathcal{B}_1$$

$$T = T_2^{-1} T_1$$

DZ

$$T_2^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Napomena

$\mathcal{B}_2 \xrightarrow{T} \mathcal{B}_1$ $\quad \mathcal{B}_{\text{kan}} \xrightarrow{T_1} \mathcal{B}_1$ $\quad \mathcal{B}_{\text{kan}} \xrightarrow{T_2} \mathcal{B}_2$

$X_{\mathcal{B}_2} = T X_{\mathcal{B}_1}$ $\quad X_{\mathcal{B}_{\text{kan}}} = T_1 X_{\mathcal{B}_1}$ $\quad X_{\mathcal{B}_{\text{kan}}} = T_2 X_{\mathcal{B}_2}$

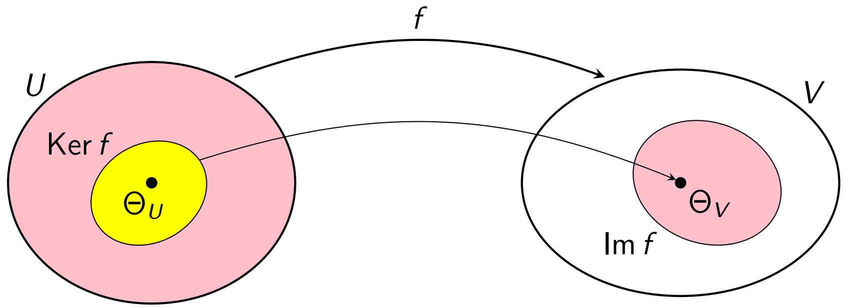
$\vec{v} = (3, -1, 2)_{\mathcal{B}_1}$

$\vec{v} = (5, 6, -1)_{\mathcal{B}_{\text{kan}}}$

$$X_{\mathcal{B}_2} = T_2^{-1} X_{\mathcal{B}_{\text{kan}}}$$

$$X_{\mathcal{B}_2} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{8} & \frac{1}{8} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{23}{8} \\ -\frac{3}{4} \end{bmatrix}$$

Linearni operator



- $f(\alpha a+\beta b)=\alpha f(a)+\beta f(b), \quad \alpha,\beta\in F,\ a,b,\in U$
- $r(f)=\dim(\operatorname{Im} f), \quad d(f)=\dim(\operatorname{Ker} f), \quad r(f)+d(f)=\dim U$
- $f: U\to V$ je injekcija $\iff d(f)=0$
- $f: U\to V$ je surjekcija $\iff r(f)=\dim V \quad (\dim V<\infty)$

Zadatak 2

Zadano je preslikavanje $h: M_2(\mathbb{R})\to\mathbb{R}^2$ s

$$h\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\big(a+d,\ a-b+c\big).$$

a) *Dokažite da je h linearni operator.*

b) *Odredite jezgru, sliku, rang i defekt operatora h.*

c) *Odredite matrični zapis operatora h u paru kanonskih baza.*

Rješenje $h: M_2(\mathbb{R})\to\mathbb{R}^2$ $\quad h\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\big(a+d,\ a-b+c\big)$

a) $h(\alpha A+\beta B)\overset{?}{=}\alpha h(A)+\beta h(B)$

$$h(\alpha A+\beta B)=h\left(\alpha\cdot\begin{bmatrix} a_1 & b_1\\ c_1 & d_1\end{bmatrix}+\beta\cdot\begin{bmatrix} a_2 & b_2\\ c_2 & d_2\end{bmatrix}\right)=$$

$$=h\left(\begin{bmatrix} \alpha a_1+\beta a_2 & \alpha b_1+\beta b_2\\ \alpha c_1+\beta c_2 & \alpha d_1+\beta d_2\end{bmatrix}\right)=$$

$$=\big((\alpha a_1+\beta a_2)+(\alpha d_1+\beta d_2),\ (\alpha a_1+\beta a_2)-(\alpha b_1+\beta b_2)+(\alpha c_1+\beta c_2)\big)=$$

$$=\big(\alpha(a_1+d_1)+\beta(a_2+d_2),\ \alpha(a_1-b_1+c_1)+\beta(a_2-b_2+c_2)\big)=$$

$$=\alpha\cdot\big(a_1+d_1,\ a_1-b_1+c_1\big)+\beta\cdot\big(a_2+d_2,\ a_2-b_2+c_2\big)=$$

$$=\alpha\cdot h\left(\begin{bmatrix} a_1 & b_1\\ c_1 & d_1\end{bmatrix}\right)+\beta\cdot h\left(\begin{bmatrix} a_2 & b_2\\ c_2 & d_2\end{bmatrix}\right)=\alpha h(A)+\beta h(B)$$

b) $h: M_2(\mathbb{R})\to\mathbb{R}^2$ $\quad h\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\big(a+d,\ a-b+c\big)$

Ker h

$$h\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\Theta_{\mathbb{R}^2}$$

$$\big(a+d,\ a-b+c\big)=(0,0)$$

$$\left.\begin{aligned} a+d&=0\\ a-b+c&=0\end{aligned}\right\}\ \begin{aligned} &\rightsquigarrow d=-a\\ &\rightsquigarrow c=-a+b\end{aligned}$$

$$\operatorname{Ker} h=\left\{\begin{bmatrix} a & b\\ -a+b & -a\end{bmatrix} : a,b\in\mathbb{R}\right\}$$

$$\mathcal{B}_{\operatorname{Ker} h}=\left\{\begin{bmatrix} 1 & 0\\ -1 & -1\end{bmatrix},\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}\right\}$$

$$\begin{bmatrix} a & b\\ -a+b & -a\end{bmatrix}=a\cdot\begin{bmatrix} 1 & 0\\ -1 & -1\end{bmatrix}+b\cdot\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$$

$d(h)=2\neq 0$ $\downarrow$ h nije injekcija

Im h $\quad r(h)+d(h)=\dim M_2(\mathbb{R})$ $\quad \mathcal{B}_{\operatorname{Im} h}=\{(1,0),(0,1)\}$

$r(h)=4-2=2 \qquad r(h)=2=\dim\mathbb{R}^2$ $\longrightarrow$ h je surjekcija $\qquad \operatorname{Im} h=\mathbb{R}^2$

c) $h : M_2(\mathbb{R}) \to \mathbb{R}^2$ $\qquad h\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d,\ a-b+c)$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \qquad \mathcal{B}_2 = \{(1,0),(0,1)\}$$

$$h\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = (1,1) = 1\cdot(1,0) + 1\cdot(0,1) \qquad H = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

$$h\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = (0,-1) = 0\cdot(1,0) + (-1)\cdot(0,1)$$

$$h\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = (0,1) = 0\cdot(1,0) + 1\cdot(0,1)$$

$$h\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = (1,0) = 1\cdot(1,0) + 0\cdot(0,1)$$

Zadatak 3

Zadano je preslikavanje $f : \mathbb{R}^4 \to \mathbb{R}^2$ *s*

$$f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v).$$

a) *Dokažite da je* f *linearni operator.*

b) *Odredite jezgru, sliku, rang i defekt operatora* f.

c) *Odredite matrični prikaz operatora* f *u paru kanonskih baza.*

d) *Odredite matrični prikaz operatora* f *u paru baza*

$$\mathcal{A} = \{(1,0,0,0),\ (1,2,0,0),\ (1,2,3,0),\ (1,2,3,4)\},$$

$$\mathcal{B} = \{(1,10),\ (1,11)\}.$$

e) *Odredite sliku vektora* $(1,0,-1,8)$.

Rješenje $\qquad f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

a) $f(\alpha a + \beta b) \stackrel{?}{=} \alpha f(a) + \beta f(b) \qquad f : \mathbb{R}^4 \to \mathbb{R}^2$

$$f(\alpha a + \beta b) = f\big(\alpha\cdot(x_1,y_1,u_1,v_1) + \beta\cdot(x_2,y_2,u_2,v_2)\big) =$$

$$= f(\alpha x_1 + \beta x_2,\ \alpha y_1 + \beta y_2,\ \alpha u_1 + \beta u_2,\ \alpha v_1 + \beta v_2) =$$

$$= \big((\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) - (\alpha u_1 + \beta u_2) - (\alpha v_1 + \beta v_2),$$

$$-(\alpha x_1 + \beta x_2) - 2(\alpha y_1 + \beta y_2) + (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2)\big) =$$

$$= \big(\alpha(x_1 + 2y_1 - u_1 - v_1) + \beta(x_2 + 2y_2 - u_2 - v_2),$$

$$\alpha(-x_1 - 2y_1 + u_1 + v_1) + \beta(-x_2 - 2y_2 + u_2 + v_2)\big) =$$

$$= \alpha\cdot(x_1 + 2y_1 - u_1 - v_1,\ -x_1 - 2y_1 + u_1 + v_1) +$$

$$+ \beta\cdot(x_2 + 2y_2 - u_2 - v_2,\ -x_2 - 2y_2 + u_2 + v_2) =$$

$$= \alpha f(x_1,y_1,u_1,v_1) + \beta f(x_2,y_2,u_2,v_2) = \alpha f(a) + \beta f(b)$$

b) $\qquad f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

Ker f $\qquad f : \mathbb{R}^4 \to \mathbb{R}^2$

$$f(x,y,u,v) = \Theta_{\mathbb{R}^2}$$

$$(x+2y-u-v,\ -x-2y+u+v) = (0,0)$$

$$\left.\begin{aligned} x+2y-u-v &= 0 \\ -x-2y+u+v &= 0 \end{aligned}\right\}$$

x	y	u	v	
(1)	2	−1	−1	0 / ·1
−1	−2	1	1	0 ← +
1	2	−1	−1	0
0	0	0	0	0
1	2	−1	−1	0

$$x+2y-u-v = 0 \rightsquigarrow x = -2y+u+v$$

$$\operatorname{Ker} f = \{(-2y+u+v, y, u, v) : y,u,v \in \mathbb{R}\}$$

$$(-2y+u+v, y, u, v) = y\cdot(-2,1,0,0) + u\cdot(1,0,1,0) + v\cdot(1,0,0,1)$$

$$\mathcal{B}_{\operatorname{Ker} f} = \{(-2,1,0,0),\ (1,0,1,0),\ (1,0,0,1)\}$$

$d(f) = 3 \neq 0$ → f nije injekcija

$$f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$$

Im f

$$f:\mathbb{R}^4 \to \mathbb{R}^2$$

$$r(f) + d(f) = \dim \mathbb{R}^4$$
$$r(f) + 3 = 4$$
$$r(f) = 1$$

$$r(f) \neq \dim \mathbb{R}^2 \longrightarrow f \text{ nije surjekcija}$$

$$(x+2y-u-v,\ -x-2y+u+v) =$$
$$= x\cdot(1,-1) + y\cdot(2,-2) + u\cdot(-1,1) + v\cdot(-1,1)$$

$$\begin{bmatrix} \textcircled{1} & 2 & -1 & -1 \\ -1 & -2 & 1 & 1 \end{bmatrix} /\cdot 1 \ (+) \sim \begin{bmatrix} \textcircled{1} & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{B}_{\mathrm{Im}\,f} = \{(1,-1)\}$$

c) $$f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$$

$$f:\mathbb{R}^4 \to \mathbb{R}^2$$

$$F_{(\mathcal{A}_{\text{kan}},\mathcal{B}_{\text{kan}})} = \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & -2 & 1 & 1 \end{bmatrix}$$

$$\mathcal{A}_{\text{kan}} = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$$
$$\mathcal{B}_{\text{kan}} = \{(1,0),(0,1)\}$$

$$f(1,0,0,0) = (1,-1) = 1\cdot(1,0) + (-1)\cdot(0,1)$$
$$f(0,1,0,0) = (2,-2) = 2\cdot(1,0) + (-2)\cdot(0,1)$$
$$f(0,0,1,0) = (-1,1) = -1\cdot(1,0) + 1\cdot(0,1)$$
$$f(0,0,0,1) = (-1,1) = -1\cdot(1,0) + 1\cdot(0,1)$$

d) **1. način**

$$f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$$

$$\mathcal{A} = \{(1,0,0,0),(1,2,0,0),(1,2,3,0),(1,2,3,4)\} \qquad f:\mathbb{R}^4 \to \mathbb{R}^2$$

$$\mathcal{B} = \{(1,10),(1,11)\}$$

$$f(1,0,0,0) = (1,-1) = \alpha_1\cdot(1,10) + \alpha_2\cdot(1,11)$$
$$\alpha_1 + \alpha_2 = 1$$
$$10\alpha_1 + 11\alpha_2 = -1$$

$$f(1,2,0,0) = (5,-5) = \beta_1\cdot(1,10) + \beta_2\cdot(1,11)$$
$$\beta_1 + \beta_2 = 5$$
$$10\beta_1 + 11\beta_2 = -5$$

$$F_{(\mathcal{A},\mathcal{B})} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{bmatrix}$$

$$f(1,2,3,0) = (2,-2) = \gamma_1\cdot(1,10) + \gamma_2\cdot(1,11)$$
$$\gamma_1 + \gamma_2 = 2$$
$$10\gamma_1 + 11\gamma_2 = -2$$

$$f(1,2,3,4) = (-2,2) = \delta_1\cdot(1,10) + \delta_2\cdot(1,11)$$
$$\delta_1 + \delta_2 = -2$$
$$10\delta_1 + 11\delta_2 = 2$$

		α_i	β_i	γ_i	δ_i	
①	1	1	5	2	−2	/·(−10)
10	11	−1	−5	−2	2	+
1	1	1	5	2	−2	+
0	①	−11	−55	−22	22	/·(−1)
1	0	12	60	24	−24	
0	1	−11	−55	−22	22	

$$\left.\begin{matrix} \alpha_1 + \alpha_2 = 1 \\ 10\alpha_1 + 11\alpha_2 = -1 \end{matrix}\right\}$$
$$\left.\begin{matrix} \beta_1 + \beta_2 = 5 \\ 10\beta_1 + 11\beta_2 = -5 \end{matrix}\right\}$$
$$\left.\begin{matrix} \gamma_1 + \gamma_2 = 2 \\ 10\gamma_1 + 11\gamma_2 = -2 \end{matrix}\right\}$$
$$\left.\begin{matrix} \delta_1 + \delta_2 = -2 \\ 10\delta_1 + 11\delta_2 = 2 \end{matrix}\right\}$$

$$F_{(\mathcal{A},\mathcal{B})} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{bmatrix}$$

$$F_{(\mathcal{A},\mathcal{B})} = \begin{bmatrix} 12 & 60 & 24 & -24 \\ -11 & -55 & -22 & 22 \end{bmatrix}$$

2. način $f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

$f: \mathbb{R}^4 \to \mathbb{R}^2$

$\mathcal{A} = \{(1,0,0,0),(1,2,0,0),(1,2,3,0),(1,2,3,4)\}$

$\mathcal{A}_{\text{kan}} = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

$\mathcal{B} = \{(1,10),(1,11)\} \quad \mathcal{B}_{\text{kan}} = \{(1,0),(0,1)\}$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$F_{(\mathcal{A},\mathcal{B})} = T^{-1} F_{(\mathcal{A}_{\text{kan}},\mathcal{B}_{\text{kan}})} S$$

$$\mathcal{A}_{\text{kan}} \xrightarrow{S} \mathcal{A}, \quad \mathcal{B}_{\text{kan}} \xrightarrow{T} \mathcal{B}$$

$$T = \begin{bmatrix} 1 & 1 \\ 10 & 11 \end{bmatrix}$$

$$F_{(\mathcal{A},\mathcal{B})} = \frac{1}{1} \begin{bmatrix} 11 & -1 \\ -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$F_{(\mathcal{A},\mathcal{B})} = \begin{bmatrix} 12 & 60 & 24 & -24 \\ -11 & -55 & -22 & 22 \end{bmatrix}$$

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e) **1. način** $f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

$f: \mathbb{R}^4 \to \mathbb{R}^2$

$x \quad y \quad u \quad v$

$f(1,0,-1,8) = (1+2\cdot 0-(-1)-8,\ -1-2\cdot 0+(-1)+8)$

$f(1,0,-1,8) = (-6,6)$

2. način

$Y_{\mathcal{B}} = F_{(\mathcal{A},\mathcal{B})} X_{\mathcal{A}}$

$$Y_{\mathcal{B}_{\text{kan}}} = F_{(\mathcal{A}_{\text{kan}},\mathcal{B}_{\text{kan}})} X_{\mathcal{A}_{\text{kan}}}$$

$$Y_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} 1 & 2 & -1 & -1 \\ -1 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 8 \end{bmatrix} \qquad Y_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$$

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3. način $f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

$f: \mathbb{R}^4 \to \mathbb{R}^2$

$(1,0,-1,8) \rightsquigarrow$ moramo pronaći koordinate u bazi $\mathcal{A}$

$Y_{\mathcal{B}} = F_{(\mathcal{A},\mathcal{B})} X_{\mathcal{A}}$

DZ

$$S^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\mathcal{A}_{\text{kan}} \xrightarrow{S} \mathcal{A}$$

$$\mathcal{A} \xrightarrow{S^{-1}} \mathcal{A}_{\text{kan}}$$

$$X_{\mathcal{A}} = S^{-1} X_{\mathcal{A}_{\text{kan}}}$$

$$X_{\mathcal{A}} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ -\frac{7}{3} \\ 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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$f(x,y,u,v) = (x+2y-u-v,\ -x-2y+u+v)$

$f: \mathbb{R}^4 \to \mathbb{R}^2$

$\mathcal{B} = \{(1,10),\ (1,11)\}$

$Y_{\mathcal{B}} = F_{(\mathcal{A},\mathcal{B})} X_{\mathcal{A}}$

$$Y_{\mathcal{B}} = \begin{bmatrix} 12 & 60 & 24 & -24 \\ -11 & -55 & -22 & 22 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{3} \\ -\frac{7}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} -72 \\ 66 \end{bmatrix}$$

$$(-72,66)_{\mathcal{B}} = -72\cdot(1,10) + 66\cdot(1,11) = (-6,6)_{\mathcal{B}_{\text{kan}}}$$

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Domaća zadaća

Zadatak 4

Zadano je preslikavanje $f : \mathbb{R}^3 \to \mathbb{R}^3$ *definirano s*

$$f(x,y,z) = (x-2y,\ z,\ x+y).$$

a) *Dokažite da je* f *linearni operator.*

b) *Odredite jezgru, sliku, rang i defekt operatora* f.

c) *Odredite matrični prikaz operatora* f *u kanonskoj bazi.*

d) *Odredite matrični prikaz operatora* f *u bazi*

$$\mathcal{B} = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}.$$

e) *Odredite sliku vektora* $(2,1,-3)$.

Rješenje

b) $\operatorname{Ker} f = \{(0,0,0)\},\ d(f) = 0,\ \operatorname{Im} f = \mathbb{R}^3,\ r(f) = 3$

Baza za $\operatorname{Ker} f$ ne postoji jer je jezgra u ovom slučaju trivijalni vektorski prostor.

$\mathcal{B}_{\operatorname{Im} f} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

Linearni operator f je izomorfizam.

c)

$$F_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

d) Zadatak riješite na dva načina: bez korištenja matrice prijelaza i pomoću matrice prijelaza.

$$F_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ 1 & 2 & 2 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_{\mathcal{B}} = T^{-1} F_{\mathcal{B}_{\text{kan}}} T \qquad \mathcal{B}_{\text{kan}} \xrightarrow{T} \mathcal{B}$$

e) 1. način (uvrštavanje u formulu kojom je zadan linearni operator)

$f(2,1,-3) = (0,-3,3)$

2. način $Y_{\mathcal{B}_{\text{kan}}} = F_{\mathcal{B}_{\text{kan}}} X_{\mathcal{B}_{\text{kan}}}$

$$Y_{\mathcal{B}_{\text{kan}}} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

3. način $Y_{\mathcal{B}} = F_{\mathcal{B}} X_{\mathcal{B}}, \quad X_{\mathcal{B}} = T^{-1} X_{\mathcal{B}_{\text{kan}}}$

$$Y_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$(3,-6,3)_{\mathcal{B}} = 3\cdot(1,0,0) + (-6)\cdot(1,1,0) + 3\cdot(1,1,1) = (0,-3,3)_{\mathcal{B}_{\text{kan}}}$$